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ON THE LARGEST PRIME FACTORS OF CONSECUTIVE INTEGERS IN SHORT INTERVALS

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ABSTRACT. For an integer $n > 1$, let $P(n)$ be the largest prime factor of n. We prove that, for $x \to \infty$, there exists a positive proportion of consecutive integers n and $n+1$ such that $P(n) < P(n+1)$ in short intervals $(x, x+y]$ with $x^{7/12} < y \leqslant x$. In particular, we have

$$\left|\{n \leqslant x : P(n) < P(n+1)\}\right| > 0.1063x.$$

This improves a previous result of La Bretèche, Pomerance and Tenenbaum.

1. INTRODUCTION

For each integer $n \geqslant 1$, let $P(n)$ denote the largest prime factor of n with the convention that $P(1) = 1$. In 1930, Dickman [2] obtained the well-known result: the following asymptotic formula

$$\Psi(x, y) = |\{n \leqslant x : P(n) \leqslant y\}| \sim x\varrho(u) \quad (u \geqslant 1) \tag{1.1}$$

holds for $x \to \infty$ with $u = \log x / \log y$ fixed, where $\varrho(u)$ is the Dickman function. Furthermore, one conjectured that the largest prime factors of consecutive integers n and $n+1$ are "independent events" and the density of integers n with $P(n) < P(n+1)$ is $1/2$. In this direction, in 1978 Erdős and Pomerance [3] proved that there exists a positive proportion of integers n with $P(n) < P(n+1)$. More precisely, they proved that

Theorem (A). *For $x \to \infty$, we have*

$$|\{n \leqslant x : P(n) < P(n+1)\}| > 0.0099x.$$

Another important problem on consecutive integers is the following. Let $\{\varepsilon_n\}_{1 \leqslant n < N}$ be a finite sequence with each $\varepsilon_n \in \{-1, 0, 1\}$, and write

$$\frac{a}{b} = \prod_{1 \leqslant n < N} \left(\frac{n}{n+1}\right)^{\varepsilon_n}, \tag{1.2}$$

where the fraction is in its smallest terms, then define $A(N)$ as the maximal value of a as $\{\varepsilon_n\}_{1 \leqslant n < N}$ runs through all possible 3^{N-1} sequences of $-1, 0, 1$. In 1988, Nicolas [7] showed that

$$\log A(N) \leqslant \left\{\frac{2}{3} + o(1)\right\} N \log N,$$

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and further a brief argument of M. Langevin is presented that

$$\log A(N) \geqslant \{\log 4 + o(1)\}N.$$

In 2005, La Bretèche, Pomerance and Tenenbaum [1] improved the lower bound of $\log A(N)$ to the same order of magnitude as the upper bound:

$$\log A(N) \geqslant \{0.107005 + o(1)\}N \log N. \tag{1.3}$$

In order to prove this lower bound, they studied the quantity

$$S(x,c) := |\{n \leqslant x : P(n) > x^{1-c} \text{ and } P(n+1) > x^{1-c}\}| \tag{1.4}$$

and proved by the sieve methods that

$$S(x,c) \leqslant 2x \int_0^c \log\left(\frac{1-v}{1-v-2c}\right)\frac{\mathrm{d}v}{1-v} + o(x) \tag{1.5}$$

for any $c \in (0, \frac{1}{5})$ and $x \to \infty$. Clearly this is of independent interest. As they indicated, (1.3) is an immediate consequence of (1.5). As another application of (1.5), they also can improve the result of Theorem (A) and obtain

Theorem (B). *For $x \to \infty$, we have*

$$|\{n \leqslant x : P(n) < P(n+1)\}| > 0.05544x.$$

In addition, in their paper they also indicated that the constant 0.05544 can be improved to 0.05866 by using more sophisticated sieve methods thanks to an observation of Fouvry, and so the constant 0.107005 in (1.3) can also be improved to 0.112945.

The purpose of this short note is to prove that there exists again a positive proportion of consecutive integers n and $n+1$ with $P(n) < P(n+1)$ when n varies in short intervals. It seems the largest prime factors of consecutive integers are also "independent events" in short intervals. Moreover, we can improve Fouvry's constant 0.05866.

Our results are as follows.

Theorem 1.

(i) *Let*

$$\tfrac{3}{5} < \theta \leqslant 1 \quad \text{and} \quad 0 < c < \min\left\{\tfrac{5\theta-3}{2},\ \theta - \tfrac{1}{2}\right\}. \tag{1.6}$$

Then for $x \to \infty$ and $y = x^\theta$, we have

$$\big|\{x < n \leqslant x + y : P(n) < P(n+1)\}\big| \geqslant \{g(\theta; c) + o_{\theta,c}(1)\}y, \tag{1.7}$$

where

$$g(\theta; c) := \log \frac{1}{1-c} - 2\int_0^c \frac{\log(1-v)^{-1}}{\theta - \frac{1}{2} - v}\mathrm{d}v. \tag{1.8}$$

(ii) *For $\frac{3}{5} < \theta \leqslant 1$, there is a unique $c(\theta) \in \big(0, \min\{\frac{5\theta-3}{2},\ \theta - \frac{1}{2}\}\big)$ such that*

$$g(\theta) := \max_{0<c<\min\{\frac{5\theta-3}{2},\, \theta-\frac{1}{2}\}} g(\theta; c) = g(\theta; c(\theta)) > 0.$$

In particular, we have $g(1) > g(1; 0.1778) > 0.1063$.

(iii) *For $x \to \infty$, we have*

$$\big|\{n \leqslant x : P(n) < P(n+1)\}\big| > 0.1063x. \tag{1.9}$$

One of the principal tools for proving Theorem 1 is Wu's mean value theorem of convolution type in short intervals [10, Theorem 2] (see Lemma 2.2 below). With the help of [9, Theorem 1] (see Lemma 2.3 below), we can also get a similar Theorem 2 for $\frac{7}{12} < \theta \leqslant 1$. Obviously, Theorem 2 allows a wider range for θ, but we have

$$\begin{cases} g(\theta) > h(\theta) & \text{if } \theta \in (\frac{3}{5} + \delta_0,\, 1], \\ g(\theta) \leqslant h(\theta) & \text{if } \theta \in (\frac{3}{5},\, \frac{3}{5} + \delta_0], \end{cases}$$

where δ_0 is a very small positive constant, and $h(\theta)$ is defined by (1.12) below. Hence Theorem 1 gives a better lower bound than Theorem 2 for $\theta \in (\frac{3}{5} + \delta_0,\, 1]$, and Theorem 2 gives a better lower bound than Theorem 1 for $\theta \in (\frac{3}{5},\, \frac{3}{5} + \delta_0]$.

Theorem 2.
(i) *Let*

$$\tfrac{7}{12} < \theta \leqslant 1.$$

Then for $x \to \infty$ and $y = x^\theta$, we have

$$\big|\{x < n \leqslant x + y : P(n) < P(n+1)\}\big| \geqslant \{h(\theta; c) + o_{\theta,c}(1)\}y,$$

where $h(\theta; c)$ is defined by

$$h(\theta; c) := \log \frac{1}{1-c} - 2\int_0^c \log \frac{(\theta - 1/2)(1-v)}{(\theta - 1/2)(1-v) - c} \cdot \frac{dv}{\theta - v}$$

if

$$0 < c < \min\{\tfrac{2\theta-1}{2\theta+3},\, \tfrac{12\theta-7}{5}\}, \quad \tfrac{3}{5} < \theta \leqslant 1, \tag{1.10}$$

and

$$h(\theta; c) := \log \frac{1}{1-c} - 2\int_0^c \log \frac{(\theta - 11/20)(1-v)}{(\theta - 11/20)(1-v) - c} \cdot \frac{dv}{\theta - v}$$

if

$$0 < c < \min\{\tfrac{20\theta-11}{20\theta+29},\, \tfrac{12\theta-7}{5}\}, \quad \tfrac{7}{12} < \theta \leqslant \tfrac{3}{5}. \tag{1.11}$$

(ii) *For $\frac{7}{12} < \theta \leqslant 1$, there is a unique $c(\theta)$ satisfying* (1.10) *and* (1.11) *such that*

$$h(\theta) := \max_c h(\theta; c) = h(\theta; c(\theta)) > 0. \tag{1.12}$$

Remark. Similarly to the proof of [1, Theorem 1.1], we can easily extend (1.3) to the short intervals by following Fouvry's argument [1]. Define $A(N, N^\theta)$ as the maximal value of a as $\{\varepsilon_n\}_{1\leqslant n<N}$ runs through all possible 3^{N-1} sequences of $-1, 0, 1$ with

$$\frac{a}{b} = \prod_{N<n\leqslant N+N^\theta} \left(\frac{n}{n+1}\right)^{\varepsilon_n} \qquad (0 < \theta \leqslant 1).$$

Then for $N \to \infty$ and θ, c satisfying (1.6), we have

$$\log A(N, N^\theta) \geqslant 2\{h(\theta; c) + o_{\theta,c}(1)\}N^\theta \log N,$$

where

$$h(\theta; c) := c - 2(1-c)\log(1-c)\log\big(1 - \tfrac{2c}{2\theta-1}\big) - 2\int_0^c \log(1-u)\log\big(1 - \tfrac{2u}{2\theta-1}\big)\mathrm{d}u.$$

For Theorem 1, we shall follow Fouvry's argument [1]. Let $y = x^\theta$ with $\frac{3}{5} < \theta \leqslant 1$. The starting point is the inequality

$$\sum_{\substack{x<n\leqslant x+y\\ P(n-1)<P(n)}} 1 \geqslant \sum_{\substack{x<n\leqslant x+y\\ P(n)>n^{1-c}}} 1 - \sum_{\substack{x<n\leqslant x+y\\ P(n-1)>P(n)>n^{1-c}}} 1 =: \mathcal{S}_A - \mathcal{S}_B \tag{1.13}$$

with $0 < c \leqslant \frac{1}{2}$. So we only need to give a lower bound of $\mathcal{S}_A$ and an upper bound of $\mathcal{S}_B$.

To estimate $\mathcal{S}_A$, we shall use the asymptotic formula about the distribution of friable numbers in short intervals.

For $\mathcal{S}_B$, it counts the number of n satisfying $n = ap+1 = bp'$ with $p > p' > n^{1-c}$, namely $a < b \leqslant n^c$. So we have

$$\begin{aligned}\mathcal{S}_B &\leqslant |\{x < n \leqslant x+y : n = ap+1 = bp', a < b \leqslant (x+y)^c\}|\\ &= \sum_{b\leqslant(x+y)^c} |\{x < n \leqslant x+y : n = ap+1 = bp', a < b\}|\\ &= \sum_{b\leqslant(x+y)^c} |\{n \in \mathscr{A}(b) : n \text{ is prime}\}|,\end{aligned}$$

where

$$\mathscr{A}(b) := \left\{\frac{ap+1}{b} : x < ap \leqslant x+y,\ a < b,\ ap+1 \equiv 0 (\operatorname{mod} b)\right\}. \tag{1.14}$$

For Theorem 1, we consider $\mathcal{S}_B$ with the condition "$a < b \leqslant (x+y)^c$" by substituting for "$a \leqslant x^c$, $b \leqslant x^c$" in Fouvry's arguments [1, Further remarks], so in fact we sieve only half of the sequence from before. This is why we can improve Fouvry's constant "0.05866" to "0.1063".

But for $A(N, N^\theta)$ in the Remark, we should consider the sum

$$\sum_{\substack{x<n\leqslant x+y\\ P(n-1)>n^{1-c},\ P(n)>n^{1-c}}} 1$$

with the condition "$a \leqslant (x+y)^c$, $b \leqslant (x+y)^c$", not the sum $\mathcal{S}_B$ with "$a < b \leqslant (x+y)^c$", if n is of the form $n = ap+1 = bp'$, so unfortunately when $\theta = 1$ we can't improve Fouvry's constant 0.112945 in Theorem (B).

Then for $\mathcal{S}_B$ we will use Rosser-Iwaniec's sieve [5,6] to sieve $\mathscr{A}(b)$ by the set of primes

$$\mathscr{P} = \{p : p \text{ is prime}\}, \tag{1.15}$$

and then give an upper bound of $\mathcal{S}_B$. In addition, we also need some generalized Bombieri-Vinogradov theorems in short intervals [10].

Throughout this paper, we denote by ε an arbitrarily small positive constant, and p, p' primes. For convenience, we write $\mathscr{L} := \log x$.

2. Lemmas

Let $\mathcal{A}$ be a finite sequence of integers, $\mathcal{P}$ a set of primes, $z \geqslant 2$ a real number and d a squarefree integer with all its prime factors belonging to $\mathcal{P}$, and denote

$$\mathcal{A}_d := \{a \in \mathcal{A} : d \mid a\}, \qquad P(z) := \prod_{p<z,\, p\in\mathcal{P}} p.$$

We shall evaluate

$$S(\mathcal{A};\mathcal{P},z) := |\{a \in \mathcal{A} :\ (a,\ P(z)) = 1\}|.$$

Assume that $|\mathcal{A}_d|$ may be written in the form

$$|\mathcal{A}_d| = \frac{\omega(d)}{d}X + r(\mathcal{A},d) \quad \text{for } d \mid P(z),$$

where X is an approximation to $|\mathcal{A}|$ independent of d, ω is a multiplicative function satisfying $0 < \omega(p) < p$ for $p \in \mathcal{P}$, $\omega(d)d^{-1}X$ is considered as a main term and $r(\mathcal{A},d)$ is an error term which we expect to be small on average over d.

The first lemma is a simple consequence of [5, Theorem 1].

Lemma 2.1. *Suppose that there exists a constant $K \geqslant 2$ such that*

$$\prod_{w\leqslant p<v}\left(1-\frac{\omega(p)}{p}\right)^{-1} < \frac{\log v}{\log w}\left(1+\frac{K}{\log w}\right)$$

for all $v > w \geqslant 2$. Then for any $D \geqslant z \geqslant 2$, we have

$$S(\mathcal{A};\mathcal{P},z) \leqslant XV(z)\left\{F(s)+O\left(\frac{1}{(\log y)^{1/3}}\right)\right\} + \sum_{d<D,\, d\mid P(z)} |r(\mathcal{A},d)|,$$

where $s := \log D/\log z$, γ is the Euler constant and

$$V(z) := \prod_{p<z,\, p\in\mathcal{P}}\left(1-\omega(p)p^{-1}\right), \qquad F(s) := 2\mathrm{e}^{\gamma}s^{-1} \quad (0 < s \leqslant 3).$$

For $(a,\ q) = 1$ and $\ell \geqslant 1$, we define

$$\pi(z;\, q, a, \ell) = \sum_{\substack{\ell p\leqslant z\\ \ell p\equiv a (\mathrm{mod}\, q)}} 1.$$

The second lemma is [10, Theorem 2].

Lemma 2.2. *Let $g(\ell)$ be an arithmetic function satisfying*

$$\sum_{\ell\leqslant x} |g(\ell)|^2\ell^{-1} \ll \mathscr{L}^{\lambda}$$

for some positive constant $\lambda > 0$. Define

$$H(z,h;\, q,a,\ell) := \pi(z+h;\, q,a,\ell) - \pi(z;\, q,a,\ell) - \frac{1}{\varphi(q)}\int_{z/\ell}^{(z+h)/\ell}\frac{dt}{\log t}. \tag{2.1}$$

Then for any $A > 0$ and $\varepsilon > 0$, there exists a constant $B = B(A,\ \lambda) > 0$ such that the estimate

$$\sum_{q\leqslant Q}\max_{(a,\,q)=1}\max_{h\leqslant y}\max_{x/2<z\leqslant x}\Bigg|\sum_{\substack{\ell\leqslant L\\ (\ell,\,q)=1}} g(\ell)H(z,h;\, q,a,\ell)\Bigg| \ll_{A,\lambda,\varepsilon} y\mathscr{L}^{-A}$$

holds uniformly for $\frac{3}{5}+\varepsilon \leqslant \theta \leqslant 1$, $Q = x^{\theta-1/2}\mathscr{L}^{-B}$ and $L = x^{(5\theta-3)/2-\varepsilon}$.

In particular, we can easily see that Lemma 2.2 covers the result of Perelli, Pintz and Salerno [8, Theorem], which states that if $\pi(z;\, q, a) := \pi(z;\, q, a, 1)$, then for any $A > 0$ we have

$$\sum_{q\leqslant Q}\max_{(a,\,q)=1}\max_{h\leqslant y}\max_{x/2<z\leqslant x}\left|\pi(z+h;\,q,a)-\pi(z;\,q,a)-\frac{1}{\varphi(q)}\int_z^{z+h}\frac{\mathrm{d}t}{\log t}\right|\ll y\mathscr{L}^{-A} \tag{2.2}$$

uniformly for $\frac{3}{5} < \theta \leqslant 1$ and $Q = x^{\theta-1/2}\mathscr{L}^{-B}$, where $B = B(A)$ is a positive constant.

The third lemma [9, Theorem 1] allows a wider range for y than [8, Theorem], which states that by an equivalent version:

Lemma 2.3. *Given any positive constants A, ε and δ, (2.2) is valid for $x^{7/12+\varepsilon} \leqslant y \leqslant x$ with $Q = yx^{-11/20-\delta}$, and for $x^{3/5}(\log x)^{2(A+64)+1} \leqslant y \leqslant x$ with $Q = yx^{-1/2}(\log x)^{(A+64)}$, where the implied constants depend only on A, ε and δ.*

The fourth lemma is [4, Theorem 3].

Lemma 2.4. *As $\Psi(x,\, y)$ is defined by (1.1), then the asymptotic formula*

$$\Psi(x+z,\, y) - \Psi(x,\, y) \sim \frac{z}{x}\Psi(x,\, y) \sim z\varrho(u) \tag{2.3}$$

holds uniformly for $1 \leqslant x/z \leqslant y^{5/12}$ and $\exp\left\{(\log\mathscr{L})^{5/3+\varepsilon}\right\} \leqslant y \leqslant x$.

3. Proofs of Theorem 1 and Theorem 2

To prove Theorem 1, recall that $\mathcal{S}_A, \mathcal{S}_B, \mathscr{A}(b)$ and $\mathscr{P}$ are defined in (1.13), (1.14) and (1.15) respectively. First, we shall give an upper bound of $\mathcal{S}_B$, and then estimate $\mathcal{S}_A$.

For $\mathcal{S}_B$, we first sieve $\mathscr{A}(b)$ by $\mathscr{P}$ and then sum over b. Let θ and c be two positive real numbers satisfying (1.6) respectively, and $z = x^{\theta-1/2}/(b\mathscr{L}^{B_0})$, where B_0 is an appropriate positive constant. So for $b \leqslant (x+y)^c$ and $d \mid P(z)$, we have

$$\begin{aligned}|\mathscr{A}_d(b)| &= \left|\left\{\frac{ap+1}{b} : x < ap \leqslant x+y,\ a < b, ap+1 \equiv 0\,(\mathrm{mod}\, bd)\right\}\right|\\ &= \sum_{a<b}\ \sum_{\substack{x<ap\leqslant x+y\\ ap+1\equiv 0(\mathrm{mod}\ bd)}} 1.\end{aligned}$$

With the notation (2.1) we can write

$$\begin{aligned}|\mathscr{A}_d(b)| &= \sum_{a<b}\ \sum_{\substack{x<ap\leqslant x+y\\ ap+1\equiv 0(\mathrm{mod}\,bd)}} 1\\ &= \sum_{\substack{a<b\\ (a,\,bd)=1}}\frac{1}{\varphi(bd)}\int_{x/a}^{(x+y)/a}\frac{\mathrm{d}t}{\log t} + \sum_{\substack{a<b\\ (a,\,bd)=1}} H(x, y;\, bd, -1, a). \qquad (3.1)\end{aligned}$$

For the first sum on the right hand side of (3.1), we have

$$\sum_{\substack{a<b\\(a,\,bd)=1}} \frac{1}{\varphi(bd)} \int_{x/a}^{(x+y)/a} \frac{\mathrm{d}t}{\log t} = \sum_{\substack{a<b\\(a,\,bd)=1}} \frac{y}{a\varphi(bd)} \left\{ \frac{1}{\log \frac{x}{a}} + O\left(\frac{1}{\log \frac{x}{a}} - \frac{1}{\log \frac{x+y}{a}} \right) \right\}$$
$$= \left\{ 1 + O\left(\frac{1}{\mathscr{L}}\right) \right\} \frac{y}{\varphi(bd)} \sum_{\substack{a<b\\(a,\,bd)=1}} \frac{1}{a\log(x/a)}.$$

By the Möbius inversion, we have

$$\sum_{\substack{a<b\\(a,\,bd)=1}} \frac{1}{\varphi(bd)} \int_{x/a}^{(x+y)/a} \frac{\mathrm{d}t}{\log t} = \left\{ 1 + O\left(\frac{1}{\mathscr{L}}\right) \right\} \frac{y}{\varphi(bd)} \sum_{a<b} \frac{1}{a\log(x/a)} \sum_{q|(a,\,bd)} \mu(q)$$
$$= \left\{ 1 + O\left(\frac{1}{\mathscr{L}}\right) \right\} \frac{y}{\varphi(bd)} \sum_{q|bd} \frac{\mu(q)}{q} \sum_{aq<b} \frac{1}{a\log(x/aq)}$$
$$= \left\{ 1 + O\left(\frac{1}{\mathscr{L}}\right) \right\} \frac{y}{\varphi(bd)} (S_1 + S_2), \tag{3.2}$$

where

$$S_1 := \sum_{q|bd,\, q<\mathscr{L}^9} \frac{\mu(q)}{q} \sum_{a<b/q} \frac{1}{a\log(x/aq)}, \qquad S_2 := \sum_{q|bd,\, q\geqslant\mathscr{L}^9} \frac{\mu(q)}{q} \sum_{a<b/q} \frac{1}{a\log(x/aq)}.$$

Denoting by $\tau(n)$ the number of divisors of n, we have

$$S_2 \ll \sum_{q|bd,\, q\geqslant\mathscr{L}^9} q^{-1} \ll \tau(bd)\mathscr{L}^{-9}. \tag{3.3}$$

And for S_1, by the partial summation we have

$$S_1 = \sum_{q|bd,\, q<\mathscr{L}^9} \frac{\mu(q)}{q} \left\{ \int_1^{b/q} \frac{\mathrm{d}t}{t\log(x/tq)} + O\left(\frac{1}{\mathscr{L}}\right) \right\}$$
$$= \sum_{q|bd,\, q<\mathscr{L}^9} \frac{\mu(q)}{q} \left\{ \log\left(\frac{\log x - \log q}{\log x - \log b} \right) + O\left(\frac{1}{\mathscr{L}}\right) \right\}$$
$$= \sum_{q|bd,\, q<\mathscr{L}^9} \frac{\mu(q)}{q} \left\{ \log\left(\frac{\log x}{\log(x/b)} \right) + O\left(\frac{\log\mathscr{L}}{\mathscr{L}}\right) \right\};$$

owing to the condition $q < \mathscr{L}^9$, we can separate the above $\log q$ from the main term. Thus we have

$$S_1 = \left(\sum_{q|bd} \frac{\mu(q)}{q} - \sum_{\substack{q|bd\\q\geqslant\mathscr{L}^9}} \frac{\mu(q)}{q} \right) \left\{ \log\left(\frac{\log x}{\log(x/b)} \right) + O\left(\frac{\log\mathscr{L}}{\mathscr{L}}\right) \right\}$$
$$= \frac{\varphi(bd)}{bd} \left\{ \log\left(\frac{\log x}{\log(x/b)} \right) + O\left(\frac{\log\mathscr{L}}{\mathscr{L}}\right) \right\} + O\left(\frac{\tau(bd)}{\mathscr{L}^9}\right). \tag{3.4}$$

So we infer from (3.1), (3.2), (3.3) and (3.4) that

$$|\mathscr{A}_d(b)| = \frac{\omega(d)}{d} X + r(\mathscr{A}(b), d)$$

with $\omega(d) = 1$ and

$$X = \frac{y}{b} \log\left(\frac{\log x}{\log(x/b)}\right)\left\{1 + O\left(\frac{\log \mathscr{L}}{\mathscr{L}}\right)\right\},$$

$$r(\mathscr{A}(b), d) = \sum_{\substack{a<b \\ (a,\, bd)=1}} H(x, y;\, bd, -1, a) + O\left(\frac{y\tau(bd)}{\varphi(bd)\mathscr{L}^9}\right).$$

Thus we can apply Lemma 2.1 with $D = z = x^{\theta-1/2}/(b\mathscr{L}^{B_0})$ to write

$$S(\mathscr{A}(b); \mathscr{P}, z) \leqslant \{1 + o(1)\}\frac{2X}{\log(x^{\theta-1/2}/b)} + \sum_{d<z,\, d|P(z)} |r(\mathscr{A}(b), d)|,$$

where we have used Mertens' formula to evaluate

$$V(z) = \prod_{p<x^{\theta-1/2}b^{-1}\mathscr{L}^{-B_0}} \left(1 - \frac{1}{p}\right) = \frac{\mathrm{e}^{-\gamma}}{\log(x^{\theta-1/2}/b)}\left\{1 + O\left(\frac{\log \mathscr{L}}{\mathscr{L}}\right)\right\}.$$

From this and the trivial inequality

$$|\{n \in \mathscr{A}(b) : n \text{ is prime}\}| \leqslant S(\mathscr{A}(b); \mathscr{P}, z) + z,$$

we deduce that

$$\mathcal{S}_B \leqslant \sum_{b\leqslant(x+y)^c} (S(\mathscr{A}(b); \mathscr{P}, z) + z) \leqslant \{1 + o(1)\}S_3 + S_4 + O\big(y\mathscr{L}^{-1}\big), \tag{3.5}$$

where

$$S_3 := \sum_{b\leqslant(x+y)^c} \frac{2y}{b\log(x^{\theta-1/2}/b)} \log\left(\frac{\log x}{\log(x/b)}\right),$$

$$S_4 := \sum_{b\leqslant(x+y)^c} \sum_{d<z} \Bigg|\sum_{\substack{a<b \\ (a,\, bd)=1}} H(x, y;\, bd, -1, a)\Bigg|,$$

and we have used the standard estimation $\sum_{m\leqslant x} \tau(m)^2/\varphi(m) \ll \mathscr{L}^4$ to bound

$$y\mathscr{L}^{-9} \sum_{b\leqslant(x+y)^c} \sum_{d<x^{\theta-1/2}/(b\mathscr{L}^{B_0})} \frac{\tau(bd)}{\varphi(bd)} \ll y\mathscr{L}^{-9} \sum_{m<x^{1/2}\mathscr{L}^{-B_0}} \frac{\tau(m)^2}{\varphi(m)} \ll y\mathscr{L}^{-5}.$$

First we evaluate the error term S_4. By the Cauchy-Schwarz inequality, it follows that

$$S_4 \leqslant \sum_{m<x^{\theta-1/2}\mathscr{L}^{-B_0}} \tau(m)\Bigg|\sum_{\substack{a<b \\ (a,\, m)=1}} H(x, y;\, m, -1, a)\Bigg| \leqslant (S_{41}S_{42})^{1/2},$$

where

$$S_{41} := \sum_{m<x^{\theta-1/2}\mathscr{L}^{-B_0}} \Bigg|\sum_{\substack{a<b \\ (a,\, m)=1}} H(x, y;\, m, -1, a)\Bigg|,$$

$$S_{42} := \sum_{m<x^{\theta-1/2}\mathscr{L}^{-B_0}} \tau(m)^2\Bigg|\sum_{\substack{a<b \\ (a,\, m)=1}} H(x, y;\, m, -1, a)\Bigg|.$$

For S_{42}, we use a trivial estimate, and for S_{41}, we shall apply Lemma 2.2 with

$$g(\ell) = \begin{cases} 1 & \text{if } \ 0 < \ell \leqslant b, \\ 0 & \text{if } \ b < \ell \leqslant x^{(5\theta-3)/2-\varepsilon}, \end{cases}$$

since $b \leqslant (x+y)^c \leqslant x^{(5\theta-3)/2-\varepsilon}$ thanks to hypothesis (1.6). So we obtain

$$\begin{aligned} S_4 &\ll \Big(\frac{y}{\mathscr{L}^A}\Big)^{1/2}\Big(\sum_{m<x^{\theta-1/2}\mathscr{L}^{-B_0}} \tau(m)^2 \sum_{a<b}\Big\{\frac{y}{am} + O(1)\Big\}\Big)^{1/2} \\ &\ll \Big(\frac{y}{\mathscr{L}^A}\Big)^{1/2}\Big(y\mathscr{L}^5 + bx^{\theta-1/2}\mathscr{L}^{-B_0+3}\Big)^{1/2} \\ &\ll \frac{y}{\mathscr{L}^{A/2-3}} \end{aligned} \tag{3.6}$$

since $y^{1/2}x^{(c+\theta-1/2)/2} \ll yx^{-(1/2-c)/2} \ll y$ with $c < \theta - \frac{1}{2} \leqslant \frac{1}{2}$.

Next we evaluate S_3. By partial summation we get

$$\begin{aligned} S_3 &= 2y\{1+o(1)\}\int_1^{(x+y)^c} \log\Big(\frac{\log x}{\log(x/t)}\Big)\frac{\mathrm{d}t}{t\log(x^{\theta-1/2}/t)} \\ &= y\bigg\{2\int_0^c \frac{\log(1-v)^{-1}}{\theta-\frac{1}{2}-v}\mathrm{d}v + o_{\theta,c}(1)\bigg\}. \end{aligned} \tag{3.7}$$

So from (3.5), (3.6) and (3.7) we obtain

$$\mathcal{S}_B \leqslant y\bigg\{2\int_0^c \frac{\log(1-v)^{-1}}{\theta-\frac{1}{2}-v}\mathrm{d}v + o_{c,\theta}(1)\bigg\}. \tag{3.8}$$

Finally we evaluate $\mathcal{S}_A$. Noticing that $\frac{3}{5} < \theta \leqslant 1$, $0 < c < \theta - \frac{1}{2} \leqslant \frac{1}{2}$ and $\varrho(u) = 1 - \log u$ for $1 \leqslant u \leqslant 2$, Lemma 2.4 gives us immediately

$$\mathcal{S}_A = y\Big\{\log\frac{1}{1-c} + o_{c,\theta}(1)\Big\}. \tag{3.9}$$

Now, the required inequality (1.7) follows from (1.13), (3.8) and (3.9).

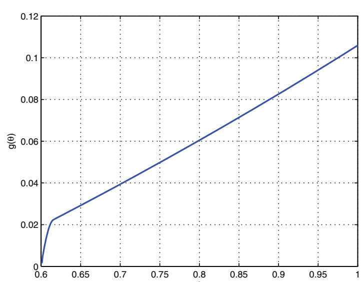

FIGURE 1. $g(\theta)$

Then for Theorem 1, part (ii), we have

$$\begin{aligned} \frac{\partial g}{\partial c}(\theta;c) &= \frac{1}{1-c} - 2\frac{\log(1-c)^{-1}}{\theta-\frac{1}{2}-c}, \\ \frac{\partial^2 g}{\partial c^2}(\theta;c) &= -\frac{1}{1-c}\Big(\frac{2}{\theta-\frac{1}{2}-c} - \frac{1}{1-c}\Big) - 2\frac{\log(1-c)^{-1}}{(\theta-\frac{1}{2}-c)^2}. \end{aligned}$$

It's obvious that $\frac{\partial^2 g}{\partial c^2}(\theta; c) < 0$ for $\frac{3}{5} < \theta \leqslant 1$ and $0 \leqslant c < \theta - \frac{1}{2}$. Thus $c \mapsto \frac{\partial g}{\partial c}(\theta; c)$ is decreasing. Since $\frac{\partial g}{\partial c}(\theta; 0) = 1$ and $\lim_{c \to (\theta - \frac{1}{2})-} \frac{\partial g}{\partial c}(\theta; c) = -\infty$, there is a unique $c(\theta) \in \big(0, \min\{\frac{5\theta-3}{2}, \theta - \frac{1}{2}\}\big)$ such that $\frac{\partial g}{\partial c}(\theta; c(\theta)) = 0$ and

$$g(\theta) := \max_{0 \leqslant c < \min\{\frac{5\theta-3}{2}, \theta - \frac{1}{2}\}} g(\theta; c) = g(\theta; c(\theta)) > g(\theta; 0) = 0$$

for $\frac{3}{5} < \theta \leqslant 1$. Figure 1 gives the graph of $g(\theta)$.

By *Mathematica* 9.0, we can find $c(1) \approx 0.1778$ and $g(1) = g(1, c(1)) > 0.1063$.

Theorem 1, part (iii), is an immediate consequence of Theorem 1, parts (i) and (ii), via a simple dyadic summation.

This completes the proof of Theorem 1.

For Theorem 2, part (i), the proof is similar to [1, Theorem 1.2] by Lemma 2.3, and for Theorem 2, part (ii), the proof is similar to Theorem 1, part (ii). Here we have omitted the details.

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