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WEAK SHADOWING PROPERTY FOR FLOWS ON ORIENTED SURFACES

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ABSTRACT. In this paper, we prove that on an oriented smooth closed surface, a vector field has the (C^1) robustly weak shadowing property if and only if it is structurally stable.

1. Introduction

The theory of the tracking of approximate trajectories (shadowing property) is an important topic of the global theory of dynamical systems. The shadowing property means that, near a sufficiently precise approximate orbit of a dynamical system, there exists an exact orbit. It is closely related to the theory of hyperbolicity of dynamics. See, for example, the monographs [8], [9] for more details.

One of the first shadowing results was proved by Anosov [1] and Bowen [2]. They showed that a diffeomorphism has the shadowing property in a neighborhood of a hyperbolic set. Later, it was proved that a structurally stable diffeomorpism has the shadowing property on the whole manifold [7], [14], [18]. For the inverse problem, Sakai proved in [17] that the C^1 interior of the set of diffeomorphisms with the shadowing property coincides with the set of structurally stable diffeomorphisms. However, structural stability is not equivalent to shadowing because there exists an example [11] of a diffeomorphism which is not structurally stable but has the shadowing property.

Other types of the shadowing property are also studied, for example, the weak shadowing property (see [9] for more details). It was proved in [12], [15], [16] that the diffeomorphism belonging to the set of diffeomorphisms on a smooth closed surface having the weak shadowing property is Ω-stable (it is not true for high dimensions), but is not structurally stable in general.

Similar questions were posed for flows generated by vector fields. As in the case of diffeomorphisms, it is well known that a vector field has the shadowing property in a neighborhood of a hyperbolic set [8], [9] and a structurally stable vector field has the shadowing property on the whole manifold [9], [10]. However, there are some essential differences between the cases of diffeomorphisms and vector fields. One of the main differences is the possibility of accumulation of recurrent orbits to a singularity in vector fields. Pilyugin and Tikhomirov gave an example of a non-structurally stable vector field containing singularities on a 4-manifold which is in the C^1-interior of the set of vector fields with the shadowing property [13].

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The problem of the description of vector fields having the shadowing property was also studied. A vector field belonging to the C^1-interior of the set of vector fields with the shadowing property is Ω-stable [4], and it is structurally stable if there is no singularity [5] or there is no non-transverse intersection of stable and unstable manifolds of two hyperbolic singularities [13]. However, it is not clear how to characterize systems with the weak shadowing property.

In the present work we study the C^1-interior of the set of vector fields on an oriented smooth closed surface with the weak shadowing property, and prove that vector fields in such a C^1-interior are structurally stable.

Recently in [6] it was shown that there exist vector fields on 2-dimensional Klein manifolds which are contained in the C^1-interior of the set of vector fields with the weak shadowing property but are not structurally stable. This example and the present result show that the behavior of the weak shadowing property is quite different for vector fields and diffeomorphisms.

2. Preliminaries and the main result

In this section, we will recall some basic properties and state the main result.

2.1. **Definitions and the main result.** Let M be a compact smooth Riemannian manifold without boundary. Denote by $\mathcal{X}^1(M)$ the set of C^1 vector fields on M. For any $X \in \mathcal{X}^1(M)$, X generates a C^1 flow

$$\phi_t = \phi_{X,t} : M \to M,\ t \in \mathbb{R}.$$

Denote by $\mathrm{Orb}(x) = \mathrm{Orb}_X(x) = \phi_{(-\infty,+\infty)}(x)$ the X-orbit of x, $\mathrm{Orb}^+(x) = \phi_{[0,+\infty)}(x)$ and $\mathrm{Orb}^-(x) = \phi_{(-\infty,0]}(x)$ the positive and the negative orbit of x.

For any $d > 0$, a map $g : \mathbb{R} \to M$ (not necessarily continuous) is called a *d-pseudo orbit*, if

$$\mathrm{dist}(g(t+\tau), \phi_\tau(g(t))) < d$$

for all $t \in \mathbb{R}$ and $\tau \in [0,1]$.

We say that a vector field $X \in \mathcal{X}^1(M)$ has the *weak shadowing property* if for any $\varepsilon > 0$, there exists a constant $d > 0$ such that, for any d-pseudo orbit g of X, there exists a point $x \in M$ satisfying

$$g(\mathbb{R}) \subset U_\varepsilon(\mathrm{Orb}(x)),$$

where $U_\varepsilon(\mathrm{Orb}(x))$ is the ε-neighborhood of the orbit of x. We write $X \in \mathrm{WS}(M)$ if $X \in \mathcal{X}^1(M)$ has the weak shadowing property. Moreover, we say that a vector field X has the C^1 *robustly weak shadowing property* if $X \in \mathrm{Int}^1(\mathrm{WS}(M))$, where $\mathrm{Int}^1(\mathrm{WS}(M))$ is the C^1-interior of $\mathrm{WS}(M)$.

Denote by $\mathrm{SS}(M)$ the set of structurally stable vector fields on M.

The main result of this paper is as follows.

Theorem 2.1. *Let M^2 be an oriented smooth closed surface. Then a vector field $X \in \mathcal{X}^1(M^2)$ has the C^1 robustly weak shadowing property if and only if X is structurally stable. That is,*

$$\mathrm{Int}^1(\mathrm{WS}(M^2)) = \mathrm{SS}(M^2).$$

2.2. **Kupka-Smale vector fields.** Recall that a vector field $X \in \mathcal{X}^1(M)$ is called a *Kupka-Smale* vector field if X satisfies

(KS1) every singularity and every periodic orbit of X is hyperbolic; and

(KS2) every intersection between stable and unstable manifolds of singularities and periodic orbits is transverse.

Denote by $\mathrm{KS}(M)$ the set of Kupka-Smale vector fields on M. The following well-known result works as the basis of the proof of Theorem 2.1.

Theorem 2.2 ([3]). $\mathrm{Int}^1(\mathrm{KS}(M)) = \mathrm{SS}(M)$.

We will prove Theorem 2.1 by using the following propositions.

Proposition 2.3. *Let M^2 be an oriented smooth closed surface. If $X \in \mathrm{Int}^1(\mathrm{WS}(M^2))$ has the C^1 robustly weak shadowing property, then X satisfies* (KS1).

Proposition 2.4. *Let M^2 be an oriented smooth closed surface. If $X \in \mathrm{Int}^1(\mathrm{WS}(M^2))$ has the C^1 robustly weak shadowing property, then X satisfies* (KS2).

Proof of Theorem 2.1. It is shown in [10] that

$$\mathrm{SS}(M) \subset \mathrm{WS}(M)$$

for any finitely dimensional manifold M. Then we have

$$\mathrm{SS}(M) \subset \mathrm{Int}^1(\mathrm{WS}(M))$$

since $\mathrm{SS}(M)$ is open. Thus we only need to prove that $\mathrm{Int}^1(\mathrm{WS}(M^2)) \subset \mathrm{SS}(M^2)$.

Let $X \in \mathrm{Int}^1(\mathrm{WS}(M^2))$ be a vector field having the C^1 robustly weak shadowing property on an oriented surface M^2. According to Propositions 2.3 and 2.4, we know that $X \in \mathrm{KS}(M^2)$ is a Kupka-Smale vector field. That is,

$$\mathrm{Int}^1(\mathrm{WS}(M^2)) \subset \mathrm{KS}(M^2).$$

Thus by Theorem 2.2 we have

$$\mathrm{Int}^1(\mathrm{WS}(M^2)) \subset \mathrm{Int}^1(\mathrm{KS}(M^2)) = \mathrm{SS}(M^2).$$

□

2.3. **The C^1 connecting lemma.** We will use the following version of the C^1 connecting lemma.

Lemma 2.5 ([19]). *Let $X \in \mathcal{X}^1(M)$ be a C^1 vector field on M, and $z \in M$ be neither singular nor periodic of X. Then for any C^1 neighborhood $\mathcal{U}$ of X, there exist three numbers $\rho > 1$, $T > 1$ and $\delta_0 > 0$ such that for any $0 < \delta \le \delta_0$ and any two points x, y outside the tube $\Delta = \bigcup_{t\in[0,T]} B(\phi_t(z), \delta)$ (or $\Delta = \bigcup_{t\in[-T,0]} B(\phi_t(z), \delta)$), if the positive X-orbit of x and the negative X-orbit of y both hit $B(z, \delta/\rho)$, then there is $Y \in \mathcal{U}$ with $Y = X$ outside Δ such that y is on the positive Y-orbit of x.*

3. Hyperbolicity of singularities and periodic orbits

From now on we always assume that M^2 is a smooth closed surface. In this section we will prove Proposition 2.3. That is, we will prove that if X is a vector field having the C^1 robustly weak shadowing property, then every singularity and every periodic orbit of X is hyperbolic.

Lemma 3.1. *If $X \in \mathrm{Int}^1(\mathrm{WS}(M^2))$, then every singularity σ is hyperbolic.*

Proof. Suppose to the contrary that there exists a vector field $X \in \mathrm{Int}^1(\mathrm{WS}(M^2))$ having a non-hyperbolic singularity σ. Note that the vector field X still belongs to the open set $\mathrm{Int}^1(\mathrm{WS}(M^2))$ under C^1 small enough perturbations. Thus up to a C^1 small perturbation, we may assume that $X \in \mathrm{Int}^1(\mathrm{WS}(M^2))$ is linear in a neighborhood $U = U_r(\sigma)$ of σ on a proper chart, where $r > 0$ is small.

Since σ is non-hyperbolic, we have that $\mathrm{D}X(\sigma)$ has an eigenvalue λ with $\mathrm{Re}(\lambda) = 0$, where $\mathrm{D}X(\sigma)$ denotes the Jacobi matrix of $X(\sigma)$. Denote by μ the other eigenvalue of $\mathrm{D}X(\sigma)$. We consider the following two cases.

Case 1. $\lambda \in \mathbb{R}$.

In this case we have $\mu \in \mathbb{R}$. Up to a C^1 small perturbation, we may assume that $\mu \neq 0$. Without loss of generality, we assume that $\mu > 0$.

Denote by E^1 and E^2 the 1-dimensional eigenspaces of $\mathrm{D}X(\sigma)$ with respect to λ and μ respectively. Changing the Riemannian metric if necessary, we assume that E^1 and E^2 are orthogonal. We introduce an orthogonal coordinate system $x = (x^1, x^2)$ in U with respect to the splitting

$$T_\sigma M^2 = E^1 \oplus E^2.$$

For any point $x = (x^1, x^2)$ in U and any $t \in \mathbb{R}$, if $\phi_{[0,t]}(x) = \{\phi_\tau(x) : \tau \in [0,t]\} \subset U$, then

$$\phi_t(x) = (x^1 e^{\lambda t}, x^2 e^{\mu t}) = (x^1, x^2 e^{\mu t}).$$

Thus for any $x = (x^1, x^2) \in U$, we have

$$\mathrm{Orb}^-(x) = \{x^1\} \times [x^2, 0) \quad \text{or} \quad \{x^1\} \times (0, x^2]. \tag{3.1}$$

Take $\varepsilon \in (0, r/8)$. Let d correspond to ε with respect to the weak shadowing property. Choose $n \in \mathbb{Z}^+$ such that $r/2n < d$.

It is easy to see that

$$g(t) = \begin{cases} (0,0) = \sigma, & t \leq 0; \\ (kr/2n, 0), & k-1 < t \leq k, k = 1, 2, \ldots, n-1; \\ (r/2, 0), & t > n-1, \end{cases}$$

is a d-pseudo orbit of X. Thus there is a point $y \in M^2$ such that

$$g(\mathbb{R}) \subset U_\varepsilon(\mathrm{Orb}(y)).$$

Particularly, there exist $y_1, y_2 \in \mathrm{Orb}(y)$ such that

$$\mathrm{dist}(y_1, (0,0)) \leq \varepsilon \quad \text{and} \quad \mathrm{dist}(y_2, (0, r/2)) \leq \varepsilon.$$

But from (3.1) we know that

$$\mathrm{Orb}^-(y_1) \cap \mathrm{Orb}^-(y_2) = \emptyset.$$

It contradicts the fact that y_1 and y_2 are contained in the same orbit.

Case 2. $\lambda \notin \mathbb{R}$.

Denote $\lambda = bi$, $b \in \mathbb{R}$. Thus the conjugated eigenvalue $\mu = -bi$.

Choose an adapted coordinate system such that the matrix of $\mathrm{D}X(\sigma)$ is

$$\mathrm{D}X(\sigma) = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}.$$

Changing the Riemannian metric if necessary, we assume that the coordinate axes are orthogonal.

Then for any point $x = (x^1, x^2) \in U$, the flow ϕ_t in U can be represented by

$$\phi_t((x^1, x^2)) = (x^1 \cos bt - x^2 \sin bt, x^1 \sin bt + x^2 \cos bt). \tag{3.2}$$

That is, every orbit in U is a periodic circle with the same period $2\pi/b$.

Take $\varepsilon \in (0, r/8)$. Let d correspond to ε with respect to the weak shadowing property. Choose $m, n \in \mathbb{Z}^+$ such that $2m\pi/b > 1$ and $r/2n < d$.

We create a d-pseudo orbit as follows:

$$g(t) = \begin{cases} (0,0) = \sigma, & t \le 0; \\ (-\frac{kr}{2n} \sin bt, \frac{kr}{2n} \cos bt), & (k-1)2m\pi/b < t \le k2m\pi/b, k{=}1, 2, \ldots, n-1; \\ (-\frac{r}{2} \sin bt, \frac{r}{2} \cos bt), & t > (n-1)2m\pi/b. \end{cases}$$

The image of the pseudo orbit g is a bunch of concentric circles.

Then there is a point $y \in M^2$ such that

$$g(\mathbb{R}) \subset U_\varepsilon(\mathrm{Orb}(y)).$$

Similar to Case 1, there exist $y_1, y_2 \in \mathrm{Orb}(y)$ such that

$$\mathrm{dist}(y_1, (0,0)) \le \varepsilon \quad \text{and} \quad \mathrm{dist}(y_2, (0, r/2)) \le \varepsilon.$$

But from (3.2) we know that

$$\mathrm{Orb}(y_1) \cap \mathrm{Orb}(y_2) = \emptyset.$$

It is a contradiction.

We have finished the proof of this lemma. □

Lemma 3.2. *If $X \in \mathrm{Int}^1(\mathrm{WS}(M^2))$, then every periodic orbit of X is hyperbolic.*

Proof. The idea of the proof is similar to Case 2 in Lemma 3.1. Suppose to the contrary that there is a vector field $X \in \mathrm{Int}^1(\mathrm{WS}(M^2))$ having a non-hyperbolic periodic orbit γ. Take $p \in \gamma$ and a local coordinate system center at p.

Take a small 1-dimensional segment I in the coordinate system containing p which is transverse to the flow ϕ_t. Denote by P the Poincaré map (the first return map) of I. Since γ is non-hyperbolic, we have that $\mathrm{D}P(p) = \pm 1$. Up to a C^1 small perturbation, we may assume that $P(x) = x$ or $P^2(x) = x$ in a small segment $I \cap U_r(p)$ for some $r > 0$ (see [13] for more details on the perturbations). Thus the orbits of the points on $I \cap U_r(p)$ is a family of closed curves.

Take $q \in I \cap U_r(p)$ with $\mathrm{dist}(p, q) = r/2$. Similar to the proof of Case 2 in Lemma 3.1, take $\varepsilon \in (0, r/8)$. For the corresponding $d > 0$ we may create a d-pseudo orbit $g(t)$ which is a bunch of closed orbits containing p and q. $g(t)$ cannot be ε-weakly shadowed by any X-orbit. This contradiction proves the lemma. □

Proof of Proposition 2.3. This proposition can be concluded immediately by Lemmas 3.1 and 3.2. □

4. Existence of homoclinic connections

Recall that a homoclinic connection Γ of a singularity σ is the closure of the orbit of a regular point which is contained in both the stable and unstable manifolds of σ. In this section, we will prove that if $X \in \mathrm{Int}^1(\mathrm{WS}(M^2))$, then there is no homoclinic connection of singularities of X.

4.1. **Local linear model.** We first give some properties of a homoclinic connection of a singularity when the vector field is linear near the singularity.

Let σ be a hyperbolic saddle type singularity of X exhibiting a homoclinic connection

$$\Gamma \subset W^{X,s}(\sigma) \cap W^{X,u}(\sigma).$$

Denote by

$$\lambda < 0 < \mu$$

the eigenvalues of $\mathrm{D}X(\sigma)$ and

$$T_\sigma M = E^s_\sigma \oplus E^u_\sigma$$

the hyperbolic splitting on σ.

The *saddle value* of σ is the sum

$$\mathrm{SV}(\sigma) = \lambda + \mu.$$

We assume that X satisfies:

A1. $\mathrm{SV}(\sigma) < 0$;
A2. X is linear in a small neighborhood $U = U_r(\sigma)$ of σ, where $r > 0$ is small enough such that $\Gamma \cap U_{2r}(\sigma)$ is contained in the local stable manifold $W^{X,s}_{\mathrm{loc}}(\sigma)$ and the local unstable manifold $W^{X,u}_{\mathrm{loc}}(\sigma)$ of σ; and
A3. E^s_σ and E^u_σ are orthogonal.

We introduce an orthogonal coordinate system $x = (x^1, x^2)$ in U with respect to the hyperbolic splitting. Then the flow ϕ_t on U can be represented by

$$\phi_t((x^1, x^2)) = (x^1 e^{\lambda t}, x^2 e^{\mu t}). \tag{4.1}$$

Assume that $O_s = (r/2, 0) \in \Gamma \cap E^s_\sigma$ and $O_u = (0, r/2) \in \Gamma \cup E^u_\sigma$. For any $a, b \in (0, r/2]$, denote by

$$I^s_a = \{r/2\} \times [-a, a] \quad \text{and} \quad I^u_b = [-b, b] \times \{r/2\}$$

two small segments in U which are orthogonal to E^s_σ and E^u_σ at O_s and O_u respectively.

Denote by $Q : I^u_b \to I^s_a$ the Poincaré map from I^u_b to I^s_a, and $\tau_Q(y)$ the minimal positive t such that $\phi_t(y) = Q(y)$. Reduce $b > 0$ if necessary, we can assume that $Q(I^u_b) \subset I^s_a$ and τ_Q is continuous in I^u_b.

We have the following lemma immediately by the differentiability.

Lemma 4.1. *There exist two positive constants $\beta \in (0, b]$ and $B > 0$ such that, for any $y \in I^u_\beta = [-\beta, \beta] \times \{r/2\}$ we have*

$$\mathrm{dist}(Q(y), O_s) \leq B \mathrm{dist}(y, O_u).$$

Denote by $P : \mathrm{Dom}(P) \subset I^s_a \to I^u_\beta$ the Poincaré map from I^s_a to I^u_β in U, where

$$\mathrm{Dom}(P) = \{x \in I^s_a : \exists\, T > 0 \text{ such that } \phi_T(x) \in I^u_\beta \text{ and } \phi_{[0,T]}(x) \subset U\}$$

is the domain of P. It is clear that

$$\mathrm{Dom}(P) = \{r/2\} \times (0, a']$$

for some $a' \in (0, a]$. Denote

$$I^s_{a',+} = \{r/2\} \times (0, a'].$$

Denote by $\tau_P(x)$ the minimal positive t such that $\phi_t(x) = P(x)$.

Lemma 4.2. *There exists a positive constant* $\alpha \in (0, a]$ *such that, for any* $x \in I^s_{\alpha,+} = \{r/2\} \times (0, \alpha]$ *we have*

$$\mathrm{dist}(Q \circ P(x), O_s) \leq \frac{1}{2}\mathrm{dist}(x, O_s).$$

Proof. According to (4.1), by an elementary calculation we have that

$$P((r/2, x^2)) = ((r/2)^{1+\frac{\lambda}{\mu}}(x^2)^{-\frac{\lambda}{\mu}}, r/2)$$

for any $x = (r/2, x^2) \in \mathrm{Dom}(P)$. Thus

$$\frac{\mathrm{dist}(P(x), O_u)}{\mathrm{dist}(x, O_s)} = \frac{(r/2)^{1+\frac{\lambda}{\mu}}(x^2)^{-\frac{\lambda}{\mu}}}{x^2} = (r/2)^{1+\frac{\lambda}{\mu}}(x^2)^{\frac{-\mathrm{SV}(\sigma)}{\mu}}.$$

Since $\mathrm{SV}(\sigma) < 0$, we have that

$$\frac{\mathrm{dist}(P(x), O_u)}{\mathrm{dist}(x, O_s)} \to 0 \quad \text{as} \quad x^2 \to 0.$$

So there is $\alpha \in (0, a]$ such that for any $x \in I^s_{\alpha,+}$ we have

$$\frac{\mathrm{dist}(P(x), O_u)}{\mathrm{dist}(x, O_s)} \leq \frac{1}{2B},$$

where B is given by Lemma 4.1. It implies that

$$\mathrm{dist}(Q \circ P(x), O_s) \leq B\mathrm{dist}(P(x), O_u) \leq \frac{1}{2}\mathrm{dist}(x, O_s)$$

for any $x \in I^s_{\alpha,+}$. □

Denote

$$I^u_{b,+} = (0, b] \times \{r/2\}.$$

Notice that M^2 is oriented. So the closed curve Γ has two sides in a neighborhood on M^2. Since $P(I^s_{a',+}) \subset I^u_{b,+}$, we have that

$$Q(I^u_{b,+}) \subset I^s_{a,+}.$$

Then Lemma 4.2 implies that for any point $x \in I^s_{\alpha,+}$, the positive orbit of x is contained in a neighborhood of Γ and it tends to Γ. (See Figure 1.)

Denote

$$U^- = \{x = (x^1, x^2) \in U : x^2 < 0\}.$$

Lemma 4.3. *Let* α *be given by Lemma 4.2. If* M^2 *is oriented, then for any* $x = (r/2, x^2) \in I^s_{\alpha,+}$ *we have that*

$$(\mathrm{Orb}^+(x) \setminus \{x\}) \cap I^s_a \subset I^s_{x^2/2,+},$$

and there exists a sequence $\{x_k\} \subset \mathrm{Orb}^+(x)$ *such that*

$$x_k \to \sigma \quad \text{as} \quad k \to +\infty.$$

Moreover, there exists a positive constant $\alpha_0 \in (0, \alpha]$ *such that*

$$\mathrm{Orb}^+(x) \cap U^- = \emptyset$$

for any $x \in I^s_{\alpha_0,+}$.

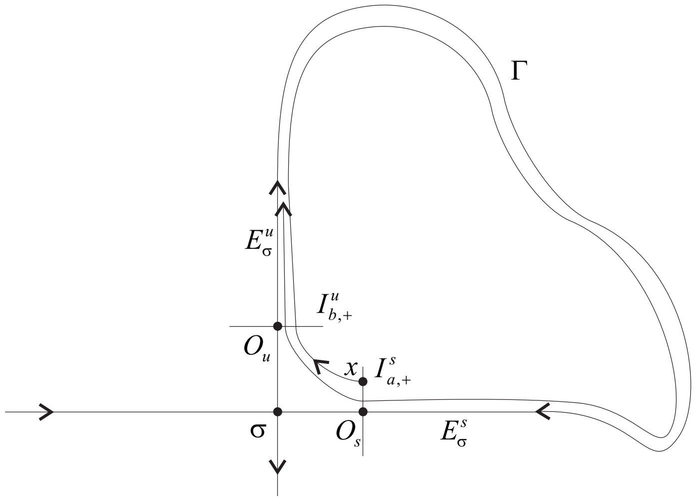

FIGURE 1. The positive orbit of $x \in I^s_{\alpha,+}$

Proof. Since M^2 is oriented, we have

$$Q(I^u_{b,+}) \subset I^s_{a,+}.$$

For any $x = (r/2, x^2) \in I^s_{\alpha,+}$, by Lemma 4.2, we have that the first return point of x on I^s_a is $Q \circ P(x)$, and

$$\mathrm{dist}(Q \circ P(x), O_s) \leq \frac{1}{2}\mathrm{dist}(x, O_s) = x^2/2.$$

Thus

$$Q \circ P(x) \in I^s_{x^2/2,+} \subset I^s_{\alpha,+}.$$

Continuing in this way, we can get that

$$(\mathrm{Orb}^+(x) \setminus \{x\}) \cap I^s_a = \{(Q \circ P)^k(x) : k = 1, 2, \ldots\} \subset I^s_{x^2/2,+},$$

and

$$\mathrm{dist}((Q \circ P)^k(x), O_s) \leq \frac{1}{2^k}\mathrm{dist}(x, O_s) = x^2/2^k,$$

for $k = 1, 2, \ldots$.

So we can choose $x_k \in \mathrm{Orb}^+((Q \circ P)^k(x))$ such that

$$x_k \to \sigma \quad \text{as} \quad k \to +\infty.$$

Moreover, since $\Gamma \cap U_{2r}(\sigma) \subset W^{X,s}_{\mathrm{loc}}(\sigma) \cup W^{X,u}_{\mathrm{loc}}(\sigma)$, there exists $\beta_0 \in (0, \beta)$ such that

$$\{\phi_{[0,\tau_Q(y)]}(y) : y \in I^u_{\beta_0,+}\} \cap U^- = \emptyset.$$

Choose $\alpha_0 \in (0, \alpha]$ such that $P(I^s_{\alpha_0,+}) \subset I^u_{\beta_0,+}$. Then we have

$$\mathrm{Orb}^+(x) \cap U^- = \emptyset$$

for any $x \in I^s_{\alpha_0,+}$. $\square$

4.2. **Non-existence of homoclinic connections.** In this subsection, we will prove that if M^2 is oriented and $X \in \mathrm{Int}^1(\mathrm{WS}(M^2))$, then X has no homoclinic connection of singularities.

Lemma 4.4. *Let M^2 be an oriented surface and $X \in \mathcal{X}^1(M^2)$ be a C^1 vector field. If there exists a singularity σ exhibiting a homoclinic connection, then $X \notin \mathrm{Int}^1(\mathrm{WS}(M^2))$.*

Proof. On the contrary, suppose that there exist a vector field $X \in \mathrm{Int}^1(\mathrm{WS}(M^2))$ and a singularity σ exhibiting a homoclinic connection.

Up to a C^1 small perturbation, we may assume that $X \in \mathrm{Int}^1(\mathrm{WS}(M^2))$ is linear in a small neighborhood $U = U_r(\sigma)$ of σ on a proper chart for some $r > 0$, and still exhibits a homoclinic connection

$$\Gamma \subset W^{X,s}(\sigma) \cap W^{X,u}(\sigma)$$

with $\Gamma \cap U_{2r}(\sigma) \subset W^{X,s}_{\mathrm{loc}}(\sigma) \cup W^{X,u}_{\mathrm{loc}}(\sigma)$. Moreover, we can assume that the saddle value $\mathrm{SV}(\sigma) \neq 0$. Without loss of generality, we assume that $\mathrm{SV}(\sigma) < 0$.

Changing the Riemannian metric if necessary, we assume that E^s_σ and E^u_σ are orthogonal.

We will prove this lemma by two steps.

Step 1. Note that X satisfies the conditions A1–A3. Below we are using the notation from Section 4.1. Let α_0 be given by Lemma 4.3.

Denote $O_u^- = (0, -r/2)$. Take a C^1 neighborhood $\mathcal{U}$ of X such that $\mathcal{U} \subset \mathrm{Int}^1(\mathrm{WS}(M^2))$. For O_u^- and $\mathcal{U}$, by Lemma 2.5, there exist numbers $\rho > 1$, $T > 1$ and $\delta_0 > 0$ with the property of the C^1 connecting lemma. Take $0 < \delta < \delta_0$ small enough such that

$$\Delta = \bigcup_{t \in [-T,0]} U_\delta(\phi_t(O_u^-)) \subset U^-.$$

Denote $p = (r/2, \alpha_0/2)$. Take $0 < \varepsilon < \min\{\delta/\rho, \alpha_0/2\}$ small enough such that for any point x in $U_\varepsilon(p)$, the connected component of $\mathrm{Orb}(x)$ in $U_\varepsilon(p)$ containing x intersects $I^s_{\alpha_0,+}$. Let d correspond to ε with respect to the weak shadowing property.

Since $p \in I^s_{\alpha_0,+}$, by Lemma 4.3 there exists a point $p' \in \mathrm{Orb}^+(p)$ such that

$$\phi_{[0,1]}(p') \subset U_{d/2}(\sigma).$$

And since $O_u^- \in W^{X,u}(\sigma)$, there is $q \in \mathrm{Orb}^-(O_u^-)$ such that

$$\phi_{[0,1]}(q) \subset U_{d/2}(\sigma).$$

Then the following map:

$$g(t) = \begin{cases} \phi_t(p'), & t < 0; \\ \phi_t(q), & t \geq 0, \end{cases}$$

is a d-pseudo orbit of X. Note that both p and O_u^- are contained in this pseudo orbit. Then there is a point y such that

$$g(\mathbb{R}) \subset U_\varepsilon(\mathrm{Orb}(y)).$$

Particularly, there exist $y_1, y_2 \in \mathrm{Orb}(y)$ such that

$$y_1 \in U_\varepsilon(p) \quad \text{and} \quad y_2 \in U_\varepsilon(O_u^-).$$

According to the choice of ε, the connected component of $\mathrm{Orb}(y_1)$ in $U_\varepsilon(p)$ containing y_1 intersects $I^s_{\alpha_0,+}$ at some point y_1'. By Lemma 4.3, we have that the positive orbit of y_1 tends to Γ and

$$U_\varepsilon(O_u^-) \cap \mathrm{Orb}^+(y_1) = \emptyset.$$

It implies that

$$y_2 \in \mathrm{Orb}^-(y_1).$$

Take $q' \in \mathrm{Orb}^-(O_u^-) \setminus \Delta$. We have that the positive orbit of q' and the negative orbit of y_1 both hit $U_\varepsilon(O_u^-) \subset U_{\delta/\rho}(O_u^-)$. Then by Lemma 2.5, there exists $\widetilde{X} \in \mathcal{U}$ with $\widetilde{X} = X$ outside Δ such that

$$y_1 \in \mathrm{Orb}^+_{\widetilde{X}}(q') \subset W^{\widetilde{X},u}(\sigma).$$

$\widetilde{X}$ still exhibits the homoclinic connection Γ of σ. The other branch of the unstable manifold of σ, $\mathrm{Orb}_{\widetilde{X}}(y_1)$, tends to Γ as $t \to +\infty$.

Step 2. Since $\widetilde{X} = X$ outside Δ, $\widetilde{X}$ is linear in a smaller neighborhood $\widetilde{U} = U_{\widetilde{r}}(\sigma)$ of σ for some $0 < \widetilde{r} < r$. Thus $\widetilde{X}$ also satisfies the conditions A1–A3. Denote

$$\widetilde{I}^s_{\widetilde{a},+} = \{\widetilde{r}/2\} \times (0, \widetilde{a}]$$

for any small $\widetilde{a} > 0$. Let $\widetilde{\alpha}$ be given by Lemma 4.2 for $\widetilde{X}$ (and $\widetilde{I}^s_{\widetilde{a},+}$).

Note that the perturbation does not change the positive orbit of y_1. Thus

$$\mathrm{Orb}^+_{\widetilde{X}}(y_1) \cap \widetilde{I}^s_{\widetilde{\alpha}/2,+} \neq \emptyset.$$

Take $p_1 = (\widetilde{r}/2, \xi) \in \mathrm{Orb}^+_{\widetilde{X}}(y_1) \cap \widetilde{I}^s_{\widetilde{\alpha}/2,+}$. Denote $p_2 = (\widetilde{r}/2, 3\xi/4)$. Take $0 < \varepsilon' < \xi/100$ small enough such that for any point x in $U_{\varepsilon'}(p_1)$ (or $U_{\varepsilon'}(p_2)$), the connected component of $\mathrm{Orb}_{\widetilde{X}}(x)$ in $U_{\varepsilon'}(p_1)$ (or $U_{\varepsilon'}(p_2)$) containing x intersects $\widetilde{I}^s_{\widetilde{\alpha},+}$. Let d' correspond to ε' with respect to the weak shadowing property.

Similarly, there exist two points $p_2' \in \mathrm{Orb}^+_{\widetilde{X}}(p_2)$ and $p_1' \in \mathrm{Orb}^-_{\widetilde{X}}(p_1)$ close enough to σ such that the map

$$g'(t) = \begin{cases} \phi_{\widetilde{X},t}(p_2'), & t < 0; \\ \phi_{\widetilde{X},t}(p_1'), & t \geq 0, \end{cases}$$

is a d'-pseudo orbit of $\widetilde{X}$.

Thus there exists a point z such that

$$g'(\mathbb{R}) \subset U_{\varepsilon'}(\mathrm{Orb}_{\widetilde{X}}(z)).$$

Particularly, there exist $z_1, z_2 \in \mathrm{Orb}_{\widetilde{X}}(z)$ such that

$$z_1 \in U_{\varepsilon'}(p_1) \quad \text{and} \quad z_2 \in U_{\varepsilon'}(p_2).$$

Let $z_1' = (\widetilde{r}/2, \xi_1)$, $z_2' = (\widetilde{r}/2, \xi_2)$ be the points of intersection of $\widetilde{I}^s_{\widetilde{\alpha},+}$ and the connected components of $\mathrm{Orb}_{\widetilde{X}}(z_1)$, $\mathrm{Orb}_{\widetilde{X}}(z_2)$ in $U_{\varepsilon'}(p_1)$, $U_{\varepsilon'}(p_2)$ containing z_1, z_2 respectively.

According to Lemma 4.3, we have that

$$(\mathrm{Orb}^+_{\widetilde{X}}(z_1') \setminus \{z_1'\}) \cap \widetilde{I}^s_{\widetilde{\alpha},+} \subset \widetilde{I}^s_{\xi_1/2,+} \quad \text{and} \quad (\mathrm{Orb}^+_{\widetilde{X}}(z_2') \setminus \{z_2'\}) \cap \widetilde{I}^s_{\widetilde{\alpha},+} \subset \widetilde{I}^s_{\xi_2/2,+}.$$

Recall that $p_1 = (\widetilde{r}/2, \xi)$, $p_2 = (\widetilde{r}/2, 3\xi/4)$ and $\varepsilon' < \xi/100$. It is easy to see that

$$z_1' \notin \widetilde{I}^s_{\xi_2/2,+} \quad \text{and} \quad z_2' \notin \widetilde{I}^s_{\xi_1/2,+},$$

which implies

$$z_1' \notin \mathrm{Orb}^+_{\widetilde{X}}(z_2') \text{ and } z_2' \notin \mathrm{Orb}^+_{\widetilde{X}}(z_1').$$

This contradiction proves this lemma. □

Remark 4.5. We emphasize that the orientability of M^2 is required in Lemma 4.4.

5. Transversality of intersections

In this section, we will prove that vector fields contained in $\mathrm{Int}^1(\mathrm{WS}(M^2))$ satisfy (KS2). That is, every intersection between stable and unstable manifolds of singularities and periodic orbits is transverse. Note that M^2 is 2-dimensional; it implies that every hyperbolic periodic orbit is a sink or a source of X. Thus we only need to consider the intersection between saddle type singularities.

5.1. **Intersections between saddle type singularities.** In what follows we will prove that there exist no separatrices joining two different saddle type singularities of $X \in \mathrm{Int}^1(\mathrm{WS}(M^2))$ on oriented surface M^2.

Lemma 5.1. *Let M^2 be oriented and $X \in \mathcal{X}^1(M^2)$. If there exists an intersection between saddle type singularities of X, then $X \notin \mathrm{Int}^1(\mathrm{WS}(M^2))$.*

Proof. Suppose on the contrary that there exists a vector field $X \in \mathrm{Int}^1(\mathrm{WS}(M^2))$ exhibiting an intersection of the stable manifold of σ_1 and the unstable manifold of σ_2, where σ_1, σ_2 are two saddle type singularities of X.

If $\sigma_1 = \sigma_2$, then we get a contradiction immediately from Lemma 4.4. In what follows we assume that $\sigma_1 \neq \sigma_2$. We will create a homoclinic connection by using the C^1 connecting lemma.

Up to a C^1 small perturbation, we may assume that $X \in \mathrm{Int}^1(\mathrm{WS}(M^2))$ is linear in two small neighborhoods $U(1) = U_r(\sigma_1)$ of σ_1 and $U(2) = U_r(\sigma_2)$ of σ_2 on a proper chart for some $r > 0$, and still exhibits an intersection

$$\Gamma \subset W^{X,s}(\sigma_1) \cap W^{X,u}(\sigma_2)$$

with

$$\Gamma \cap U(1) \subset W^{X,s}_{\mathrm{loc}}(\sigma_1) \text{ and } \Gamma \cap U(2) \subset W^{X,u}_{\mathrm{loc}}(\sigma_2).$$

Changing the Riemannian metric if necessary, we assume that $E^s_{\sigma_1}$ and $E^u_{\sigma_1}$, $E^s_{\sigma_2}$ and $E^u_{\sigma_2}$ are orthogonal respectively. We introduce two orthogonal coordinate systems in $U(1)$ and $U(2)$ with respect to the hyperbolic splittings

$$T_{\sigma_1}M = E^s_{\sigma_1} \oplus E^u_{\sigma_1} \text{ and } T_{\sigma_2}M = E^s_{\sigma_2} \oplus E^u_{\sigma_2}.$$

Below we first introduce some notation. (Also see Figure 2.)

Denote by

$$O^+_s(i), O^-_s(i) \in W^{X,s}_{\mathrm{loc}}(\sigma_i) \text{ and } O^+_u(i), O^-_u(i) \in W^{X,u}_{\mathrm{loc}}(\sigma_i)$$

the points in $U(i)$ whose coordinates are $(r/2, 0)$, $(-r/2, 0)$ and $(0, r/2)$, $(0, -r/2)$ respectively, $i = 1, 2$. Without loss of generality, assume that $O^+_s(1)$ and $O^+_u(2)$ are contained in Γ.

Take small segments

$$I^{s,\pm}_{a_i}(i) = \{\pm r/2\} \times [-a_i, a_i] \subset U(i)$$

which are orthogonal to $E^u_{\sigma_i}$ at $O^\pm_s(i)$, and

$$I^{u,\pm}_{a_i}(i) = [-a_i, a_i] \times \{\pm r/2\} \subset U(i)$$

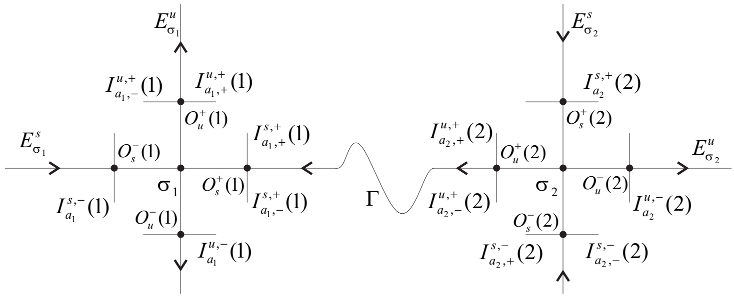

FIGURE 2. The orthogonal coordinate systems

which are orthogonal to $E^s_{\sigma_i}$ at $O^{\pm}_u(i)$ respectively, where $a_1, a_2 \in (0, r/2)$.

Denote

$$I^{s,\pm}_{a_i,+}(i) = \{\pm r/2\} \times (0, a_i], \quad I^{s,\pm}_{a_i,-}(i) = \{\pm r/2\} \times [-a_i, 0),$$

and

$$I^{u,\pm}_{a_i,+}(i) = (0, a_i] \times \{\pm r/2\}, \quad I^{u,\pm}_{a_i,-}(i) = [-a_i, 0) \times \{\pm r/2\},$$

for $i = 1, 2$.

Denote by $P_\Gamma : I^{u,+}_{a_2}(2) \to I^{s,+}_{a_1}(1)$ the Poincaré map from $I^{u,+}_{a_2}(2)$ to $I^{s,+}_{a_1}(1)$, and $\tau_\Gamma(y)$ the minimal positive t such that $\phi_t(y) = P_\Gamma(y)$. Reduce $a_1 > 0$ and $a_2 > 0$ if necessary; we can assume that

$$P_\Gamma(I^{u,+}_{a_2}(2)) = I^{s,+}_{a_1}(1)$$

and τ_Γ is continuous in $I^{u,+}_{a_2}(2)$. It is clear that $P_\Gamma(I^{u,+}_{a_2,+}(2)) = I^{s,+}_{a_1,+}(1)$ or $I^{s,+}_{a_1,-}(1)$. Without loss of generality, assume that

$$P_\Gamma(I^{u,+}_{a_2,+}(2)) = I^{s,+}_{a_1,+}(1).$$

Take a C^1 neighborhood $\mathcal{U}$ of X such that $\mathcal{U} \subset \mathrm{Int}^1(\mathrm{WS}(M^2))$. For each point of the eight points $\{O^{\pm}_c(i) : c = s, u; i = 1, 2\}$ and $\mathcal{U}$, by Lemma 2.5, there exist numbers $\rho(O^{\pm}_c(i)) > 1$, $T(O^{\pm}_c(i)) > 1$ and $\delta_0(O^{\pm}_c(i)) > 0$ with the property of the C^1 connecting lemma. Let $\rho = \max\{\rho(O^{\pm}_c(i))\}$, $T = \max\{T(O^{\pm}_c(i))\}$ and $\delta_0 = \min\{\delta_0(O^{\pm}_c(i))\}$.

Take $0 < \delta < \max\{\delta_0, a_1, a_2\}$ small enough such that the eight tubes

$$\Delta(O^{\pm}_s(i)) = \bigcup_{t \in [0,T]} U_\delta(\phi_t(O^{\pm}_s(i))), \quad \Delta(O^{\pm}_u(i)) = \bigcup_{t \in [-T,0]} U_\delta(\phi_t(O^{\pm}_u(i)))$$

and $\{\sigma_1\}$, $\{\sigma_2\}$ are pairwise disjoint.

Claim. There exist $y_1 \in I^{u,+}_{\delta/\rho}(i) \cup I^{u,-}_{\delta/\rho}(i)$ and $y_2 \in I^{s,+}_{\delta/\rho}(i) \cup I^{s,-}_{\delta/\rho}(i)$, for $i = 1$ or 2, such that $y_2 \in \mathrm{Orb}^+(y_1)$.

Choose $\varepsilon > 0$ small enough such that:

(C1) $\varepsilon < \delta/\rho$;

(C2) for any x in $U_\varepsilon(O^+_s(1))$, the connected component of $\mathrm{Orb}(x)$ in $U_\varepsilon(O^+_s(1))$ containing x intersects $I^{s,+}_{a_1}(1)$; a similar property holds for the other seven points;

(C3) for any x in $I^{s,+}_{\varepsilon,+}(1)$ (or $I^{s,+}_{\varepsilon,-}(1)$), there exists τ such that $\phi_\tau(x) \in I^{u,+}_{\delta/\rho,+}(1)$ (or $\phi_\tau(x) \in I^{u,-}_{\delta/\rho,+}(1)$) and $\phi_{[0,\tau]}(x) \subset U(1)$; a similar property holds for the other seven points.

Take $\varepsilon' \in (0,\varepsilon)$ such that

$$P_\Gamma(x) \in I^{s,+}_{\varepsilon}(1)$$

for any $x \in I^{u,+}_{\varepsilon'}(2)$. Let d correspond to ε' with respect to the weak shadowing property.

Take four points $p_1 \in \mathrm{Orb}^+(O^+_s(1))$, $p_2 \in \mathrm{Orb}^+(O^-_s(2))$, $q_1 \in \mathrm{Orb}^-(O^+_u(1))$ and $q_2 = \phi_{-\tau}(p_1) \in \mathrm{Orb}^-(O^+_u(2))$ close enough to σ_1 or σ_2 such that the following map:

$$g(t) = \begin{cases} \phi_t(p_2), & t < 0; \\ \phi_t(q_2), & 0 \le t < \tau; \\ \phi_t(q_1), & t \ge \tau, \end{cases}$$

is a d-pseudo orbit of X. Thus there is a point y such that

$$g(\mathbb{R}) \subset U_{\varepsilon'}(\mathrm{Orb}(y)).$$

According to the condition (C2), there exists $y_1 \in \mathrm{Orb}(y) \cap I^{u,+}_{\varepsilon'}(2)$ and $y_2 \in \mathrm{Orb}(y) \cap I^{s,-}_{\varepsilon}(2)$. We assume that

$$y_1 \notin I^{u,+}_{\varepsilon,-}(2).$$

If $y_1 \in I^{u,+}_{\varepsilon',-}(2)$, we may consider $\widetilde{y}_1 = P_\Gamma(y_1) \in I^{s,+}_{\varepsilon,-}(1)$ and the vector field $-X$. The proof is similar.

If $y_2 \in \mathrm{Orb}^+(y_1)$, then the proof is finished. We assume that $y_1 \in \mathrm{Orb}^+(y_2)$. It implies that $y_1 \neq O^+_u(2)$ and $y_2 \neq O^-_s(2)$ since $O^+_u(2) \in W^{X,u}_{\mathrm{loc}}(\sigma_2)$ and $O^-_s(2) \in W^{X,s}_{\mathrm{loc}}(\sigma_2)$. Thus

$$y_1 \in I^{u,+}_{\varepsilon,+}(2) \quad \text{and} \quad y_2 \in I^{s,-}_{\varepsilon,+}(2) \cup I^{s,-}_{\varepsilon,-}(2).$$

Without loss of generality, we assume that $y_2 \in I^{s,-}_{\varepsilon,+}(2)$.

By the condition (C3) we have that there exist $y_1' = \phi_{\tau_2}(y_2)$ and $y_2' = \phi_{-\tau_1}(y_1)$ such that

$$y_1' \in I^{u,+}_{\delta/\rho,-}(2), \quad y_2' \in I^{s,+}_{\delta/\rho,+}(2),$$

and

$$\phi_{[0,\tau_2]}(y_2), \phi_{[-\tau_1,0]}(y_1) \subset U(2).$$

It is easy to see that $y_2' \in \mathrm{Orb}^+(y_1')$. This claim is proved.

Now we start to create a homoclinic connection by perturbations. Let y_1 and $y_2 \in \mathrm{Orb}^+(y_1)$ be given by the Claim. Assume that

$$y_1 \in I^{u,+}_{\delta/\rho}(2) \quad \text{and} \quad y_2 \in I^{s,-}_{\delta/\rho}(2).$$

(The argument for other cases is the same.)

Take two points

$$x_1 \in \mathrm{Orb}^-(O^+_u(2)) \setminus \Delta(O^+_u(2)) \quad \text{and} \quad x_2 \in \mathrm{Orb}^+(O^-_s(2)) \setminus \Delta(O^-_s(2)).$$

Then the positive X-orbit of x_1 and the negative X-orbit of y_2 both hit $U_{\delta/\rho}(O^+_u(2))$ (at $O^+_u(2)$ and y_1). By Lemma 2.5, there is $Y \in \mathcal{U}$ with $Y = X$ outside $\Delta(O^+_u(2))$ such that y_2 is on the positive Y-orbit of x_1. Note that the positive X-orbit of $O^-_s(2)$ is unchanged under the perturbation. Thus the positive Y-orbit of x_1 and the negative Y-orbit of x_1 both hit $U_{\delta/\rho}(O^-_s(2))$ (at y_2 and $O^-_s(2)$). By Lemma 2.5 again, there is $Z \in \mathcal{U}$ (since $\Delta(O^+_u(2))$ and $\Delta(O^-_s(2))$ are disjoint; we have that

Z is still contained in $\mathcal{U}$) with $Z = Y$ outside $\Delta(O_s^-(2))$ such that x_2 is on the positive Z-orbit of x_1. Note that $\mathrm{Orb}^-(x_1)$ and $\mathrm{Orb}^+(x_2)$ are unchanged. Thus we get a homoclinic connection $\mathrm{Orb}_Z(x_1)$ of σ_2.

It contradicts Lemma 4.4. We finished the proof. □

5.2. **Proof of Proposition 2.4.** Suppose on the contrary that there exists a vector field $X \in \mathrm{Int}^1(\mathrm{WS}(M^2))$ which has a non-transverse intersection

$$W^{X,s}(p) \cap W^{X,u}(q) \neq \emptyset,$$

where p and q are singularities or periodic orbits of X.

Since $\dim M^2 = 2$, for any hyperbolic periodic orbit γ of X we have $\dim W^{X,s}(\gamma) = 2$ or $\dim W^{X,u}(\gamma) = 2$. It implies that any intersection between γ and singularities or periodic orbits is transverse. Thus both p and q are singularities.

For the same reason, we have that $\dim W^{X,s}(p) = \dim W^{X,s}(q) = 1$.

If $p = q$, then by Lemma 4.4 we have $X \notin \mathrm{Int}^1(\mathrm{WS}(M^2))$. It is a contradiction. If $p \neq q$, we can also get a contradiction according to Lemma 5.1. We finished the proof of Proposition 2.4. □

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