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Proceedings of the American Mathematical Society
DOI: 10.1090/proc/13404

Accepted Manuscript

A characterization of $\mu-$equicontinuity for topological dynamical systems

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Abstract

Two different notions of measure theoretical equicontinuity ($\mu-$equicontinuity) for topological dynamical systems with respect to Borel probability measures appeared in [14] and [16]. We show that if the probability space satisfies Lebesgue's density theorem and Vitali's covering theorem (for example a Cantor set or a subset of $\mathbb{R}^d$) then both notions are equivalent. To show this we characterize Lusin measurable maps using $\mu-$continuity points. As a corollary we also obtain a new characterization of $\mu-$mean equicontinuity.

1 Introduction

A topological dynamical system (TDS), (X, T), is a continuous transformation T on a compact metric space X. A class of rigid systems is the family of equicontinuous TDS. A TDS is equicontinuous if the family $\{T^i\}$ is equicontinuous. Equivalently we can define equicontinuity using equicontinuity points: $x \in X$ is an equicontinuous point if *for every* y close to x we have that $T^i(x), T^i(y)$ stay close for all i. A TDS is equicontinuous if and only if every point is an equicontinuous point. Sensitivity (or sensitive dependence on initial conditions) is considered a weak form of chaos. Auslander-Yorke [2] showed that a minimal TDS is either sensitive or equicontinuous. .

Equicontinuity is a very strong property and different attempts have been made to weaken this property. We focus on the ones made with the use of Borel probability measures.

While studying cellular automata (a subclass of TDSs), Gilman [14][13] introduced the concept of $\mu-$equicontinuity points: $x \in X$ is a $\mu-$equicontinuous point if for *most* y close to x we have that $T^i(x), T^i(y)$ stay close for all $i \in \mathbb{Z}_+$. He also defined $\mu-$sensitivity and he showed that if T is a cellular automaton and μ a shift$-$ergodic measure (not necessarily invariant with respect to T) then T is either $\mu-$sensitive or the set of $\mu-$equicontinuity points has measure one. With exactly the same approach one can define $\mu-$equicontinuity points for TDS with respect to measures that are not necessarily ergodic or invariant under any transformation.

In [5], Cadre and Jacob introduced $\mu-$pairwise sensitivity ($\mu-$sensitivity in this paper) for TDSs. This notion was characterized for ergodic measures by Huang, Lu and Ye in [16]. It was shown that a TDS with an ergodic measure μ is either $\mu-$pairwise sensitive or $\mu-$equicontinuous; (X, T) is $\mu-$equicontinuous if for every $\varepsilon > 0$ there exists a compact set M such that $\mu(M) > 1 - \varepsilon$ and $T\restriction_M$ is equicontinuous. They showed that every ergodic $\mu-$equicontinuous TDS has discrete spectrum.

For the main results of this paper μ need not be invariant. We show that if (X, μ) satisfies Lebesgue's density theorem and Vitali's covering theorem (for example when $X \subset \mathbb{R}^d$ or when X is a Cantor space) then (X, T) is $\mu-$equicontinuous if and only if almost every point is a $\mu-$equicontinuous point - Theorem 9 (note that satisfying Lebesgue's density theorem and Vitali's covering theorem are properties that depend on the metric of the space and not on the topology). As a consequence we get that if μ is ergodic then (X, T) is $\mu-$pairwise sensitive if and only if there are no $\mu-$equicontinuity points (Theorem 30). We also define a subclass of $\mu-$equicontinuous systems that rely on a local periodicity notion and we prove they have discrete rational spectrum when μ is ergodic.

In Section 2 we define and study topological and measure theoretical forms of equicontinuity and local periodicity ($\mu-$equicontinuity and $\mu - LP$). To characterize $\mu-$equicontinuity we study $\mu-$continuity, which is a property of functions between metric measure spaces (not a dynamical concept). In Section 3, we discuss $\mu-$equicontinuity with respect to invariant measures; we show ergodic $\mu - LP$ TDS on Cantor spaces have rational spectrum and we characterize $\mu-$sensitivity.

In [10] Fomin introduced a weaker form of equicontinuity called mean-L-stable or mean equicontinuity. Several recent papers have studied this notion (e.g. [19], [7], [12]). In [12], $\mu-$mean equicontinuity is introduced and Theorem 16 (of this paper) is applied to characterize $\mu-$mean equicontinuity (Theorem 21 in [12]), also it is shown that if μ is an ergodic measure of a TDS (X, T) then (X, T) is $\mu-$mean equicontinuous if and only if μ has discrete spectrum.

2 Equicontinuity and local periodicity

Throughout this paper X will denote a compact metric space with metric d. For $x \in X$ we represent the **balls centred in** x with $B_n(x) := \{z \mid d(x, z) \leq 1/n\}$.

A **topological dynamical system (TDS)** is a pair (X, T) where X is a compact metric space and $T : X \to X$ is a continuous transformation.

Two TDS (X_1, T_2), (X_2, T_2) are **conjugate (topologically)** if there exists a homeomorphism $f : X_1 \to X_2$ such that $T_2 \circ f = f \circ T_1$.

An important kind of TDSs are TDSs on Cantor spaces.

The positive integers are denoted with $\mathbb{N}$ and the non-negative integers with $\mathbb{Z}_+$. The $n-$**window**, $W_n \subset \mathbb{Z}_+$, is defined as $W_n = [0, n]$. Let $\mathcal{A}$ be a finite set. For $W \subset \mathbb{Z}_+$ and $x \in \mathcal{A}^{\mathbb{Z}_+}$, $x_W \in \mathcal{A}^W$ denotes the restriction of x to W, in particular x_i represents the value of x at the ith position. We endow $\mathcal{A}^{\mathbb{Z}_+}$ with the metric given by $d(x,y) = \frac{1}{m}$, where m is the largest integer such that $x_{W_m} = y_{W_m}$; we will call this a **Cantor space**. It is known that Cantor spaces are zero dimensional, i.e. there exists a countable base of clopen sets.

Another important class of systems are Euclidean TDSs. We endow the set $\mathbb{R}^d$ with the Euclidean metric. We say (X,T) is **Euclidean** if $X \subset \mathbb{R}^d$.

2.1 Equicontinuity

Mathematical definitions of chaos have been widely studied. Many of them require the system to be sensitive to initial conditions. A TDS (X,T) is **sensitive (or has sensitive dependence to initial conditions)** if there exists $\varepsilon > 0$ such that for every open set $A \subset X$ there exist $x, y \in A$ and $i \in \mathbb{Z}_+$ such that $d(T^i x, T^i y) > \varepsilon$. On the other hand, equicontinuity represents predictable behaviour. As we mentioned a TDS is equicontinuous if $\{T^i\}$ is an equicontinuous family, and a minimal TDS is either sensitive or equicontinuous [1].

We can also define equicontinuity by defining equicontinuity points.

Definition 1 *Let (X,T) be a TDS. We define the* ***orbit metric*** *d_o on X as $d_o(x,y) := \sup_{i\geq 0}\{d(T^i x, T^i y)\}$, and the* ***orbit balls*** *as*

$$\begin{aligned} B^o_m(x) &= \{y \mid d_o(x,y) \leq 1/m\} \\ &= \{y \mid d(T^i(x), T^i(y) \leq 1/m \ \forall i \in \mathbb{N}\}. \end{aligned}$$

Let (X,T) be a TDS. A point x is an ***equicontinuous point*** *of T if for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $B_n(x) \subset B^o_m(x)$.*

A TDS (X,T) is equicontinuous if and only if every $x \in X$ is an equicontinuous point. A TDS is equicontinuous if and only if it is uniformly equicontinuous, i.e. for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $B_n(x) \subset B^o_m(x)$ for every $x \in X$.

Definition 2 *Let $M \subset X$. We say $T\mid_M$ is* ***equicontinuous*** *if for all $x \in M$ and $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $B_n(x) \cap M \subset B^o_m(x)$.*

2.2 $\mu-$Equicontinuity

We will use μ to denote Borel probability measures on X (μ does not need to be invariant under T).

Based on [14] we define $\mu-$equicontinuity points.

Definition 3 *A point x is a $\mu-$**equicontinuous point** of (X,T) if for all $m \in \mathbb{N}$ we have*

$$\lim_{n\to\infty} \frac{\mu(B^o_m(x) \cap B_n(x))}{\mu(B_n(x))} = 1.$$

If x is an equicontinuous point in the support of μ then x is a $\mu-$equicontinuous point.

The following definition appeared in [16].

Definition 4 *A TDS (X,T) is $\mu-$**equicontinuity** if for every $\varepsilon > 0$ there exists a compact set M such that $\mu(M) > 1-\varepsilon$ and $T\mid_M$ is equicontinuous.*

If (X,T) is an equicontinuous TDS and μ is a Borel probability measure on X then (X,T) is $\mu-$equicontinuous. There exists a sensitive TDS on $\{0,1\}^{\mathbb{Z}}$ such that (X,T) is $\mu-$equicontinuous for a natural family of measures μ (Example 2.26 [11]).

Definition 5 *We say (X,μ) satisfies **Lebesgue's density theorem** if for every Borel set A we have that*

$$\lim_{n\to\infty}\frac{\mu(A\cap B_n(x))}{\mu(B_n(x))}=1 \quad \text{for a.e. } x\in A. \tag{1}$$

The original Lebesgue's density theorem applies to $\mathbb{R}^d$ and the Lebesgue measure. If $X\subset\mathbb{R}^d$ and μ is a Borel probability measure then (X,μ) satisfies Lebesgue's density theorem (see Remark 2.4 in [17]).

Theorem 6 (Levy's zero-one law [8] pg.262) *Let Σ be a sigma-algebra on a set Ω and P a probability measure. Let $\{\mathcal{F}_n\}\subset\Sigma$ be a filtration of sigma-algebras, that is, a sequence of sigma-algebras $\{\mathcal{F}_n\}$, such that $\mathcal{F}_i\subset\mathcal{F}_{i+1}$ for all i. If D is an event in $\mathcal{F}_\infty$ (the smallest sigma-algebra that contains every $\mathcal{F}_n$) then $P(D\mid\mathcal{F}_n)\to 1_D$ almost surely.*

Corollary 7 *Let X be a Cantor space and μ a Borel probability measure. Then (X,μ) satisfies Lebesgue's density theorem.*

Proof. Let $n\in\mathbb{N}$, and $\mathcal{F}_n$ the smallest sigma-algebra that contains all the balls $\{B_{n'}(x)\}$ with $n'\leq n$. It is easy to see that $\mathcal{F}_\infty$ is the Borel sigma-algebra on X. The desired result is a direct application from Theorem 6. ■

There exist Borel probability spaces that do not satisfy Lebesgue's density theorem (e.g. Example 5.6 in [17]).

Definition 8 *We say (X,μ) is **Vitali (**or **satisfies Vitali's covering theorem)** if for every Borel set $A\subset M$ and every $N,\varepsilon>0$, there exists a finite subset $F\subset A$ and $n(x)\geq N$ (defined for every $x\in F$) such that $\left\{B_{n(x)}(x)\right\}_{x\in F}$ are disjoint and $\mu(A\setminus\cup_{x\in F}B_{n(x)}(x))\leq\varepsilon$.*

If $X\subset\mathbb{R}^d$ and μ is a Borel probability measure then (X,μ) is Vitali ([15] pg.8). This result is known as Vitali's covering theorem.

Using a clopen base it is not hard to see that if X is a Cantor space and μ a Borel probability measure then (X,μ) satisfies Vitali's covering theorem. For more conditions see Chapter 1 in [15].

The following theorem will be proved at the end of the following subsection.

Theorem 9 *Let (X,T) be a TDS and μ a Borel probability measure on X.*

(a) If (X,μ) satisfies Lebesgue's density theorem and (X,T) is $\mu-$equicontinuous then almost every $x \in X$ is a $\mu-$equicontinuous point.

(b) If (X,μ) is Vitali and almost every $x \in X$ is a $\mu-$equicontinuous point then (X,T) is $\mu-$equicontinuous.

(c) (X,T) is $\mu-$equicontinuous if and only if there exists $X' \subset X$ such that $\mu(X') = 1$ and (X', d_0) is separable.

2.3 $\mu-$Continuity

Definition 10 *Given a TDS (X,T) we denote the change of metric projection map as $f_T : (X,d) \to (X,d_o)$.*

The function f_T^{-1} is always continuous. A point $x \in X$ is an equicontinuous point of (X,T) if and only if it is a continuity point of f_T. Thus (X,T) is equicontinuous if and only if f_T is continuous.

Definition 11 *Let X,Y be metric spaces and μ a Borel probability measure on X. A function $f : X \to Y$ is $\mu-$**Lusin** (or **Lusin measurable**) if for every $\varepsilon > 0$ there exists a compact set $M \subset X$ such that $\mu(M) > 1-\varepsilon$ and $f\mid_M$ is continuous.*

This implies that a TDS (X,T) is $\mu-$equicontinuous if and only if f_T is $\mu-$Lusin.

We will define a measure theoretic notion of continuity point ($\mu-$continuity point) that satisfies the following property: $x \in X$ is a $\mu-$equicontinuous point of T if and only if x is a $\mu-$continuity point of f_T. We will show that if (X,μ) is Vitali and almost every $x \in X$ is a $\mu-$continuity point of f then f is $\mu-$Lusin; we show the converse is true if (X,μ) satisfies Lebesgue's density theorem (Theorem 16).

In this subsection X will denote a compact metric space, μ a Borel probability measure on X, and Y a metric space with metric d_Y (and balls $B_m^Y(y) := \{z \in Y : d_Y(y,z) \le 1/m\}$).

Definition 12 *In some cases we can talk about the measure of not necessarily measurable sets. Let $A \subset X$. We say A has **full measure** if A contains a measurable subset with measure one. We say $\mu(A) < \varepsilon$, if there exists a measurable set $A' \supset A$ such that $\mu(A') < \varepsilon$.*

Definition 13 *A set $A \subset X$ is $\mu-$**measurable** if A is in the the completion of μ.*

*The function $f : X \to Y$ is $\mu-$**measurable** if for every Borel set $D \subset Y$, we have that $f^{-1}(D)$ is $\mu-$measurable.*

*A point $x \in X$ is a $\mu-$**continuity point** if for every $m \in \mathbb{N}$ we have that $1_{f^{-1}(B_m^Y(f(x)))}$ is $\mu-$measurable and*

$$\lim_{n\to\infty} \frac{\mu\left[f^{-1}(B_m^Y(f(x))) \cap B_n(x)\right]}{\mu(B_n(x))} = 1. \qquad (2)$$

*If $\mu-$almost every $x \in X$ is a $\mu-$continuity point we say f is $\mu-$**continuous.***

Every $\mu-$Lusin function is $\mu-$measurable. Lusin's theorem states the converse is true if Y is separable (note that (X, d_0) is not necessarily separable). This fact is generalized in the following theorem.

Theorem 14 ([21] pg. 145) *Let $f : X \to Y$ be a function and μ a Borel probability measure on X such that there exists $X' \subset X$ such that $\mu(X') = 1$ and $f(X')$ is separable. The function f is $\mu-$Lusin if and only if for every ball B, $f^{-1}(B)$ is $\mu-$measurable.*

Lemma 15 *Let (X,μ) be Vitali and $f : X \to Y$ $\mu-$continuous. For every $m \in \mathbb{N}$ and $\varepsilon > 0$ there exists a finite set $F_{m,\varepsilon}$ and a value $n(x)$ for each $x \in F_{m,\varepsilon}$ such that $\left\{B_{n(x)}(x)\right\}_{x\in F_{m,\varepsilon}}$ are disjoint, and $\mu(\cup_{x\in F_{m,\varepsilon}}\left[f^{-1}(B^Y_m(f(x))) \cap B_{n(x)}(x)\right]) > 1-\varepsilon$.*

Proof. Let $\varepsilon > 0$. For every $\mu-$continuous point x and $m \in \mathbb{N}$ there exists N_x such that $\frac{\mu(f^{-1}(B^Y_m(f(x)))\cap B_n(x))}{\mu(B_n(x))} > 1-\varepsilon$ for every $n \geq N_x$. Let

$$A_i := \{x \mid x \text{ is a } \mu-\text{continuity point}, \ N_x \leq i\}.$$

Using Fubini's theorem one can check that $\frac{\mu(f^{-1}(B^Y_m(f(x)))\cap B_n(x))}{\mu(B_n(x))}$ is a measurable function and hence the sets A_i are also $\mu-$measurable. Since $\cup_{i\in\mathbb{N}}A_i$ contains the set of $\mu-$continuity points, there exists N such that $\mu(X \setminus A_N) < \varepsilon$. Since (X,μ) is Vitali there exists a finite set of points $F_{m,\varepsilon} \subset A_N$ and a function $n(x) \geq N$ such that $\left\{B_{n(x)}(x)\right\}_{x\in F_{m,\varepsilon}}$ are disjoint and $\mu(\cup_{x\in F_{m,\varepsilon}}B_{n(x)}(x)) > \mu(A_N)-\varepsilon$. This implies $\mu(\cup_{x\in F_{m,\varepsilon}}B_{n(x)}(x)) > 1-2\varepsilon$. Since $F_{m,\varepsilon} \subset A_N$, we obtain

$$\mu(\cup_{x\in F_{m,\varepsilon}}\left[f^{-1}(B^Y_m(f(x))) \cap B_{n(x)}(x)\right]) > 1-3\varepsilon.$$

■

Theorem 16 *Let $f : X \to Y$ be a map and μ be a Borel probability measure on X.*

(a) If (X,μ) is Vitali and f is $\mu-$continuous then f is $\mu-$Lusin.

(b) If (X,μ) satisfies Lebesgue's density theorem and f is $\mu-$Lusin then f is $\mu-$continuous.

Proof. (a)

Let $m \in \mathbb{N}$ and $\varepsilon > 0$. Since (X,μ) is Vitali and f is $\mu-$continuous we can use Lemma 15; hence there exists a finite set of points $F_{m,\varepsilon/2^m}$ and a value $n(x)$ for every $x \in F_{m,\varepsilon/2^m}$ such that

$$\mu(\cup_{x\in F_{m,\varepsilon/2^m}}\left[f^{-1}(B^Y_m(f(x))) \cap B_{n(x)}(x)\right]) > 1-\varepsilon/2^m$$

with $\left\{B_{n(x)}(x)\right\}_{x\in F_{m,\varepsilon/2^m}}$ disjoint.

For every $x \in F_{m,\varepsilon/2^m}$ we define

$$G_x := f^{-1}(B^Y_m(f(x))) \cap B_{n(x)}(x), \text{ and}$$
$$M_m := \cup_{x \in F_{m,\varepsilon/2^m}} G_x$$

Note that $\mu(M_m) > 1 - \varepsilon/2^m$. If x_1,$x_2 \in M_m$ and are sufficiently close then there exists $x \in F_{m,\varepsilon/2^m}$ such that x_1,$x_2 \in G_x$. This implies that $f(x_1), f(x_2) \in B^Y_m(f(x))$ and hence $d_Y(f(x_1), f(x_2)) \leq 2m$.

Let

$$M := \cap_{m \in \mathbb{N}} M_m.$$

We have that $\mu(M) > 1-\varepsilon$. Also using the previous argument it follows that $f\mid_M$is continuous. The regularity of the Borel measure gives us the existence of the compact set.

(b)

Since f is $\mu-$Lusin we have that it is $\mu-$measurable.

Let $\varepsilon > 0$. There exists a compact set $M \subset X$ such that $\mu(M) > 1 - \varepsilon$ and $f\mid_M$ is continuous. Using Lebesgue's density theorem we have

$$\lim_{n\to\infty} \frac{\mu(M \cap B_n(x))}{\mu(B_n(x))} = 1 \text{ for a.e. } x \in M.$$

Let $m \in \mathbb{N}$. Since $f\mid_M$ is continuous then for sufficiently large n and almost every $x \in M$, $M \cap B_n(x) \subset f^{-1}(B^Y_m(f(x)))$, and so for almost every x

$$\lim_{n\to\infty} \frac{\mu(f^{-1}(B^Y_m(f(x))) \cap B_n(x))}{\mu(B_n(x))} = 1.$$

This implies the set of $\mu-$continuity points has measure larger than $1 - \varepsilon$. We conclude the desired result. ■

Proof of Theorem 9. For (a) and (b) apply Theorem 16 with $f = f_T$, $(Y, d_Y) = (X, d_o)$ and $B^Y_m = B^o_m$.

(X, T) is $\mu-$equicontinuous if and only if there exists $X' \subset X$ such that $\mu(X') = 1$ and (X', d_0) is separable.

(c)

Assume that there exists $X' \subset X$ such that $\mu(X') = 1$ and (X', d_0) is separable. Using Theorem 14 with $f = f_T$, $(Y, d_Y) = (X, d_o)$ and $B^Y_m = B^o_m$ we obtain that f_T is $\mu-$Lusin and hence (X, T) is $\mu-$equicontinuous.

If (X, T) is $\mu-$equicontinuous then f_T is $\mu-$Lusin; therefore for every $\kappa > 0$ there exists a compact set $M_\kappa \subset X$ such that $\mu(M_\kappa) > 1 - \kappa$ and $f_T\mid_{M_\kappa}$ is continuous. Let $X' = \cup_{n\in\mathbb{N}} M_{1/n}$. We have that $\mu(X') = 1$ and since (X', d_o) is the union of compact sets it is separable.

Note that orbit balls are countable intersections of balls, hence measurable. ■

The concept of approximate continuity for metric measure spaces has a similar flavour to $\mu-$continuity. It also emulates locally the definition of continuity using measures. The definition of an approximate continuity point is stronger than (2) (in Definition 13) but it does not assume the measurability of any set. A classical result states that if the image of a function is separable then approximate continuity and measurability are equivalent (Theorem 2.9.13 [9]).

Similar notions as (2) have been studied. For example a result by Sierpinski states that every function $f : \mathbb{R} \to \mathbb{R}$ satisfies (2) with respect to $\mu = \nu^*$, where ν^* is the outer measure defined by Lebesgue's measure (Theorem 2.6.2 [18]).

2.4 Local periodicity

In this subsection we study local periodicity and $\mu - LEP$ systems, a subclass of $\mu-$equicontinuous systems.

Definition 17 *Let (X, T) be a TDS, $x \in X$ and $m \in \mathbb{N}$. We define $LP_m(T)$ as the set of points x such that there exists $p \in \mathbb{N}$ with*

$$d(T^i x, T^{i+jp}x) \leq 1/m \quad \text{for every } i, j \in \mathbb{Z}_+.$$

We define $LEP_m(T)$ as the set of points, x such that there exists $p, q \in \mathbb{N}$ with

$$d(T^i x, T^{i+jp}x) \leq 1/m \quad \text{for every } i \geq q, j \in \mathbb{Z}_+.$$

For $x \in LEP_m(T)$, $p_m(x)$ denotes the smallest possible p and $pp_m(x)$ the smallest possible q for $p_m(x)$. We also define

$$LP(T) := \cap_{m \in \mathbb{N}} LP_m(T)$$
$$LEP(T) := \cap_{m \in \mathbb{N}} LEP_m(T).$$

*A TDS (X, T) is **locally eventually periodic** (LEP) if $LEP(T) = X$ and **locally periodic** (LP) if $LP(T) = X$.*

Remark 18 *If X is a Cantor space then $LP_m(T)$ is the set of points x such that $(T^i x)_{W_m}$ is periodic (with respect to i, see Figure 1).*

Example 19 *Let $S = (s_0, s_1, ...)$ be a finite or infinite sequence of natural numbers. The $S-$**adic odometer** is the $+(1, 0, ...)$ (with carrying) map defined on the compact set $D = \prod_{i \geq 0} \mathbb{Z}_{s_i}$ (for a survey on odometers see [6]).*

Odometers are locally periodic.

Subshifts are a special kind of TDS on symbolic spaces where T is the left shift, σ (see [20] for definitions). It is not hard to see that if (X, σ) is a subshift then $x \in LP(\sigma)$ if and only if x is periodic (i.e. x_i is periodic for $i \geq 0$).

A closely related concept is regular recurrence. A point $x \in X$ is **regularly recurrent** if for every m there exists $p > 0$ such that $d(x, T^{ip}x) \leq 1/m$ for every $i \in \mathbb{N}$. Every LP point is regularly recurrent, but regularly recurrent points are not necessarily even LEP (for example Toeplitz subshifts [6]). LEP points are not necessarily regularly recurrent (the point 1000... in a one-sided subshift). If every point of a minimal TDS is regularly recurrent then it is conjugate to an odometer [3], and hence LP.

Equicontinuity and local periodicity are related concepts.

Using results from [4] it is easy to show that if X is the unit interval and (X,T) an equicontinuous TDS then (X,T) is LEP. Using results from [22] it is easy to see that if X is the unit circle and (X,T) an equicontinuous TDS then (X,T) is either an irrational rotation or LEP.

Suppose X a subshift and $T : X \to X$ a continuous shift-commuting transformation (these TDS are known as cellular automata or shift endomorphisms). In [11] it is shown that if (X,T) is LP then it is equicontinuous, that if (X,T) is equicontinuous then it is LEP, and that there exist non-equicontinuous LEP systems.

Definition 20 *Let (X,T) be a TDS and μ a Borel probability measure on X. If $\mu(LEP(T)) = 1$ we say (X,T) is $\mu-$**locally eventually periodic** ($\mu-$**LEP**) and μ is $T-$**locally eventually periodic** ($\mathbf{T}-\mathbf{LEP}$). We define $\mu - \mathbf{LP}$ and $\mathbf{T} - \mathbf{LP}$ analogously.*

Lemma 21 *Let $m \in \mathbb{N}$ and $\varepsilon > 0$. If (X,T) is $\mu-LEP$ then there exist positive integers p_ε^m and q_ε^m such that $\mu(Y_\varepsilon^m) > 1 - \varepsilon$, where*

$$Y_\varepsilon^m := \{x \mid x \in LEP(T),\ \text{with } p_m(x) \le p_\varepsilon^m \text{ and } pp_m(x) \le q_\varepsilon^m\}\,. \qquad (3)$$

Proof. Let

$$Y := \cup_{s,t\in\mathbb{N}} \{x \mid x \in LEP(T),\ \text{with } p_m(x) \le s \text{ and } pp_m(x) \le t\}\,.$$

Since (X,T) is $\mu - LEP$ we have that $\mu(Y) = 1$. Monotonicity of the measure gives the desired result. ■

Definition 22 *Let (X,T) be a $\mu - LEP$ TDS, $m \in \mathbb{N}$, and $\varepsilon > 0$. We will use p_ε^m and q_ε^m to denote a particular choice of integers that satisfy the conditions of Lemma 21 and that satisfy $p_\varepsilon^m \to \infty$ and $q_\varepsilon^m \to \infty$, as $\varepsilon \to 0$. We define Y_ε^m as in (3).*

Proposition 23 *Let (X,T) be a TDS. If (X,T) is $\mu - LEP$ then (X,T) is $\mu-$equicontinuous.*

Proof. Let $\varepsilon > 0$ and $M := \cap_{m\in\mathbb{N}} Y_{\varepsilon/2^m}^m$ (hence $\mu(M) > 1 - \varepsilon$).

Let $m \in \mathbb{N}$ and $x \in Y_{\varepsilon/2^m}^m$. There exists K such that if $d(x,y) < 1/K$ then $d(T^i x, T^i y) < 1/m$ for $0 \le i \le (p_\varepsilon^m)^2 + q_\varepsilon^m$. Let $x, y \in Y_\varepsilon^m$, $d(x,y) < 1/K$, $p = p_m(x)p_m(y)$ and $i \ge p + q_\varepsilon^m$. We can express $i = q_\varepsilon^m + j \cdot p + k$ with $j \in \mathbb{N}$ and $k \le p$. Using the fact that $x, y \in Y_\varepsilon^m$ and $q_\varepsilon^m + k \le (p_\varepsilon^m)^2 + q_\varepsilon^m$ we obtain

$$\begin{aligned} d(T^i x, T^i y) &\le d(T^i x, T^{q_\varepsilon^m + k} x) + d(T^{q_\varepsilon^m + k} x, T^{q_\varepsilon^m + k} y) + d(T^{q_\varepsilon^m + k} y, T^i y) \\ &\le 3/m. \end{aligned}$$

This means that $B_K(x) \cap Y_\varepsilon^m \subset B^o_{m/3}(x)$ and hence $T\mid_M$ is equicontinuous. ■

The converse of this proposition is not true in general. An irrational rotation on the circle is equicontinuous, and hence $\mu-$equicontinuous, but contains no LEP points so it is not $\mu - LEP$. For cellular automata there are conditions (for example if X is a shift of finite type and μ a shift$-$invariant measure) under which $\mu-$equicontinuity implies $\mu - LEP$ [11].

3 Invariant measures and spectral properties

In this section we are interested in properties of invariant measures of topological dynamical systems.

We say (M, T, μ) is a **measure preserving transformation (MPT)** if (M, μ) is a probability measure space, and $T : M \to M$ is a measure preserving transformation (measurable and $T\mu = \mu$). We say (M, T, μ) is **ergodic** if it is a MPT and every invariant set has measure 0 or 1.

Two measure preserving transformations (M_1, T_1, μ_1) and (M_2, T_2, μ_2) are **isomorphic (measurably)** if there exists 1-1 bi-measurable preserving function $f : (X_1, \mu_1) \to (X_2, \mu_2)$, and that satisfies $T_2 \circ f = f \circ T_1$.

The spectral theory of ergodic systems is particularly useful for studying rigid transformations. We will give the definitions and state the most important results. For more details and proofs see [23].

A measure preserving transformation T on a probability measure space (M, μ) generates a unitary linear operator on the Hilbert space $L^2(M, \mu)$, by $U_T : f \mapsto f \circ T$, known as the **Koopman operator**. The spectrum of the Koopman operator is called the **spectrum** of the measure preserving transformation. We say the spectrum of (M, T, μ) is **discrete (or pure point)** if there exists an orthonormal basis for $L^2(M, \mu)$ which consists of eigenfunctions of the Koopman operator. The spectrum is **rational** if the eigenvalues are complex roots of unity. Classical results by Halmos and Von Neumann state that two ergodic MPT with discrete spectrum have the same group of eigenvalues if and only if they are isomorphic, and that an ergodic MPT has discrete spectrum if and only if it is isomorphic to a rotation on a compact metric abelian group.

Theorem 24 ([16]) *Let (X, μ, T) be an ergodic $\mu-$equicontinuous TDS. Then (X, μ, T) has discrete spectrum.*

The converse is not true. For example Sturmian subshifts with their unique invariant measure have discrete spectrum and it is not hard to see they have no $\mu-$equicontinuity points. Nonetheless using a weaker version of $\mu-$equicontinuity ($\mu-$mean equicontinuity) it is possible to characterize discrete spectrum for ergodic TDS (see [12]).

In the next subsection we show that ergodic $\mu - LEP$ TDS on Cantor spaces have rational spectrum.

3.1 Local periodicity

Lemma 25 *Let (X,T) be a $\mu-LEP$ TDS and $m \in \mathbb{N}$. If $x \in LEP_m(T) \diagdown LP_m(T)$ then*

$$\lim_{n\to\infty} T^n \mu(B_m^o(x)) = 0.$$

Proof. Let $\varepsilon > 0$, and $x \in LEP_m(T) \diagdown LP_m(T)$. The periods in Y_ε^m are bounded so if n is sufficiently large then $T^{-n}(B_m^o(x)) \cap Y_\varepsilon^m = \emptyset$; hence

$$\lim_{n\to\infty} T^n \mu(B_m^o(x)) < \varepsilon.$$

■

Proposition 26 *Let (X,T) be a $\mu - LEP$ TDS. If $T\mu = \mu$ then T is $\mu - LP$.*

Proof. Since (X,T) is $\mu-$equicontinuous we have there exists $X' \subset X$ such that X' is d_o-separable and $\mu(X') = 1$ (Theorem 9). If $x \in LEP(T) \diagdown LP(T)$ then by Lemma 25 there exists m such that $T^n\mu(B_m^o(x)) \to 0$. Using the fact that μ is an invariant measure we get that $\mu(B_m^o(x)) = T^n\mu(B_m^o(x)) = 0$. This implies $\mu(X' \cap LEP(T) \diagdown LP(T)) = 0$. Therefore $\mu(LP(T)) = 1$. ■

Theorem 27 *Let X be a Cantor space, (X,T) a TDS and μ an invariant probability measure. If (X,T) is $\mu - LEP$ then (X,T,μ) has discrete rational spectrum.*

Proof. Using Proposition 26 we have that (X,T) is $\mu - LP$.

Let $m \in \mathbb{N}$, $y \in LP(T)$ and $i = \sqrt{-1}$. We define $\lambda_{m,y} := e^{2\pi i/p_m(y)} \in \mathbb{C}$ and

$$f_{m,y,k} := \sum_{j=0}^{p_m(y)-1} \lambda_{m,y}^{j\cdot k} \cdot 1_{B_m^o(T^j y)} \in L^2(X,\mu).$$

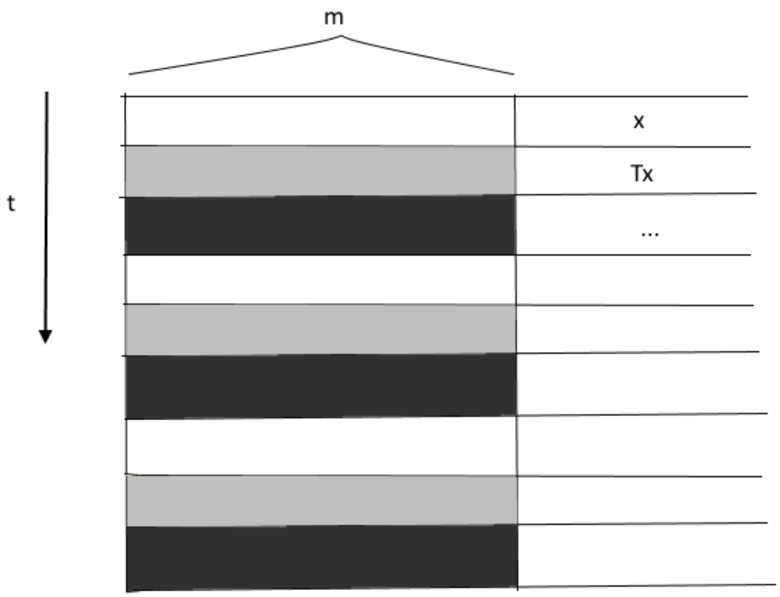
Using the fact that (X,T) is $\mu - LP$, $y \in LP(T)$ and X is a Cantor space (see Remark 18) we have that $B_m^o(T^{p_m(y)}y) = B_m^o(y)$ and

$$U_T 1_{B_m^o(T^j y)} = \begin{cases} 1_{B_m^o(T^{j-1}y)} & \text{if } 1 \le j < p_m(y) - 1 \\ 1_{B_m^o(T^{p_m(y)-1}y)} & \text{if } j = 0 \end{cases}.$$

This implies that

$$\begin{aligned} U_T f_{m,y,k} &= \sum_{j=0}^{p_m(x)-1} \lambda_{m,y}^{j\cdot k} \cdot 1_{B_m^o(T^{j-1}y)} \\ &= \sum_{j=0}^{p_m(y)-1} \lambda_{m,y}^{(j+1)\cdot k} \cdot 1_{B_m^o(T^j y)} \\ &= \lambda_{m,y}^k f_{m,y,k} \end{aligned}$$

Figure 1: An LP point on a Cantor set.



Thus $f_{m,y,k}$ is an eigenfunction corresponding to the eigenvalue $\lambda^k_{m,y}$, which is a complex root of unity.

Considering that

$$\sum_{k=0}^{p_m(y)-1} \lambda^{j\cdot k}_{m,y} = \left\{ \begin{array}{cc} 0 & \text{if } j>0 \\ p_m(y) & \text{if } j=0 \end{array} \right. ,$$

we obtain

$$\frac{1}{p_m(y)} \sum_{k=0}^{p_m(y)-1} f_{m,y,k} = 1_{B^o_m(y)}.$$

This means $1_{B^o_m(y)} \in Span\left\{f_{m,y,k}\right\}_k$.

Let $n \in \mathbb{N}$, and $x \in X$. Since (X,T) is $\mu-$ LP and X is a Cantor set there exists a sequence $\{y_i\} \subset LP(T)$ such that $B^o_n(y_i)$ are disjoint and $\mu(B_n(x)) = \mu(\cup B^o_n(y_i))$. This means $1_{B_n(x)}$ can be approximated in L^2 by elements in

$$Span\left\{1_{B^o_m(y)} : m \in \mathbb{N} \text{ and } y \in LP(T)\right\}.$$

Since the closure of $Span\left\{1_{B_n(x)} : n \in \mathbb{N},\, x \in X\right\}$ is $L^2(X,\mu)$ we conclude the closure of $Span\left\{f_{m,y,k}\right\}_{m,y,k}$ is $L^2(X,\mu)$. ■

Corollary 28 *Let X be a Cantor space, (X,T) a TDS and μ an ergodic probability measure. If (X,T) is $\mu-LEP$ then (X,T,μ) is isomorphic to an odometer*

Proof. The set of odometers cover all the possible rational spectrums (see [6]). We obtain the following corollary. Two MPTs with discrete spectrum are isomorphic if and only if they are spectrally isomorphic. ■

This result is particularly useful for cellular automata, as $\mu - LEP$ invariant measures appear naturally as the limit measures of $\mu-$equicontinuous CA; for more information see [11].

3.2 $\mu-$Sensitivity

The following notion of measure theoretical sensitivity was introduced in [5], and it was characterized in [16]. We provide another characterization.

Definition 29 ([5][16]) *Let (X, T) be a TDS and μ an invariant measure. We define the set $S(\varepsilon) := \left\{(x, y) \in X^2 : \exists n > 0 \text{ such that } d(T^n x, T^n y) \geq \varepsilon\right\}$.*

*We say (X, T) is $\mu-$**sensitive** (or $\mu-$**pairwise sensitive**) if there exists $\varepsilon > 0$ such that $\mu \times \mu(S(\varepsilon)) = 1$.*

In [16] it was shown that if μ is ergodic then either (X, T) is $\mu-$sensitive or $\mu-$equicontinuous.

Theorem 30 *Let (X, μ) be a Vitali space that satisfies Lebesgue's density theorem (X, T) be a TDS and μ an ergodic measure. Then (X, T) is $\mu-$sensitive if and only if X contains no $\mu-$equicontinuity points.*

Proof. Let E_μ be the set of $\mu-$equicontinuity points.

By Theorem 9 and the previous comment we have that T is $\mu-$sensitive or E_μ has measure one.

If $E_\mu = \varnothing$ then (X, T) is $\mu-$sensitive.

On the other hand note that if $x \in E_\mu$ then for all $\varepsilon = 1/m > 0$ we have that $\mu(B^o_m(x)) > 0$. This means that $\mu \times \mu(S(\varepsilon)) < 1$ for all $\varepsilon > 0$. ■

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