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GRADIENT ESTIMATES OF MEAN CURVATURE EQUATIONS WITH SEMI-LINEAR OBLIQUE BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we consider the semi-linear oblique boundary value problem for the prescribed mean curvature equation. We find a suitable auxiliary function and use the maximum principle to get the gradient estimate. As a consequence, we obtain the corresponding existence theorem for a class of mean curvature equations with semi-linear oblique derivative problems.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n \geq 2$ with $\partial\Omega \in C^3$, and γ be the inward unit normal vector to $\partial\Omega$. We consider the following semi-linear oblique boundary value problem:

$$\mathtt{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f(x,u) \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial \beta} = \psi(x,u) \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where β is a C^1 unit vector field: $\partial\Omega \to \mathbb{R}^n$; $f : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ and $\psi : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ are given functions. β is called strictly oblique if for some positive constant β_0

$$\beta \cdot \gamma \geq \beta_0 > 0. \tag{1.3}$$

We assume that there exists a positive constant β_1 such that

$$|D\beta| \leq \beta_1. \tag{1.4}$$

Obviously, when $\beta = \gamma$, (1.2) corresponds to the Neumann boundary value condition.

A gradient estimate for the prescribed mean curvature equation has been extensively studied. The interior gradient estimate for the minimal surface equation was obtained in the case of two variables by Finn [3]. Bombieri, De Giorgi and M. Miranda [1] obtained the estimate for the high dimensional case. For the general mean curvature equation, such an estimate had also been obtained by Ladyzhenskaya and Ural′tseva [10], Trudinger [20] and Simon [17]. All their methods used

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the test function argument and the resulting Sobolev inequality. For the maximum principle proof, Guan-Spruck [6], Korevaar [8], Trudinger [21], and Wang [23] gave a new proof for the mean curvature equation via the standard Bernstein technique. The Dirichlet problem for the prescribed mean curvature equation had been studied by Jenkins-Serrin [7] and Serrin [16]. More detailed history could be found in Gilbarg and Trudinger's book [5]; see also Li [11] for fully nonlinear elliptic equations.

For the mean curvature equation with capillary problems, Ural′ceva [22], Simon-Spruck [18] and Gerhardt [4] got the boundary gradient estimate and obtained existence theorem on the positive gravity case respectively. They obtained these estimates also via the test function technique. Spruck [19] used the maximum principle to obtain the boundary gradient estimate in two dimensions for the positive gravity capillary problems. Korevaar [9] got the gradient estimates for the positive gravity case in the high dimensional case. In [12], Lieberman got the maximum principle proof for the gradient estimates on the general quasi-linear elliptic equation with capillary boundary value problems.

Recently, for the mean curvature equation with Neumann problems, Ma-Xu [15] got the maximum principle proof for the gradient estimate and consequently obtained an existence result. Ma-Qiu [14] obtained the existence result for Hessian equations with Neumann problems.

For the following oblique boundary value problem:

$$v^{q-1}\frac{\partial u}{\partial \gamma} + \psi(x,u) = 0 \quad \text{on} \quad \partial\Omega, \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $n \geq 2$, γ is the inward unit normal vector to $\partial\Omega$, $q \geq 0$ and $v = \sqrt{1+|Du|^2}$.

Lieberman ([13, page 360]) got the gradient estimates for $q > 1$ or $q = 0$ for general quasi-linear equations. When $q = 0$, it corresponds to the capillary problem. When $q = 1$, it corresponds to the Neumann boundary value problem. In [24], the first author used the maximum principle to obtain a new unified proof of gradient estimates for the problem (1.1), (1.5) with $q > 1$ or $q = 0$ or $q = 1$ for $n = 2, 3$ cases.

Now let's state our main gradient estimates.

Theorem 1.1. *Suppose $u \in C^2(\overline{\Omega}) \cap C^3(\Omega)$ is a bounded solution to the oblique derivative problems* (1.1)-(1.2). *Assume $f(x,z) \in C^1(\overline{\Omega} \times [-M_0, M_0])$, $\psi(x,z) \in C^3(\overline{\Omega} \times [-M_0, M_0])$, and assume that there exist positive constants M_0, L_1, L_2 such that*

$$|u| \leq M_0 \quad in \quad \overline{\Omega}, \tag{1.6}$$

$$f_z(x,z) \geq 0 \quad in \quad \overline{\Omega} \times [-M_0, M_0], \tag{1.7}$$

$$|f(x,z)| + |f_x(x,z)| \leq L_1 \quad in \quad \overline{\Omega} \times [-M_0, M_0], \tag{1.8}$$

$$|\psi(x,z)|_{C^3(\overline{\Omega}\times[-M_0,M_0])} \leq L_2. \tag{1.9}$$

Then there exists a small positive constant μ_0 such that we have the following estimate:

$$\sup_{\overline{\Omega}_{\mu_0}} |Du| \leq \max\{M_1, M_2\},$$

where M_1 is a positive constant depending only on n, μ_0, M_0, L_1, which is from the *interior gradient estimates; M_2 is a positive constant depending only on $n, \Omega, \mu_0, \beta_0, \beta_1, M_0, L_1, L_2$, and $d(x) = \mathrm{dist}(x, \partial\Omega), \Omega_{\mu_0} = \{x \in \Omega : d(x) < \mu_0\}$ and μ_0 depends on Ω, L_2.*

From the standard bounded estimates in Concus-Finn [2] (see also Spruck [19]), Ma-Xu [15], interior gradient estimates [5], and Theorem 1.1 for the prescribed mean curvature equation, we can get the following existence theorem for the oblique boundary value problem of mean curvature equations.

Corollary 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n \geq 2$, $\partial\Omega \in C^3$, and β a C^1 unit oblique vector field to $\partial\Omega$ which satisfies* (1.3). *If $\psi \in C^3(\overline{\Omega})$ is a given function, then the following boundary value problem:*

$$\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = u \quad in \quad \Omega, \tag{1.10}$$

$$\frac{\partial u}{\partial \beta} = \psi(x) \quad on \quad \partial\Omega, \tag{1.11}$$

exists with a unique solution $u \in C^2(\overline{\Omega})$.

In the entire paper we follow the Einstein summation convention: all repeated indices from 1 to n denote summation. The rest of the paper is organized as follows. The proof of the main Theorem 1.1 has been divided into two sections. In section 2, we first give some notation and the boundary gradient bound. We prove the gradient estimate near the boundary in section 3 and complete the proof of Theorem 1.1.

2. Some preliminaries and a boundary gradient estimate

The important interior gradient estimate is from Gilbarg-Trudinger [5].

Remark 2.1 ([5]). If $u \in C^3(\Omega)$ is a bounded solution for the equation (1.1) with (1.6), and if $f \in C^1(\overline{\Omega} \times [-M_0, M_0])$ satisfies the conditions (1.7)-(1.8), then for any subdomain $\Omega' \Subset \Omega$, we have

$$\sup_{\Omega'} |Du| \leq M_1,$$

where M_1 is a positive constant depending only on $n, M_0, \mathtt{dist}(\Omega', \partial\Omega), L_1$.

We denote by Ω a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $\partial\Omega \in C^3$, set

$$d(x) = \mathtt{dist}(x, \partial\Omega),$$

and

$$\Omega_\mu = \{x \in \Omega : d(x) < \mu\}.$$

Then it is well known that there exists a positive constant $\mu_1 > 0$ such that $d(x) \in C^3(\overline{\Omega}_{\mu_1})$. As in Simon-Spruck [18] or Lieberman [13, page 331], we can take $\gamma = Dd$ in Ω_{μ_1} and note that γ is a $C^2(\overline{\Omega}_{\mu_1})$ vector field. As mentioned in [12] and the book [13], we also have the following formulas:

$$\begin{aligned} |D\gamma| + |D^2\gamma| \leq& C(n, \Omega) \quad \text{in} \quad \Omega_{\mu_1}, \\ \gamma^i D_j \gamma^i = 0, \gamma^i D_i \gamma^j =& 0, \ |\gamma| = 1 \quad \text{in} \quad \Omega_{\mu_1}. \end{aligned} \tag{2.1}$$

As in [13], we define

$$c^{ij} =\delta_{ij} - \gamma^i\gamma^j \quad \text{in} \quad \Omega_{\mu_1}, \tag{2.2}$$

and for a vector $\zeta \in R^n$, we write ζ' for the vector with $i-$th component $c^{ij}\zeta_j$. So

$$|D'u|^2 =c^{ij}u_iu_j. \tag{2.3}$$

Following Ma-Xu's technique in [15], we use the maximum principle for the following auxiliary function in $\overline{\Omega}_{\mu_0}$ to get our estimates, where $0 < \mu_0 < \mu_1$ is a sufficiently small number which we shall decide later.

Now we present three cases to complete the proof of Theorem 1.1.

Case 1. If $\varphi(x)$ attains its maximum at $x_0 \in \partial\Omega$, then we shall get the bound of $|Du|(x_0)$.

Case 2. If $\varphi(x)$ attains its maximum at $x_0 \in \partial\Omega_{\mu_0} \cap \Omega$, then we shall get the estimates via the interior gradient bound in Remark 2.1.

Case 3. If $\varphi(x)$ attains its maximum at $x_0 \in \Omega_{\mu_0}$, in this case for the sufficiently small constant $\mu_0 > 0$, then we can use the maximum principle to get the bound of $|Du|(x_0)$.

Setting $w = u - \psi(x, u)d$, we choose the following auxiliary function Φ:

$$\Phi(x) = \log|Dw|^2 e^{1+M_0+u}e^{\alpha_0 d}, \quad x \in \overline{\Omega}_{\mu_0}$$

where $\alpha_0 = \frac{C_0L_2+C_0+1}{\beta_0}$, C_0 is a positive constant depending only on $n, \Omega, \beta_0, \beta_1$. Let

$$\varphi(x) = \log\Phi(x) = \log\log|Dw|^2 + h(u) + g(d), \tag{2.4}$$

where

$$h(u) = 1 + M_0 + u, \quad g(d) = \alpha_0 d. \tag{2.5}$$

Note that the following constants C which appear in different lines in the whole paper are different depending only on $n, \Omega,\ \beta_0, \beta_1, \mu_0, M_0, L_1, L_2$.

In this section, we shall first give the boundary gradient bounds on $\partial\Omega$ to prove Case 1.

We differentiate φ along the strictly oblique direction

$$\frac{\partial\varphi}{\partial\beta} =\frac{(|Dw|^2)_i\beta^i}{|Dw|^2\log|Dw|^2} + h'u_\beta + g'\beta\cdot\gamma. \tag{2.6}$$

Since

$$w_i =u_i - \psi_u u_i d - \psi_{x_i}d - \psi\gamma^i, \tag{2.7}$$

$$|Dw|^2 =|D'w|^2 + w_\gamma^2, \tag{2.8}$$

we have

$$w_\gamma =u_\gamma - \psi_u u_\gamma d - \psi_{x_i}\gamma^i d - \psi = 0 \quad \text{on} \quad \partial\Omega, \tag{2.9}$$

$$(|Dw|^2)_i =(|D'w|^2)_i \quad \text{on} \quad \partial\Omega. \tag{2.10}$$

Applying (2.1), (2.3) and (2.10), it follows that

$$\begin{aligned}(|Dw|^2)_i\beta^i &= 2c^{kl}w_{ki}w_l\beta^i + (c^{kl})_i w_k w_l\beta^i\\ &= 2c^{kl}u_{ki}u_l\beta^i - 2c^{kl}u_l D_k\psi\beta\cdot\gamma - 2\psi c^{kl}u_l(\gamma^k)_i\beta^i\\ &\quad + (c^{kl})_i\beta^i(u_k - \psi\gamma^k)(u_l - \psi\gamma^l),\end{aligned} \tag{2.11}$$

where

$$D_k\psi = \psi_{x_k} + \psi_u u_k.$$

Differentiating (1.2) with respect to the tangential direction, we have

$$c^{kl}(u_\beta)_k = c^{kl} D_k\psi. \tag{2.12}$$

It follows that

$$c^{kl}u_{ik}\beta^i = - c^{kl}u_i(\beta^i)_k + c^{kl}D_k\psi. \tag{2.13}$$

Inserting (2.13) into (2.11) and combining (1.2), (2.6), we have

$$\begin{aligned}
&|Dw|^2\log|Dw|^2\frac{\partial\varphi}{\partial\beta}(x_0)\\
=&(g'(0)\beta\cdot\gamma + h'\psi)|Dw|^2\log|Dw|^2 - 2c^{kl}u_lu_i(\beta^i)_k + 2(1-\beta\cdot\gamma)c^{kl}u_lD_k\psi\\
&- 2\psi c^{kl}u_l(\gamma^k)_i\beta^i + (c^{kl})_i\beta^i(u_k-\psi\gamma^k)(u_l-\psi\gamma^l).
\end{aligned} \tag{2.14}$$

From (2.7) and the mean value inequality, we obtain

$$\begin{aligned}
|Dw|^2 =& \sum_{i=1}^n (u_i - \psi\gamma^i)^2 \le 2|Du|^2 + 2\psi^2 \quad \text{on} \quad \partial\Omega,\\
|Du|^2 =& \sum_{i=1}^n (w_i + \psi\gamma^i)^2 \le 2|Dw|^2 + 2\psi^2 \quad \text{on} \quad \partial\Omega.
\end{aligned} \tag{2.15}$$

Assume $|Du|(x_0) \ge \sqrt{100 + 5|\psi|^2_{C^0(\overline{\Omega}\times[-M_0,M_0])}}$, otherwise we get the estimates. At x_0, from (2.15) and the assumption, we have

$$\frac{3}{10}|Du|^2 \le |Dw|^2 \le \frac{12}{5}|Du|^2, \quad |Dw|^2 \ge 50. \tag{2.16}$$

Inserting (2.16) into (2.14), we have

$$\begin{aligned}
|Dw|^2\log|Dw|^2\frac{\partial\varphi}{\partial\beta}(x_0) \ge& (\alpha_0\beta_0 - C_0L_2 - C_0)|Dw|^2\log|Dw|^2\\
=&|Dw|^2\log|Dw|^2\\
>&0.
\end{aligned} \tag{2.17}$$

On the other hand, from $\beta\cdot\gamma \ge \beta_0 > 0$, we have

$$\frac{\partial\varphi}{\partial\beta}(x_0) \le 0.$$

It is a contradiction to (2.17).

Then we have

$$|Du|(x_0) \le \sqrt{100 + 5|\psi|^2_{C^0(\overline{\Omega}\times[-M_0,M_0])}}. \tag{2.18}$$

□

3. Gradient estimate near the boundary

In this section, we will get the gradient estimate in Ω_{μ_0} to prove Case 3. Precisely, if $\varphi(x)$ attains its maximum at $x_0 \in \Omega_{\mu_0}$ for the sufficiently small constant $\mu_0 > 0$, then we can use the maximum principle to get the bound of $|Du|(x_0)$. In this case, x_0 is a critical point of φ. We choose the different coordinate from Ma-Xu [15]. By rotating the coordinate system suitably, we may assume that $w_i(x_0) = 0$, $2 \le i \le n$ and $w_1(x_0) = |Dw| > 0$. And we can further assume that the matrix $(w_{ij}(x_0))(2 \le i, j \le n)$ is diagonal. Let $\mu_2 \le \frac{1}{100L_2}$ such that

$$|\psi_u|\mu_2 \le \frac{1}{100}; \quad \text{then} \quad \frac{99}{100} \le 1 - \psi_u\mu_2 \le \frac{101}{100}. \tag{3.1}$$

We can choose

$$\mu_0 = \frac{1}{2}\min\{\mu_1, \mu_2, 1\}.$$

In order to simplify the calculations, we let

$$w = u - G, \quad G = \psi(x, u)d.$$

Then we have

$$w_k = (1 - G_u)u_k - G_{x_k}. \tag{3.2}$$

Since at x_0,

$$|Du|^2 = u_1^2 + \sum_{2\le i\le n} u_i^2, \tag{3.3}$$

$$(1 - G_u)u_i = G_{x_i} = \psi_{x_i}d + \psi\gamma^i, \quad i = 2, \dots, n, \tag{3.4}$$

$$w_1 = (1 - G_u)u_1 - G_{x_1} = (1 - G_u)u_1 - \psi_{x_1}d - \psi\gamma^1. \tag{3.5}$$

So from the above relation, at x_0, we can assume

$$u_1 \ge 200(1 + |\psi|_{C^1(\overline{\Omega}\times[-M_0,M_0])}). \tag{3.6}$$

Then

$$\frac{19}{20}u_1 \le w_1 \le \frac{21}{20}u_1, \quad \frac{91}{100}w_1^2 \le |Du|^2 \le \frac{111}{100}w_1^2, \tag{3.7}$$

and by the choice of μ_0 and (3.1), we have

$$\frac{99}{100} \le 1 - G_u \le \frac{101}{100}. \tag{3.8}$$

From the above choices, we shall get the gradient estimate in Ω_{μ_0} with two steps and then complete the proof of Theorem 1.1. All the calculations will be done at the fixed point x_0.

Step 1. We first compute the first and the second derivatives of φ and get the formula (3.32).

Taking the first derivatives of φ,

$$\varphi_i = \frac{(|Dw|^2)_i}{|Dw|^2\log|Dw|^2} + h'u_i + g'\gamma^i. \tag{3.9}$$

From $\varphi_i(x_0) = 0$, we have

$$(|Dw|^2)_i = -|Dw|^2\log|Dw|^2(h'u_i + g'\gamma^i). \tag{3.10}$$

Take the derivatives again for φ_i,

$$\begin{aligned}\varphi_{ij} =&\frac{(|Dw|^2)_{ij}}{|Dw|^2\log|Dw|^2}-(1+\log|Dw|^2)\frac{(|Dw|^2)_i(|Dw|^2)_j}{(|Dw|^2\log|Dw|^2)^2}\\ &+h'u_{ij}+h''u_iu_j+g''\gamma^i\gamma^j+g'(\gamma^i)_j.\end{aligned}\tag{3.11}$$

Using (3.10), it follows that

$$\begin{aligned}\varphi_{ij} =& \frac{(|Dw|^2)_{ij}}{|Dw|^2\log|Dw|^2}+h'u_{ij}+\big[h''-(1+\log|Dw|^2)h'^2\big]u_iu_j\\ &+\big[g''-(1+\log|Dw|^2)g'^2\big]\gamma^i\gamma^j-(1+\log|Dw|^2)h'g'(\gamma^iu_j\\ &+\gamma^ju_i)+g'(\gamma^i)_j.\end{aligned}\tag{3.12}$$

Then we get

$$0\ge a^{ij}\varphi_{ij} =: I_1+I_2,\tag{3.13}$$

where

$$a^{ij}(Du)=v^2\delta_{ij}-u_iu_j,\quad v=(1+|Du|^2)^{\frac{1}{2}},\quad a^{ij}u_iu_j=|Du|^2,\tag{3.14}$$

$$I_1=\frac{1}{|Dw|^2\log|Dw|^2}a^{ij}(|Dw|^2)_{ij},\tag{3.15}$$

and

$$\begin{aligned}I_2 =&a^{ij}\Big\{h'u_{ij}+\big[h''-(1+\log|Dw|^2)h'^2\big]u_iu_j+\big[g''-(1+\log|Dw|^2)g'^2\big]\gamma^i\gamma^j\\ &-2(1+\log|Dw|^2)h'g'\gamma^iu_j+g'(\gamma^i)_j\Big\}.\end{aligned}\tag{3.16}$$

By (3.14), the equation (1.1) is equivalent to the following equation:

$$a^{ij}u_{ij}=f(x,u)v^3\quad\text{in}\quad\Omega.\tag{3.17}$$

Now we first treat I_2. From the choice of the coordinate and the equations (3.17), (3.14), (2.5), we have

$$I_2 \ge fv^3-2(a^{ij}u_iu_j+\alpha_0^2a^{ij}\gamma^i\gamma^j)\log w_1-Cu_1^2.\tag{3.18}$$

Next, we calculate I_1 and get the formula (3.32).

Taking the first derivatives of $|Dw|^2$, we have

$$(|Dw|^2)_i =2w_1w_{1i}.\tag{3.19}$$

Taking the derivatives of $|Dw|^2$ once more, we have

$$(|Dw|^2)_{ij} =2w_1w_{1ij}+2w_{ki}w_{kj}.\tag{3.20}$$

By (3.15) and (3.20), we can rewrite I_1 as

$$I_1 =\frac{1}{w_1\log w_1}a^{ij}w_{ij1}+\frac{1}{w_1^2\log w_1}a^{ij}w_{ki}w_{kj} =: I_{11}+I_{12}.\tag{3.21}$$

In the following, we shall deal with I_{11} and I_{12} respectively.

For the term I_{11}: Taking the first and second derivatives of (3.6), then we have

$$w_{ki} =(1 - G_u)u_{ki} - G_{uu}u_k u_i - G_{ux_i}u_k - G_{ux_k}u_i - G_{x_k x_i}, \tag{3.22}$$

$$\begin{aligned} w_{kij} =&(1 - G_u)u_{kij} - G_{uu}(u_{ki}u_j + u_{kj}u_i + u_{ij}u_k) \\ &- G_{ux_i}u_{kj} - G_{ux_j}u_{ki} - G_{ux_k}u_{ij} \\ &- G_{uuu}u_k u_i u_j - G_{uux_i}u_j u_k - G_{uux_j}u_i u_k - G_{uux_k}u_i u_j \\ &- G_{ux_i x_j}u_k - G_{ux_k x_j}u_i - G_{ux_i x_k}u_j - G_{x_i x_j x_k}. \end{aligned} \tag{3.23}$$

So from the choice of the coordinate and the equations (3.17), (3.14), we have

$$a^{ij}w_{ij1} \geq (1 - G_u)a^{ij}u_{ij1} - 2a^{ij}(G_{uu}u_i + G_{ux_i})u_{1j} - (G_{uu}u_1 + G_{ux_1})fv^3 - Cu_1^3. \tag{3.24}$$

Differentiating (3.17), we have

$$a^{ij}u_{ij1} = - a^{ij}_{p_l}u_{l1}u_{ij} + v^3 D_1 f + 3fv^2 v_1. \tag{3.25}$$

From (3.14), we have

$$a^{ij}_{p_l} =2u_l\delta_{ij} - \delta_{il}u_j - \delta_{jl}u_i. \tag{3.26}$$

By the definition of v, we have

$$vv_1 =u_k u_{k1}. \tag{3.27}$$

Since

$$D_1 f =f_u u_1 + f_{x_1}, \tag{3.28}$$

from (3.26)-(3.28) and (3.17), we have

$$a^{ij}u_{ij1} =\frac{2}{v^2}a^{ij}u_{il}u_{1j}u_l + f_u v^3 u_1 + f_{x_1}v^3 + fvu_k u_{k1}. \tag{3.29}$$

Inserting (3.29) into (3.24) and combining (3.21), from $f_u \geq 0$, we have

$$\begin{aligned} I_{11} \geq& \frac{1}{w_1 \log w_1}\Big\{\frac{2(1 - G_u)}{v^2}a^{ij}u_{il}u_{1j}u_l + (1 - G_u)fvu_j u_{1j} - 2a^{ij}(G_{uu}u_i + G_{ux_i})u_{1j} \\ &- (G_{uu}u_1 + G_{ux_1})fv^3 - Cu_1^3\Big\}. \end{aligned} \tag{3.30}$$

For the term I_{12}: applying (3.14) and (3.22), we have

$$I_{12} \geq \frac{1}{w_1^2 \log w_1}a^{ij}w_{1i}w_{1j} + \frac{1}{w_1^2 \log w_1}\sum_{2\leq i\leq n} a^{ii}w_{ii}^2. \tag{3.31}$$

Combining (3.30), (3.31), (3.21) and (3.18), (3.13), we can obtain the following formula:

$$\begin{aligned} 0 \geq& a^{ij}\varphi_{ij}(x_0) \\ \geq& \frac{1}{w_1^2 \log w_1}\Big\{\frac{2(1 - G_u)w_1}{v^2}a^{ij}u_{il}u_{1j}u_l + a^{ij}w_{1i}w_{1j} + \sum_{2\leq i\leq n} a^{ii}w_{ii}^2 \\ &+ (1 - G_u)fvw_1 u_j u_{1j} - 2w_1 a^{ij}(G_{uu}u_i + G_{ux_i})u_{1j}\Big\} \\ &+ fv^3 - \frac{1}{w_1 \log w_1}(G_{uu}u_1 + G_{ux_1})fv^3 \\ &- 2(a^{ij}u_i u_j + \alpha_0^2 a^{ij}\gamma^i\gamma^j)\log w_1 - Cu_1^2. \end{aligned} \tag{3.32}$$

Step 2. In this step we shall treat the quadratic terms in (3.32) and finish the proof of Theorem 1.1. By (3.10) and (3.19), we have

$$w_{1i} = - w_1 \log w_1 (h' u_i + g' \gamma^i), \qquad i = 1, 2, \ldots, n. \tag{3.33}$$

By (3.22), it follows that

$$\begin{aligned}(1 - G_u) u_{1i} = & - w_1 \log w_1 (h' u_i + g' \gamma^i) + (G_{uu} u_1 + G_{ux_1}) u_i + (u_1 G_{ux_i} + G_{x_1 x_i}), \\ & \qquad i = 1, 2, \ldots, n.\end{aligned} \tag{3.34}$$

Now we use the formulas (3.33)-(3.34) to treat the quadratic terms in (3.32). We obtain

$$\begin{aligned}
&\frac{2(1-G_u)w_1}{v^2} a^{ij} u_{il} u_{1j} u_l + a^{ij} w_{1i} w_{1j} + \sum_{2 \le i \le n} a^{ii} w_{ii}^2 \\
\ge & \frac{2(1-G_u)w_1 u_1}{v^2} a^{ij} u_{1i} u_{1j} + a^{ij} w_{1i} w_{1j} + \frac{2(1-G_u)w_1}{v^2} \sum_{2 \le l \le n} a^{ij} u_{il} u_{1j} u_l \\
& + \sum_{2 \le i \le n} a^{ii} w_{ii}^2 \\
\ge & \Big[\frac{2 w_1 u_1}{(1-G_u) v^2} + 1 \Big] (a^{ij} u_i u_j + \alpha_0^2 a^{ij} \gamma^i \gamma^j) w_1^2 \log^2 w_1 - C w_1^2 \log w_1,
\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}
&(1 - G_u) f v u_j u_{1j} - 2 a^{ij} (G_{uu} u_i + G_{ux_i}) u_{1j} \\
\ge & - f v |Du|^2 w_1 \log w_1 + f v |Du|^2 (G_{uu} u_1 + G_{ux_1}) - C u_1^3 \log w_1.
\end{aligned} \tag{3.36}$$

Inserting (3.35) and (3.36) into (3.32), we have

$$0 \ge a^{ij} \varphi_{ij} \ge \frac{1}{8} u_1^2 \log w_1 - C u_1^2. \tag{3.37}$$

Then there exists a positive constant C_3 such that

$$|Du|(x_0) \le C_3, \quad x_0 \in \Omega_{\mu_0}. \tag{3.38}$$

Finally, since x_0 is the maximum point, then we have $\varphi(x) \le \varphi(x_0)$, $\forall\, x \in \overline{\Omega}_{\mu_0}$. From the estimate (2.18) in section 2 and (3.38) and Remark 2.1, we obtain

$$\sup_{\overline{\Omega}_{\mu_0}} |Du| \le \max\{M_1, M_2\},$$

where the positive constant M_1 depends only on n, μ_0, M_0, L_1; and M_2 depends only on $n, \Omega, \mu_0, \beta_0, \beta_1, M_0, L_1, L_2$.

Thus we complete the proof of Theorem 1.1. □

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