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INTERVAL DECOMPOSITION OF INFINITE ZIGZAG PERSISTENCE MODULES

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Abstract. We show that every pointwise finite-dimensional infinite zigzag persistence module decomposes into a direct sum of interval persistence modules.

1. Introduction

The results in this paper are motivated by a construction in topological data analysis called *levelset persistence* [3]. The setup is as follows: let X be a topological space and $\phi\colon X \to \mathbb{R}$ a continuous map. For $U \subseteq \mathbb{R}$, we adapt the notation $X_U = \phi^{-1}(U)$, and for $t \in \mathbb{R}$, let X_t denote the fiber over t. We assume that ϕ satisfies the following: there exists a set $S = \{s_i\}_{i\in\mathbb{Z}} \subset \mathbb{R}$, where $s_i < s_{i+1}$ and $\lim_{i\to\pm\infty} s_i = \pm\infty$, such that:

(1) The inclusion $X_{s_i} \hookrightarrow X_{(s_{i-1},s_{i+1})}$ is a homotopy equivalence.
(2) The restriction $\phi|_{X\setminus\phi^{-1}(S)}\colon X\setminus\phi^{-1}(S) \to \mathbb{R}\setminus S$ is a locally trivial fibration. That is, there exists an open interval I around every $t \in \mathbb{R}\setminus S$, such that $\phi|_{X_I}\colon X_I \to I$ is a trivial fiber bundle with fiber X_t.

Observe that the fibers X_t and $X_{t'}$ are homeomorphic for $t, t' \in (s_i, s_{i+1})$. Let $t \in (s_i, s_{i+1})$ and consider the composition $X_t \hookrightarrow X_{(s_{i-1},s_{i+1})} \to X_{s_i}$, where the latter morphism is a homotopy inverse of the inclusion $X_{s_i} \hookrightarrow X_{(s_{i-1},s_{i+1})}$. Similarly we get a morphism $X_t \to X_{s_{i+1}}$. Thus, by choosing values $\{t_i\}_{i\in\mathbb{Z}}$ satisfying $s_i < t_i < s_{i+1} < t_{i+1}$, the following zigzag completely characterizes the evolution of fibers along ϕ up to homotopy

$$\cdots \leftarrow X_{t_{i-1}} \to X_{s_i} \leftarrow X_{t_i} \to X_{s_{i+1}} \leftarrow X_{t_{i+1}} \to \cdots .$$

By applying homology with coefficients in a field we get a zigzag of vector spaces called the *levelset zigzag persistence of* ϕ. Previous work on levelset persistence has been done under the inconvenient assumption that S is finite.

A thorough introduction to zigzags (and other directed graphs) in topological data analysis can be found in Oudot's recent book [8].

1.1. Background.

A $\mathbf{P}$-*indexed persistence module* M is a functor $M\colon \mathbf{P} \to \mathbf{Vec}$ where $\mathbf{P}$ is a poset category and $\mathbf{Vec}$ is the category of vector spaces over a fixed field $\mathbb{F}$. We say that M is pointwise finite-dimensional (p.f.d.) if $\dim M_p < \infty$ for all $p \in \mathbf{P}$. Every persistence module considered from this point on, and up until Example 2.9 is assumed to be p.f.d., and will be referred to simply as a

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persistence module. To further avoid confusion we let **vec** denote the category of finite-dimensional vector spaces over $\mathbb{F}$.

A *discrete persistence module* is a **Z**-indexed persistence module where **Z** is the integer viewed as a poset category. It was proved by Webb [11] that a discrete persistence module admits a decomposition into a direct sum of *interval persistence modules*. This was later generalized by Crawley-Boevey [6] to persistence modules indexed over the category of real numbers. Another type of persistence module is the *zigzag* persistence module considered by Carlsson and de Silva [4]. Such persistence modules also decompose into intervals, a fact well known to representation theorists. In this note we generalize this result to infinite zigzags, or, in the language of representation theory, to locally finite-dimensional representations of A_∞^∞ with arbitrarily ordered arrows.

In the same way **R**-indexed persistence can be viewed as a continuous extension of **Z**-index persistence, it is natural to wonder whether a continuous extension exists for zigzag persistence. Motivated by a generalization of levelset persistence introduced in [2], Oudot and Cochoy [5] introduced a special type of $\mathbf{R}^2$-indexed persistence, which can be viewed as such an extension. Their decomposition theorem implies the main theorem of this paper, albeit at a much greater level of technicality.

As outlined above, zigzag persistence modules arise in topological data analysis. While investigating stability properties of such constructions [2] it proved itself convenient to work with infinite zigzag persistence modules. Not only did it allow for a greater class of functions, but it also led to notational simplifications. Moreover, an immediate consequence of this paper is a new proof of the interval decomposability of discrete persistence modules. Finally, it should be emphasized that Proposition 2.8 appears in more general form in Section 6 (Covering Theory) of Ringel's Izmir Notes [9].

2. Zigzag persistence modules

A *zigzag persistence module* is a sequence of (finite-dimensional) vector spaces and linear maps indexed by the integers

$$V\colon \cdots \leftrightarrow V_{-1} \leftrightarrow V_0 \leftrightarrow V_1 \leftrightarrow \cdots$$

where $\leftrightarrow$ denotes an arrow of type $\leftarrow$ or $\rightarrow$. This is a generalization of discrete persistence modules for which arrows point in the same direction. In this note we will restrict ourselves to zigzag persistence modules of the form

$$V\colon \cdots \to V_{-1} \leftarrow V_0 \to V_1 \leftarrow \cdots,$$

i.e., where we have sinks at all odd numbers and sources at even numbers. Any other zigzag persistence module can be understood from such a zigzag by adding appropriate isomorphisms.

In the language introduced above, a zigzag persistence module is a **ZZ**-indexed persistence module where **ZZ** is the category with objects the integers $\mathbb{Z}$, together with morphisms $i \to i-1$ and $i \to i+1$ for all even numbers i. We shall denote the morphisms $V(i \to i-1)$ and $V(i \to i+1)$ by g_i and f_i, respectively. For integers $s \leq t$ we define the *restriction of* V *to* $[s,t]$ to be the persistence module $V|_{[s,t]}\colon \mathbf{ZZ}|_{[s,t]} \to \mathbf{vec}$ where $\mathbf{ZZ}|_{[s,t]}$ is the full subcategory of **ZZ** with objects $\{i\colon s \leq i \leq t\}$. Conversely, for $V\colon \mathbf{ZZ}|_{[s,t]} \to \mathbf{vec}$, define $E^{s,t}(V)\colon \mathbf{ZZ} \to \mathbf{vec}$ by $E^{s,t}(V)_i = V_s$ for $i \leq s$, $E^{s,t}(V)_i = V_t$ for $i \geq t$, $E^{s,t}(V)|_{[s,t]} = V|_{[s,t]}$, together with the obvious identity morphisms. Observe that $E^{s,t}$ is a functor and that $E^{s,t}(V \oplus W) = E^{s,t}(V) \oplus E^{s,t}(W)$.

For $a \leq b \in \mathbb{Z} \cup \{\pm\infty\}$ define the *interval (zigzag) persistence module* $I^{[a,b]} \colon \mathbf{ZZ} \to \mathbf{vec}$ on objects by

$$I_i^{[a,b]} = \begin{cases} \mathbb{F} & \text{if } a \leq i \leq b, \\ 0 & \text{otherwise,} \end{cases}$$

and which assigns the identity morphism to any morphism connecting two non-zero vector spaces. Note that we have adopted the convention $-\infty < i < +\infty$ for all $i \in \mathbb{Z}$.

For zigzag persistence modules $U, W \colon \mathbf{ZZ} \to \mathbf{vec}$, define their *direct sum* $U \oplus W$ to be the persistence module defined on objects by $(U \oplus W)_i = U_i \oplus W_i$ and on morphisms by $(U \oplus W)(\alpha) = U(\alpha) \oplus W(\alpha)$. We say that V is *decomposable* if there exist non-zero U, W such that $V \cong U \oplus W$. If no such decomposition exists, then V is *indecomposable.*

A weaker form of indecomposability is indecomposability over an interval. Let $s \leq t$ be integers; $W \colon \mathbf{ZZ} \to \mathbf{vec}$ is $[s,t]$*-indecomposable* if for any decomposition $W = W^1 \oplus W^2$, then either $W_i^1 = 0$ for all $i \in [s,t]$, or $W_i^2 = 0$ for all $i \in [s,t]$. Moreover, an $[s,t]$*-decomposition* of V is a decomposition $V = \bigoplus_{j \in \mathcal{J}} W^j$ such that W^j is $[s,t]$-indecomposable for all $j \in \mathcal{J}$. It is not hard to see that such a decomposition exists for every $[s,t]$: if V is $[s,t]$-indecomposable, then we are done. Otherwise, decompose V and inductively choose an $[s,t]$-decomposition for each of its summands. Since the sum of dimensions $\dim V_s + \ldots + \dim V_t$ is finite, this process must terminate after a finite number of steps.

Lemma 2.1. *V is indecomposable if and only if V is $[-k,k]$-indecomposable for all non-negative integers k.*

Proof. $\Leftarrow$: Assume that $V = U \oplus W$ for non-trivial U and W. Then there exist indices i_1 and i_2 such that $U_{i_1} \neq 0$ and $W_{i_2} \neq 0$. This contradicts that V is $[-\max(|i_1|, |i_2|), \max(|i_1|, |i_2|)]$-indecomposable. $\Rightarrow$: This follows by definition. □

The ensuing theorem is a special case of Gabriel's theorem [7]. For an elementary, self-contained proof, see [10].

Theorem 2.2. *Let $0 \neq V \colon \mathbf{ZZ}|_{[s,t]} \to \mathbf{vec}$. There exists a finite set of intervals $\{[a_i, b_i]\}_{i=1}^n$, $s \leq a_i \leq b_i \leq t$, such that*

$$V \cong \bigoplus_{i=1}^{n} I^{[a_i,b_i]}|_{[s,t]}.$$

Moreover, the theorem of Azumaya-Krull-Remak-Schmidt [1] asserts that such an interval decomposition is unique up to re-indexing of the intervals.

The goal of this paper is to prove the following generalization:

Theorem 2.3. *Any non-zero p.f.d. zigzag persistence module decomposes as a direct sum of interval modules.*

This theorem will be an immediate consequence of Propositions 2.7 and 2.8.

2.1. **The only indecomposables are interval persistence modules.** First we prove that every indecomposable zigzag persistence module is an interval module, and then we show that every zigzag persistence module decomposes into a direct sum of indecomposables.

Lemma 2.4. *Let $V\colon \mathbf{ZZ} \to \mathbf{vec}$. If there exists an interval $[s,t]$ such that the interval decomposition of $V|_{[s,t]}$ includes an interval $I^{[a_j,b_j]}|_{[s,t]}$ with $s < a_j$ and $b_j < t$, then V is decomposable.*

Proof. To simplify notation we shall assume that both s and t are even, and that $a_j = s+1$ and $b_j = t-1$. The former can be achieved by increasing the interval $[s,t]$ and the latter is purely a cosmetic assumption to make the commutative diagram below smaller. Let

$$\phi\colon V|_{[s,t]} \to U \oplus I^{[a_j,b_j]}|_{[s,t]}$$

be an isomorphism and let $\tilde{g}_i$ and $\tilde{f}_i$ denote the linear maps of the zigzag persistence module $U \oplus I^{[a_j,b_j]}|_{[s,t]}$. The following commutative diagram shows that this decomposition extends to a decomposition $V \cong X \oplus I^{[a_j,b_j]}$ where $X|_{[s,t]} = U$:

$$\begin{tikzcd}
\cdots V_{s-1} \arrow[d, "="] & V_s \arrow[l, "g_s"'] \arrow[r, "f_s"] \arrow[d, "\phi_s"', "\cong"] & V_{s+1} \arrow[d, "\phi_{s+1}"', "\cong"] & \cdots \arrow[l, "g_{s+2}"'] \arrow[r, "f_{t-2}"] & V_{t-1} \arrow[d, "\phi_{t-1}"', "\cong"] & V_t \arrow[l, "g_t"'] \arrow[r, "f_t"] \arrow[d, "\phi_t"', "\cong"] & V_{t+1} \cdots \arrow[d, "="] \\
\cdots V_{s-1} & U_s \arrow[l, "\tilde{g}_s"'] \arrow[r, "\tilde{f}_s"] & U_{s+1} \oplus I^{[a_j,b_j]}_{s+1} & \cdots \arrow[l, "\tilde{g}_{s+2}"'] \arrow[r, "\tilde{f}_{t-2}"] & U_{t-1} \oplus I^{[a_j,b_j]}_{t-1} & U_t \arrow[l, "\tilde{g}_t"'] \arrow[r, "\tilde{f}_t"] & V_{t+1} \cdots
\end{tikzcd}$$

Here we have defined

$$\tilde{g}_s = g_s \circ \phi_s^{-1}, \qquad \tilde{f}_t = f_t \circ \phi_t^{-1},$$

and $\tilde{f}_i = f_i$, $\tilde{g}_i = g_i$ and $U_i = V_i$ for all $i \leq s-1$ and $i \geq t+1$. ☐

Lemma 2.5. *Let $V\colon \mathbf{ZZ} \to \mathbf{vec}$ be indecomposable; then there exists a $t \geq 0$ such that f_i is injective and g_i is surjective for all $i \geq t$. Dually, there exists an $s \leq 0$ such that f_i is surjective and g_i is injective for all $i \leq s$.*

Proof. Let $\dim V_0 = N$; we shall show that there can be at most N morphisms f_i, $i \geq 0$, that are non-injective.

Assume that $\ker f_i \neq 0$ for an even integer $i \geq 0$ and look at the interval decomposition of $V|_{[0,i+1]}$. Since $\ker f_i \neq 0$, there exists an interval $I^{[a,i]}$ in the interval decomposition of $V|_{[0,i+1]}$. By indecomposability of V and Lemma 2.4 it follows that $a = 0$. Thus, if there are M indices $0 \leq i_1 < i_2 < \ldots < i_M$ such that $\ker f_{i_j} \neq 0$, then the interval decomposition of $V|_{[0,i_M+1]}$ has at least M intervals supported on 0, implying that $M \leq \dim V_0 = N$.

The setting with g_i non-surjective is completely analogous. ☐

Lemma 2.6. *Let $g_i\colon V_i \to V_{i-1}$ be a surjection and $f_i\colon V_i \to V_{i+1}$ an injection. If $V_{i-1} = U_{i-1} \oplus W_{i-1}$, then we can choose decompositions $V_i = U_i \oplus W_i$ and $V_{i+1} = U_{i+1} \oplus W_{i+1}$ such that*

$$U_i = g_i^{-1}(U_{i-1}), \qquad g_i(U_i) = U_{i-1}, \qquad g_i(W_i) \subseteq W_{i-1},$$
$$f_i(U_i) = U_{i+1}, \qquad f_i(W_i) \subseteq W_{i+1}.$$

Proof. Let $U_i = g_i^{-1}(U_{i-1})$ and $\widetilde{W} = g_i^{-1}(W_{i-1})$. We may choose a basis $\mathcal{B}$ for $\widetilde{W}$ which includes a basis for $\ker g_i$. Let W_i be the subspace spanned by the elements of $\mathcal{B}$ that are not in the kernel of g_i. It follows from the surjectivity of g that $V_i = U_i \oplus W_i$. Similarly, let $U_{i+1} = f_i(U_i)$ and $\widetilde{W} = f_i(W_i)$. The injectivity of f_i ensures that $\widetilde{W}$ can be extended to an internal complement W_{i+1} of $f_i(U_i)$ in V_{i+1}. $\square$

We are now ready to prove the first of our two needed results.

Proposition 2.7. *Let* $0 \neq V\colon \mathbf{ZZ} \to \mathbf{vec}$ *be indecomposable. Then* V *is an interval persistence module.*

Proof. Let $m \geq 0$ be an index such that f_i is injective and g_i is surjective for all $i \geq m$. We shall show that f_i and g_i are isomorphisms. Assume for the sake of contradiction that f_i is not surjective and decompose $V_{i+1} = U_{i+1} \oplus \operatorname{im} f_i$ where U_{i+1} is a non-trivial internal complement of $\operatorname{im} f_i$. This yields the following sequence of vector spaces and linear maps:

$$\begin{tikzcd}
\cdots V_i \arrow[r, "f_i"'] \arrow[d, "="'] & V_{i+1} \arrow[d, "="'] & V_{i+2} \arrow[l, "g_{i+2}"'] \arrow[r, "f_{i+2}"] \arrow[d, "="'] & V_{i+3} \arrow[d, "="'] & \cdots \arrow[l, "g_{i+4}"'] \arrow[d, "="'] \\
\cdots V_i \arrow[r, "f_i"] & U_{i+1} \oplus \operatorname{im} f_i & g_{i+2}^{-1}(U_{i+1}) \oplus U_{i+2} \arrow[l, "g_{i+2}"'] \arrow[r, "f_{i+2}"] & f_{i+2}(g_{i+2}^{-1}(U_{i+1})) \oplus U_{i+3} & \cdots \arrow[l, "g_{i+4}"']
\end{tikzcd}$$

where we have used Lemma 2.6 together with the fact that g_{i+2l} is surjective and f_{i+2l} is injective for all $l \geq 1$. Since the process can be continued indefinitely, this contradicts that V is indecomposable. Similarly, we must have that g_i is an isomorphism for all $i \geq m$. Dually there exists an m' such that f_i and g_i are isomorphisms for all $i \leq m'$. In particular, we get the following chain of isomorphisms:

$$V \cong E^{m',m}(V|_{[m',m]}) \cong E^{m',m}\left(\bigoplus_{j=1}^{n} I^{[a_j,b_j]}|_{[m',m]}\right) = \bigoplus_{j=1}^{n}\left(E^{m',m}\left(I^{[a_j,b_j]}|_{[m',m]}\right)\right)$$

where the second isomorphism follows from Theorem 2.2. Thus, $n = 1$ and

$$V \cong E^{m',m}\left(I^{[a_1,b_1]}|_{[m',m]}\right).$$

$\square$

2.2. **Decomposition into indecomposables.** The next thing we need to show is that every zigzag persistence module decomposes into a direct sum of indecomposables. This is a special case of the first theorem in Section 6 of [9] and the proof below is an adaptation of the corresponding proof.

Proposition 2.8. *Any non-zero p.f.d. zigzag persistence module* V *decomposes into a direct sum of indecomposables.*

Proof. We shall inductively define a $[-k,k]$-decomposition of V for every $k \geq 0$.

Start by choosing a $[0,0]$-decomposition of $V \cong \bigoplus_{j_0 \in [m]} V^{(j_0)}$ where $[m] = \{0,\ldots,m\}$. The idea is to choose a $[-1,1]$-decomposition of $V^{(j_0)}$, and then, for every summand in the $[-1,1]$-decomposition of $V^{(j_0)}$, choose a $[-2,2]$-decomposition, and so forth. To illustrate the first step, let $j_0 \in [m]$ be as above and let

$$V^{(j_0)} \cong \bigoplus_{j_1 \in [m_{j_0}]} V^{(j_0,j_1)}$$

be a $[-1,1]$-decomposition of $V^{(j_0)}$. We parameterize the $[-1,1]$-indecomposables by a pair of indices (j_0, j_1) under the convention that $V^{(j_0,j_1)}$ is the j_1-th $[-1,1]$-indecomposable in a $[-1,1]$-decomposition of $V^{(j_0)}$. Hence, the summands in a $[-2,2]$-decomposition of $V^{(j_0,j_1)}$ will be denoted by $V^{(j_0,j_1,j_2)}$ where $j_2 \in [m_{(j_0,j_1)}]$, and so forth.

Inductively, for every $(k+1)$-tuple $(j_0,\ldots,j_k)$ satisfying

$$j_i \in [m_{(j_0,\ldots,j_{i-1})}] \qquad \text{for all} \qquad 1 \le i \le k, \tag{2.1}$$

choose a $[-(k+1),(k+1)]$-decomposition

$$V^{(j_0,\ldots,j_k)} \cong \bigoplus_{j_{k+1}\in[m_{(j_0,\ldots,j_k)}]} V^{(j_0,\ldots,j_k,j_{k+1})},$$

which in turn yields a $[-(k+1),(k+1)]$-decomposition of V

$$V \cong \bigoplus_{j_0\in[m]} \bigoplus_{j_1\in[m_{j_0}]} \cdots \bigoplus_{j_{k+1}\in[m_{(j_0,\ldots,j_k)}]} V^{(j_0,\ldots,j_{k+1})}.$$

Let I be the set of all infinite sequences $s = (j_0, j_1, \ldots)$ such that the restriction to $(j_0,\ldots,j_k)$ satisfies (2.1) for every k, and for every $s \in I$ define

$$V^s = V^{(j_0)} \cap V^{(j_0,j_1)} \cap V^{(j_0,j_1,j_2)} \cap \cdots.$$

It is not hard to see that $V_i^s = V_i^{(j_0,\ldots,j_k)}$ for every $-k \le i \le k$. In particular, V^s is $[-k,k]$-indecomposable for all $k \ge 0$ and thus indecomposable by Lemma 2.1. Also, by the same observation, it follows that $V_i \cong \bigoplus_{s\in I} V_i^s$ for all i.

Note that there can be sequences $s \in I$ such that $V^s = 0$. To give a proper direct sum decomposition of V we let $I' \subseteq I$ be the set of all sequences $s \in I$ such that $V^s \neq 0$. Hence,

$$V \cong \bigoplus_{s\in I'} V^s.$$

□

To conclude this paper we provide an example showing that the assumption $\dim V_i < \infty$ for all $i \in \mathbb{Z}$ is crucial. The example is due to Michael Lesnick.

Example 2.9. Let $V' = \bigoplus_{k=1}^{\infty} I^{[-k,0]}$ and define V by the following properties: V restricted to the non-positive integers equals V', $V_1 = \mathbb{F}$, $V_i = 0$ for $i > 1$, and f_0 restricted to any of the $I_0^{[-k,0]}$ above is the identity.

Assume that V decomposes into a direct sum of interval persistence modules and let $I^{[a,1]}$ be the single interval summand that is non-zero at index $i = 1$. Since $f_0\left(I_0^{[-k,0]}\right) = I_1^{[a,1]}$ for all $k \ge 1$ we must have that $I_{-k}^{[a,1]}$ is non-zero for all $k \ge 1$. Or, in other words, it must be of the form $I^{[-\infty,a]}$. This is not possible as the restriction of V to non-positive integers equals a direct sum of interval persistence modules that are non-zero on a finite number of indices. Hence, there cannot be an interval persistence module containing V_1.

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