PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 145, Number 8, August 2017, Pages 3645–3656
http://dx.doi.org/10.1090/proc/13508
Article electronically published on February 22, 2017

SINGULAR DECOMPOSITIONS OF A CAP PRODUCT

DAVID CHATAUR, MARTINTXO SARALEGI-ARANGUREN, AND DANIEL TANRÉ

(Communicated by Michael A. Mandell)

Abstract. In the case of a compact orientable pseudomanifold, a well-known theorem of M. Goresky and R. MacPherson says that the cap product with a fundamental class factorizes through the intersection homology groups. In this work, we show that this classical cap product is compatible with a cap product in intersection (co)homology that we have previously introduced. If the pseudomanifold is also normal, for any commutative ring of coefficients, the existence of a classical Poincaré duality isomorphism is equivalent to the existence of an isomorphism between the intersection homology groups corresponding to the zero and the top perversities.

Let X be a compact oriented pseudomanifold and $[X] \in H_n(X;\mathbb{Z})$ be its fundamental class. In [7], M. Goresky and R. MacPherson prove that the *Poincaré duality map* defined by the cap product $-\cap[X]\colon H^k(X;\mathbb{Z}) \to H_{n-k}(X;\mathbb{Z})$ can be factorized as

$$H^k(X;\mathbb{Z}) \xrightarrow{\alpha^{\overline{p}}} H^{\overline{p}}_{n-k}(X;\mathbb{Z}) \xrightarrow{\beta^{\overline{p}}} H_{n-k}(X;\mathbb{Z}), \tag{1}$$

where the groups $H^{\overline{p}}_i(X;\mathbb{Z})$ are the intersection homology groups for the perversity $\overline{p}$. The study of the Poincaré duality map via a filtration on homology classes is also considered in the thesis of C. McCrory [10], [11], using a Zeeman's spectral sequence.

In [5, Section 8.1.6], G. Friedman asks for a factorization of the Poincaré duality map through a cap product defined in the context of an intersection cohomology recalled in Section 4. He proves it with a restriction on the torsion part of the intersection cohomology. In this work we answer positively, without restriction and for any commutative ring R of coefficients, in the context of a cohomology $H^*_{\mathrm{TW},\overline{p}}(-)$ obtained via a simplicial blow-up with an intersection cap product, $-\cap[X]\colon H^k_{\mathrm{TW},\overline{p}}(X;R) \xrightarrow{\cong} H^{\overline{p}}_{n-k}(X;R)$, defined in [2, Section 11] and recalled in Section 2. Roughly, our main result consists of the fact that this "intersection cap product" corresponds to the "classical cap product". This property can be expressed as the commutativity of the next diagram.

Received by the editors June 14, 2016 and, in revised form, September 21, 2016.
2010 *Mathematics Subject Classification.* Primary 55N33, 57P10, 57N80.
Key words and phrases. Intersection homology, cap product, Poincaré duality.
This research was supported through the program "Research in Pairs" at the Mathematisches Forschungsinstitut Oberwolfach in 2016. The authors thank the MFO for its generosity and hospitality.
The third author was also supported by the MINECO grant MTM2016-78647-P and the ANR-11-LABX-0007-01 "CEMPI".

Theorem A. *Let X be a compact oriented n-dimensional pseudomanifold. For any perversity $\overline{p}$, there exists a commutative diagram*

$$\begin{tikzcd}
H^k(X;R) \arrow[r, "-\cap[X]"] \arrow[d, "\mathcal{M}^*_{\overline{p}}"'] \arrow[dr, "\alpha^{\overline{p}}"] & H_{n-k}(X;R) \\
H^k_{\mathrm{TW},\overline{p}}(X;R) \arrow[r, "-\cap[X]", "\cong"'] & H^{\overline{p}}_{n-k}(X;R). \arrow[u, "\beta^{\overline{p}}"']
\end{tikzcd} \tag{2}$$

In [7], the spaces and maps of (1) appear in the piecewise linear setting. In the previous statement we were working with singular homology and cohomology. However, we keep the same letter $\alpha^{\overline{p}}$. The morphism $\beta^{\overline{p}}$ is generated by the inclusion of the corresponding complexes, and the morphism $\mathcal{M}^*_{\overline{p}}$ is defined in Section 3.

For a normal compact oriented n-dimensional pseudomanifold, the specification of the previous statement to the constant perversity with value 0 gives a commutative diagram, where $\overline{t}$ is the top perversity defined by $\overline{t}(i) = i - 2$:

$$\begin{tikzcd}
H^k(X;R) \arrow[r, "-\cap[X]"] \arrow[dd, "\mathcal{M}^*_{\overline{0}}"', "\cong"] & H_{n-k}(X;R) \\
 & H^{\overline{t}}_{n-k}(X;R) \arrow[u, "\cong", "\beta^{\overline{t}}"'] \\
H^k_{\mathrm{TW},\overline{0}}(X;R) \arrow[r, "-\cap[X]", "\cong"'] & H^{\overline{0}}_{n-k}(X;R). \arrow[u, "\beta^{\overline{0},\overline{t}}"']
\end{tikzcd} \tag{3}$$

As a consequence, we have the next characterization, expressed in Goresky-MacPherson intersection homology. This extends to any commutative ring of coefficients the criterion established in [7] and [5] .

Theorem B. *Let X be a normal compact oriented n-dimensional pseudomanifold. Then the following conditions are equivalent.*

(i) The Poincaré duality map $-\cap[X]\colon H^k(X;R) \to H_{n-k}(X;R)$ is an isomorphism.

(ii) The natural map $\beta^{\overline{0},\overline{t}}\colon H^{\overline{0}}_{n-k}(X;R) \to H^{\overline{t}}_{n-k}(X;R)$, induced by the canonical inclusion of the corresponding complexes, is an isomorphism.

We have chosen the setting of the original perversities of Goresky and MacPherson [7]. However, the previous results remain true in more general situations.

In the last section, we quote the existence of a cup product structure on $H^*_{\mathrm{TW},\overline{\bullet}}(-)$ and detail how it combines with this factorization. We are also looking for a factorization involving an intersection cohomology defined from the dual of intersection chains (see [6]) and denoted $H^*_{\mathrm{GM},\overline{p}}(-)$. In the case of a locally torsion free pseudomanifold (see [8]), in particular if R is a field, a factorization as in (2) and (3) has been established by Friedman in [5, Section 8.1.6] for that cohomology. In this book, Friedman asks also for such factorization through a cap product without this restriction on torsion. In the last section, we give an example showing that such factorization with $H^*_{\mathrm{GM},\overline{p}}(-)$ may not exist.

The coefficients for homology and cohomology are taken in a commutative ring R (with unity), and we do not mention it explicitly in the rest of this work.

The degree of an element x of a graded module is denoted $|x|$. All the maps β, with subscript or superscript, are induced by canonical inclusions of complexes. For any topological space Y, we denote by $\mathtt{c}Y = Y \times [0,1]/Y \times \{0\}$ the cone on Y and by $\mathring{\mathtt{c}}Y = Y \times [0,1[/Y \times \{0\}$ the open cone on Y.

1. Background on intersection homology and cohomology

We recall the basic definitions and properties we need, sending the reader to [7], [5], [1] or [2] for more details.

Definition 1.1. An n-dimensional *pseudomanifold* is a topological space X, filtered by closed subsets,

$$X_{-1} = \emptyset \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-2} = X_{n-1} \subsetneq X_n = X,$$

such that, for any $i \in \{0, \dots, n\}$, $X_i \backslash X_{i-1}$ is an i-dimensional topological manifold or the empty set. Moreover, for each point $x \in X_i \backslash X_{i-1}$, $i \neq n$, there exist

(i) an open neighborhood V of x in X, endowed with the induced filtration,
(ii) an open neighborhood U of x in $X_i \backslash X_{i-1}$,
(iii) a compact pseudomanifold L of dimension $n-i-1$, whose cone $\mathring{\mathtt{c}}L$ is endowed with the filtration $(\mathring{\mathtt{c}}L)_i = \mathring{\mathtt{c}}L_{i-1}$,
(iv) a homeomorphism, $\varphi\colon U \times \mathring{\mathtt{c}}L \to V$, such that
  (a) $\varphi(u, \mathtt{v}) = u$, for any $u \in U$, where $\mathtt{v}$ is the apex of the cone $\mathring{\mathtt{c}}L$,
  (b) $\varphi(U \times \mathring{\mathtt{c}}L_j) = V \cap X_{i+j+1}$, for all $j \in \{0, \dots, n-i-1\}$.

The pseudomanifold L is called a *link* of x. The pseudomanifold X is called *normal* if its links are connected.

As in [7], a *perversity* is a map $\overline{p}\colon \mathbb{N} \to \mathbb{Z}$ such that $\overline{p}(0) = \overline{p}(1) = \overline{p}(2) = 0$ and $\overline{p}(i) \leq \overline{p}(i+1) \leq \overline{p}(i) + 1$ for all $i \geq 2$. Among them, we quote the null perversity $\overline{0}$ constant with value 0 and the *top perversity* defined by $\overline{t}(i) = i - 2$. For any perversity $\overline{p}$, the perversity $D\overline{p} := \overline{t} - \overline{p}$ is called the *complementary perversity* of $\overline{p}$.

In this work, we compute the intersection homology of a pseudomanifold X via *filtered simplices.* They are singular simplices $\sigma\colon \Delta \to X$ such that Δ admits a decomposition in join products, $\Delta = \Delta_0 * \cdots * \Delta_n$ with $\sigma^{-1}X_i = \Delta_0 * \cdots * \Delta_i$. The *perverse degree* of σ is defined by $\|\sigma\| = (\|\sigma\|_0, \dots, \|\sigma\|_n)$ with $\|\sigma\|_i = \dim(\Delta_0 * \cdots * \Delta_{n-i})$. A filtered simplex is called $\overline{p}$-*allowable* if

$$\|\sigma\|_i \leq \dim \Delta - i + \overline{p}(i), \tag{4}$$

for any $i \in \{0, \dots, n\}$. A singular (filtered) chain ξ is $\overline{p}$-allowable if it can be written as a linear combination of $\overline{p}$-allowable filtered simplices, and of $\overline{p}$-*intersection* if ξ and $\partial\xi$ are $\overline{p}$-allowable. We denote by $C_*^{\overline{p}}(X)$ the complex of singular (filtered) chains of $\overline{p}$-intersection. In [3, Théorème A], we have proved that $C_*^{\overline{p}}(X)$ is quasi-isomorphic to the singular intersection chain complex introduced by H. King in [9].

Given a euclidean simplex Δ, we denote by $N_*(\Delta)$ and $N^*(\Delta)$ the associated simplicial chain and cochain complexes. For each face F of Δ, we write $\mathbf{1}_F$ for the cochain of $N^*(\Delta)$ taking the value 1 on F and 0 otherwise. If F is a face of Δ, we denote by $(F, 0)$ the same face viewed as a face of the cone $\mathtt{c}\Delta = \Delta * [\mathtt{v}]$ and by $(F, 1)$ the face $\mathtt{c}F$ of $\mathtt{c}\Delta$. By extension, we use also the notation $(\emptyset, 1) = \mathtt{c}\emptyset = [\mathtt{v}]$ for the apex. The corresponding cochains are denoted $\mathbf{1}_{(F,\varepsilon)}$ for $\varepsilon = 0$ or 1.

A filtered simplex $\sigma\colon \Delta = \Delta_0 * \cdots * \Delta_n \to X$ is called *regular* if $\Delta_n \neq \emptyset$. The cochain complex we use for cohomology is built on the blow-ups of regular filtered simplices. More precisely, we first set

$$\widetilde{N}^*_\sigma = \widetilde{N}^*(\Delta) = N^*(\mathtt{c}\Delta_0) \otimes \cdots \otimes N^*(\mathtt{c}\Delta_{n-1}) \otimes N^*(\Delta_n).$$

With the previous convention, a basis of $\widetilde{N}^*(\Delta)$ is composed of elements of the form $\mathbf{1}_{(F,\varepsilon)} = \mathbf{1}_{(F_0,\varepsilon_0)} \otimes \cdots \otimes \mathbf{1}_{(F_{n-1},\varepsilon_{n-1})} \otimes \mathbf{1}_{F_n} \in \widetilde{N}^*(\Delta)$, where $\varepsilon_i \in \{0,1\}$ and F_i is a face of Δ_i for $i \in \{1,\ldots,n\}$ or the empty set with $\varepsilon_i = 1$ if $i < n$. We set $|\mathbf{1}_{(F,\varepsilon)}|_{>s} = \sum_{i>s}(\dim F_i + \varepsilon_i)$, with the convention $\dim\emptyset = -1$.

Definition 1.2. Let ℓ be an element of $\{1,\ldots,n\}$ and $\mathbf{1}_{(F,\varepsilon)} \in \widetilde{N}^*(\Delta)$. The *$\ell$-perverse degree* of $\mathbf{1}_{(F,\varepsilon)} \in N^*(\Delta)$ is

$$\|\mathbf{1}_{(F,\varepsilon)}\|_\ell = \left\{ \begin{array}{cll} -\infty & \text{if} & \varepsilon_{n-\ell} = 1, \\ |\mathbf{1}_{(F,\varepsilon)}|_{>n-\ell} & \text{if} & \varepsilon_{n-\ell} = 0. \end{array} \right.$$

In the general case of a cochain $\omega = \sum_b \lambda_b\, \mathbf{1}_{(F_b,\varepsilon_b)} \in \widetilde{N}^*(\Delta)$ with $\lambda_b \neq 0$ for all b, the *ℓ-perverse degree* is

$$\|\omega\|_\ell = \max_b \|\mathbf{1}_{(F_b,\varepsilon_b)}\|_\ell.$$

By convention, we set $\|0\|_\ell = -\infty$.

If $\delta_\ell\colon \Delta' \to \Delta$ is a face operator (i.e. an inclusion of a face of codimension 1) we denote by $\partial_\ell\sigma$ the filtered simplex defined by $\partial_\ell\sigma = \sigma \circ \delta_\ell\colon \Delta' \to X$. The *Thom-Whitney complex* of X is the cochain complex $\widetilde{N}^*(X)$ composed of the elements ω, associating to each regular filtered simplex $\sigma\colon \Delta_0 * \cdots * \Delta_n \to X$ an element $\omega_\sigma \in \widetilde{N}^*_\sigma$ such that $\delta^*_\ell(\omega_\sigma) = \omega_{\partial_\ell\sigma}$, for any face operator $\delta_\ell\colon \Delta' \to \Delta$ with $\Delta'_n \neq \emptyset$. (Here $\Delta' = \Delta'_0 * \cdots * \Delta'_n$ is the induced filtration.) The differential $d\omega$ is defined by $(d\omega)_\sigma = d(\omega_\sigma)$. The *$\ell$-perverse degree* of $\omega \in \widetilde{N}^*(X)$ is the supremum of all the $\|\omega_\sigma\|_\ell$ for all regular filtered simplices $\sigma\colon \Delta \to X$.

A *cochain $\omega \in \widetilde{N}^*(X)$ is $\overline{p}$-allowable* if $\|\omega\|_\ell \leq \overline{p}(\ell)$ for any $\ell \in \{1,\ldots,n\}$, and of $\overline{p}$-intersection if ω and $d\omega$ are $\overline{p}$-allowable. We denote by $\widetilde{N}^*_{\overline{p}}(X)$ the complex of $\overline{p}$-intersection cochains and by $H^*_{\mathrm{TW},\overline{p}}(X)$ its homology called *Thom-Whitney cohomology* (henceforth *TW-cohomology)* of X for the perversity $\overline{p}$.

2. Cap product and intersection homology

We first recall the definition and some basic properties of a cap product in intersection (co)homology already introduced in [2, Section 11].

Let $\Delta = [e_0,\ldots,e_m]$ be a euclidean simplex. We denote by $[\Delta]$ its face of maximal dimension. The classical cap product

$$-\cap[\Delta]\colon N^*(\Delta) \to N_{m-*}(\Delta)$$

is defined by

$$\mathbf{1}_F \cap [\Delta] = \left\{ \begin{array}{cl} [e_r,\ldots,e_m] & \text{if } F = [e_0,\ldots,e_r], \text{ for any } r \in \{0,\ldots,m\}, \\ 0 & \text{otherwise.} \end{array} \right.$$

We extend it to filtered simplices $\Delta = \Delta_0 * \cdots * \Delta_n$ as follows.

Let $\widetilde{\Delta} = \mathtt{c}\Delta_0 \times \cdots \times \mathtt{c}\Delta_{n-1} \times \Delta_n$. If $\mathbf{1}_{(F,\varepsilon)} = \mathbf{1}_{(F_0,\varepsilon_0)} \otimes \cdots \otimes \mathbf{1}_{(F_{n-1},\varepsilon_{n-1})} \otimes \mathbf{1}_{F_n} \in \widetilde{N}^*(\Delta)$, we define:

$$\begin{aligned} \mathbf{1}_{(F,\varepsilon)} \,\widetilde{\cap}\, \widetilde{\Delta} &= (-1)^{\nu(F,\varepsilon,\Delta)} (\mathbf{1}_{(F_0,\varepsilon_0)} \cap \mathtt{c}[\Delta_0]) \otimes \cdots \otimes (\mathbf{1}_{F_n} \cap [\Delta_n]) \\ &\in \widetilde{N}_*(\Delta) := N_*(\mathtt{c}\Delta_0) \otimes \cdots \otimes N_*(\mathtt{c}\Delta_{n-1}) \otimes N_*(\Delta_n), \end{aligned} \tag{5}$$

where $\nu(F,\varepsilon,\Delta) = \sum_{j=0}^{n-1} (\dim \Delta_j + 1) \left(\sum_{i=j+1}^{n} |(F_i,\varepsilon_i)|\right)$, with the convention $\varepsilon_n = 0$.

We now define a morphism, $\mu_*^{\Delta}\colon \widetilde{N}_*(\Delta) \to N_*(\Delta)$ by describing it on the elements $(F,\varepsilon) = (F_0,\varepsilon_0) \otimes \cdots \otimes (F_{n-1},\varepsilon_{n-1}) \otimes F_n$. Let ℓ be the smallest integer, j, such that $\varepsilon_j = 0$. We set

$$\mu_*^{\Delta}(F,\varepsilon) = \begin{cases} F_0 * \cdots * F_\ell & \text{if } \dim(F,\varepsilon) = \dim(F_0 * \cdots * F_\ell), \\ \quad 0 & \text{otherwise.} \end{cases} \tag{6}$$

(Note that $\mu_*^{\Delta}(F,\varepsilon) \neq 0$ if and only if all pairs (F_i,ε_i) are either $(\text{vertex}, 0)$ or $(\emptyset, 1)$ for any $i > \ell$.) If this image is not equal to zero, the application μ_*^{Δ} consists of a replacement of the tensor product $(F_0,\varepsilon_0) \otimes \cdots \otimes (F_{n-1},\varepsilon_{n-1}) \otimes F_n$ by the join $F_0 * \cdots * F_\ell$. Therefore, if we set $\nabla = \Delta_1 * \cdots * \Delta_n$, the application μ_*^{Δ} can be decomposed as

$$N_*(\mathtt{c}\Delta_0) \otimes \widetilde{N}_*(\nabla) \xrightarrow{\mathrm{id} \otimes \mu_*^{\nabla}} N_*(\mathtt{c}\Delta_0) \otimes N_*(\nabla) \xrightarrow{\mu_*^{\Delta_0 * \nabla}} N_*(\Delta). \tag{7}$$

Moreover, the application $\mu_*^{\Delta}\colon \widetilde{N}_*(\Delta) \to N_*(\Delta)$ is a chain map [2, Proposition 11.10] which allows the next local and global definitions of the intersection cap product.

Definition 2.1. Let $\Delta = \Delta_0 * \cdots * \Delta_n$ be a regular filtered simplex of dimension m. The *intersection cap product* $- \cap \widetilde{\Delta}\colon \widetilde{N}^*(\Delta) \to N_{m-*}(\Delta)$ is defined by

$$\omega \cap \widetilde{\Delta} = \mu_*^{\Delta}(\omega \,\widetilde{\cap}\, \widetilde{\Delta}).$$

In short, we have introduced three maps, called cap products:

- the classical one, $- \cap [\Delta]\colon N^*(\Delta) \to N_{m-*}(\Delta)$,
- its extension at the blow-up level, $- \,\widetilde{\cap}\, \widetilde{\Delta}\colon \widetilde{N}^*(\Delta) \to \widetilde{N}_{m-*}(\Delta)$,
- and finally the projection on chains, $- \cap \widetilde{\Delta}\colon \widetilde{N}^*(\Delta) \to N_{m-*}(\Delta)$, which is our intersection cap product. We note that the apexes of the cones do not appear in an intersection cap product since it is an element of $N_*(\Delta)$.

Definition 2.2. Let X be a pseudomanifold, $\omega \in \widetilde{N}^*(X)$ and $\sigma\colon \Delta_\sigma \to X$ a filtered simplex. *The intersection cap product* $- \cap -\colon \widetilde{N}^k(X) \otimes C_{k+j}(X) \to C_j(X)$ is defined by the linear extension of

$$\omega \cap \sigma = \begin{cases} \sigma_*(\omega_\sigma \cap \widetilde{\Delta}_\sigma) & \text{if } \sigma \text{ is regular}, \\ \quad 0 & \text{otherwise.} \end{cases}$$

Proposition 2.3 ([2, Proposition 11.16])**.** *Let X be an n-dimensional pseudomanifold and let $\overline{p}$, $\overline{q}$ be two perversities. The cap product defines a chain map*

$$- \cap -\colon \widetilde{N}^k_{\overline{p}}(X) \otimes C^{\overline{q}}_{k+j}(X) \to C^{\overline{p}+\overline{q}}_j(X).$$

Remark 2.4. Let $i \leq n-1$ and suppose $\Delta_i \neq \emptyset$. The cone vertex $\mathtt{v}_i$ belongs to the face $(F_i, 1)$ of $\mathtt{c}\Delta_i$, and, by convention, we write it as the last summit of $(F_i,1)$, i.e. $(F_i,1) = [a_{j_0}, \ldots, a_{j_r}, \mathtt{v}_i]$ with $F_i = [a_{j_0}, \ldots, a_{j_r}]$. If $F_i \neq \emptyset$, we denote by G_i the face of Δ_i such that the cup product satisfies $\mathbf{1}_{F_i} \cup \mathbf{1}_{G_i} = \mathbf{1}_{\Delta_i}$. From the definition of the cap product we have

$$\mathbf{1}_{(F_i,\varepsilon_i)} \cap [\mathtt{c}\Delta_i] = \begin{cases} (G_i,1) & \text{if } \varepsilon_i = 0, \\ [\mathtt{v}_i] & \text{if } \varepsilon_i = 1 \text{ and } F_i = \Delta_i, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Similarly, if $\Delta_i = \emptyset$ we have $\mathbf{1}_{(\emptyset,1)} \cap [\mathtt{c}\emptyset] = [\mathtt{v}_i]$.

3. Proofs of Theorems A and B

To define the morphism $\mathcal{M}^*_{\overline{p}}$ of (2), we consider the morphism dual of μ^Δ_* denoted $\mu^*_\Delta \colon N^*(\Delta) \to \widetilde{N}^*(\Delta)$. Let

$$\widetilde{N}^*_{\overline{0}}(\Delta) = \left\{\omega \in \widetilde{N}^*(\Delta) \mid \|\omega\|_\ell \leq 0 \text{ and } \|d\omega\|_\ell \leq 0 \text{ for all } \ell \in \{1,\dots,n\}\right\}.$$

The next result is in the spirit of a theorem of Verona; see [13].

Proposition 3.1. *Let $\Delta = \Delta_0 * \cdots * \Delta_n$ be a regular filtered simplex. Then the chain map*

$$\mu^*_\Delta \colon N^*(\Delta) \to \widetilde{N}^*_{\overline{0}}(\Delta) \subset \widetilde{N}^*(\Delta)$$

is an isomorphism.

Proof. Let $F = F_0 * \cdots * F_s$ be a face of Δ with $s \leq n$ and $F_s \neq \emptyset$. By definition of μ^*_Δ, we have

$$\mu^*_\Delta(\mathbf{1}_F) = \sum_{(a_{j_{s+1}},\dots,a_{j_n})} (-1)^{\nu(F)} \mathbf{1}_{(F_0,1)} \otimes \cdots \otimes \mathbf{1}_{(F_{s-1},1)} \otimes \mathbf{1}_{(F_s,0)} \otimes \mathbf{1}_{[a_{j_{s+1}}]} \otimes \cdots \otimes \mathbf{1}_{[a_{j_n}]}, \tag{9}$$

where the a_{j_i}'s run over the vertices of $\mathtt{c}\Delta_i$ if $i \in \{s+1,\dots,n-1\}$ and a_{j_n} over the vertices of Δ_n. The sign is defined by

$$\left(\mathbf{1}_{(F_0,1)} \otimes \cdots \otimes \mathbf{1}_{(F_{s-1},1)} \otimes \mathbf{1}_{(F_s,0)}\right)([\mathtt{c}F_0] \otimes \cdots \otimes [\mathtt{c}F_{s-1}] \otimes [F_s]) = (-1)^{\nu(F)}. \tag{10}$$

From Definition 1.2, we observe that $\|\mu^*_\Delta(\mathbf{1}_F)\|_\ell \leq 0$ for any $\ell \in \{1,\dots,n\}$ and the injectivity of μ^*_Δ.

Consider now a cochain $\omega \in \widetilde{N}^*_{\overline{0}}(\Delta)$. We have to prove that ω belongs to the image of μ^*_Δ. Since $\|\omega\|_\ell \leq 0$ for each $\ell \in \{1,\dots,n\}$, we may write ω as a sum of

$$\omega_F = \sum_{(a_{j_{s+1}},\dots,a_{j_n})} \lambda^F_{(a_{j_{s+1}},\dots,a_{j_n})} \mathbf{1}_{(F_0,1)} \otimes \cdots \otimes \mathbf{1}_{(F_{s-1},1)} \otimes \mathbf{1}_{(F_s,0)} \otimes \mathbf{1}_{[a_{j_{s+1}}]} \otimes \cdots \otimes \mathbf{1}_{[a_{j_n}]}$$

where $\lambda^F_{(a_{j_{s+1}},\dots,a_{j_n})} \in R$ and $F = F_0 * \cdots * F_s$ with $F_s \neq \emptyset$ runs over the faces of Δ. Since $\omega \in \widetilde{N}^*_{\overline{0}}(\Delta)$, we have $\|d\omega\|_\ell \leq 0$ for any $\ell \in \{1,\dots,n\}$. Let $F = F_0 * \cdots * F_s$ with $F_s \neq \emptyset$ being fixed. Since $d\mathbf{1}_{[a_i]}$ is of degree 1, for having $\|d\omega\|_\ell \leq 0$ for any $\ell \in \{1,\dots,n\}$, we must have

$$\sum_{(a_{j_{s+1}},\dots,a_{j_n})} \lambda^F_{(a_{j_{s+1}},\dots,a_{j_n})} (-1)^{|F|+s+1} \mathbf{1}_{(F_0,1)} \otimes \cdots \otimes \mathbf{1}_{(F_s,0)} \otimes d(\mathbf{1}_{[a_{j_{s+1}}]} \otimes \cdots \otimes \mathbf{1}_{[a_{j_n}]}) = 0,$$

which implies that

$$\sum_{(a_{j_{s+1}},\dots,a_{j_n})} \lambda^F_{(a_{j_{s+1}},\dots,a_{j_n})} d(\mathbf{1}_{[a_{j_{s+1}}]} \otimes \cdots \otimes \mathbf{1}_{[a_{j_n}]}) = 0.$$

As, up to a multiplicative constant, there exists only one cocycle in degree zero in this tensor product, all the coefficients are equal; i.e. there exists $\lambda_F \in R$ such that

$$\lambda_F = \lambda^F_{(a_{j_{s+1}},\dots,a_{j_n})},$$

for any $(n-s)$-tuple of vertices $(a_{j_{s+1}},\dots,a_{j_n})$. Therefore, we may write

$$\omega = \mu^*_\Delta\left(\sum_{F\lhd\Delta}\lambda_F\mathbf{1}_F\right),$$

and ω is in the image of μ^*_Δ. □

If $\omega \in \widetilde{N}^*(\Delta)$ is the image by μ^*_Δ of a cochain $c \in N^*(\Delta)$, the intersection cap product coincides with the usual one.

Proposition 3.2. *Let $\Delta = \Delta_0 * \cdots * \Delta_n$ be a regular filtered simplex of dimension m. For each cochain $c \in N^*(\Delta)$, we have*

$$\mu^*_\Delta(c) \cap \widetilde{\Delta} = c \cap [\Delta],$$

where $c \cap [\Delta]$ comes from the usual cap product, $-\cap -\colon N^*(\Delta)\otimes N_m(\Delta) \to N_{m-*}(\Delta)$.

Proof. The result is clear for $n = 0$. By using (7), it is sufficient to prove the result for $\Delta = \Delta_0 * \Delta_1$. Let $\mathbf{1}_{F_0*F_1} \in N^*(\Delta)$. We use Remark 2.4 in the next determinations.
– We denote by G_1 the face of Δ_1 such that the cup product $\mathbf{1}_{F_1}\cup\mathbf{1}_{G_1}$ is equal to $\mathbf{1}_{\Delta_1}$.
– The cap product with $\mathbf{1}_{(F_0,1)}$ is not equal to zero only if $F_0 = \Delta_0$. In this case, we set $G_0 = \emptyset$. If $\varepsilon_0 = 0$, we denote by G_0 the face of Δ_0 such that $\mathbf{1}_{F_0}\cup\mathbf{1}_{G_0} = \mathbf{1}_{\Delta_0}$.

We prove the statement by considering the various possibilities.

• Suppose $F_1 \neq \emptyset$. As $\left(\mathbf{1}_{(F_0,1)}\otimes\mathbf{1}_{F_1}\right)([\mathtt{c}F_0]\otimes[F_1]) = (-1)^{|\mathtt{c}F_0|\,|F_1|}$, we deduce from (9) and (10) that

$$\mu^*_\Delta\left(\mathbf{1}_{F_0*F_1}\right) = (-1)^{|F_1|\,|\mathtt{c}F_0|}\mathbf{1}_{(F_0,1)}\otimes\mathbf{1}_{F_1}$$

and

$$\begin{aligned}
\mu^*_\Delta(\mathbf{1}_{F_0*F_1})\cap(\mathtt{c}\Delta_0\times\Delta_1) &= \mu^\Delta_*(\mu^*_\Delta(\mathbf{1}_{F_0*F_1})\,\widetilde{\cap}\,(\mathtt{c}\Delta_0\times\Delta_1))\\
&= (-1)^{|F_1||\mathtt{c}F_0|}\mu^\Delta_*((\mathbf{1}_{(F_0,1)}\otimes\mathbf{1}_{F_1})\,\widetilde{\cap}\,(\mathtt{c}\Delta_0\times\Delta_1))\\
&= (-1)^{|F_1|(|\mathtt{c}F_0|+|\mathtt{c}\Delta_0|)}\mu^\Delta_*(\mathbf{1}_{(F_0,1)}\cap[\mathtt{c}\Delta_0])\\
&\quad\otimes(\mathbf{1}_{F_1}\cap[\Delta_1])\\
&=_{(1)} \begin{cases}\mu^\Delta_*([\mathtt{v}_0]\otimes G_1) & \text{if } F_0=\Delta_0,\\ 0 & \text{otherwise,}\end{cases}\\
&= \begin{cases} G_1 & \text{if } F_0=\Delta_0,\\ 0 & \text{otherwise,}\end{cases}\\
&= \mathbf{1}_{F_0*F_1}\cap[\Delta_0*\Delta_1].
\end{aligned}$$

• If $F_1=\emptyset$, we have $\mu^*_\Delta\,(\mathbf{1}_{F_0*\emptyset})=\sum_{[a_{j_1}]\lhd\Delta_1}\mathbf{1}_{(F_0,0)}\otimes\mathbf{1}_{[a_{j_1}]}$ and

$$\begin{aligned}
\mu^*_\Delta(\mathbf{1}_{F_0})\cap(c\Delta_0\times\Delta_1) &= \mu^\Delta_*(\mu^*_\Delta(\mathbf{1}_{F_0})\,\widetilde{\cap}\,(\mathsf{c}\Delta_0\times\Delta_1))\\
&= \mu^\Delta_*\left((\sum_{[a_{j_1}]\lhd\Delta_1}\mathbf{1}_{(F_0,0)}\otimes\mathbf{1}_{[a_{j_1}]})\,\widetilde{\cap}\,(\mathsf{c}\Delta_0\times\Delta_1)\right)\\
&= \mu^\Delta_*\left(\sum_{[a_{j_1}]\lhd\Delta_1}(\mathbf{1}_{(F_0,0)}\cap[\mathsf{c}\Delta_0])\otimes(\mathbf{1}_{[a_{j_1}]}\cap[\Delta_1])\right)\\
&=_{(1)} \mu^\Delta_*((G_0,1)\otimes[\Delta_1])=G_0*\Delta_1=\mathbf{1}_{F_0}\cap[\Delta_0*\Delta_1].
\end{aligned}$$

(The two equalities $=_{(1)}$ are consequences of Remark 2.4.) ☐

Let us recall that $C^*(X)$ is the complex of filtered singular cochains with coefficients in R.

Proposition 3.3. *Let X be a normal compact pseudomanifold. Then, the operator $\mathcal{M}_{\overline{0}}\colon C^*(X)\to\widetilde{N}^*_{\overline{0}}(X)$, defined by $\mathcal{M}_{\overline{0}}(c)_\sigma=\mu^*_\Delta(\sigma^*(c))$ for any regular filtered simplex $\sigma\colon\Delta\to X$, is a chain map which induces an isomorphism*

$$\mathcal{M}^*_{\overline{0}}\colon H^*(X)\xrightarrow{\cong}H^*_{\mathrm{TW},\overline{0}}(X).$$

We denote by $\mathcal{M}_{\overline{p}}\colon C^*(X)\to\widetilde{N}^*_{\overline{p}}(X)$ the composition of $\mathcal{M}_{\overline{0}}$ with the canonical inclusion of complexes and by $\mathcal{M}^*_{\overline{p}}\colon H^*(X)\to H^*_{\mathrm{TW},\overline{p}}(X)$ the induced morphism.

Proof. The maps μ^Δ_* being compatible with restrictions, the map $\mathcal{M}_{\overline{0}}$ is well defined. Its compatibility with the differentials is a consequence of [2, Proposition 11.10], and its behaviour with perversities a consequence of Proposition 3.1. For proving the isomorphism, we use a method similar to an argument of H. King in [9]; see also [5, Section 5.1]. We have to check the hypotheses of [2, Proposition 8.1].

• The first hypothesis is the existence of Mayer-Vietoris sequences. This is clear for $C^*(-)$ and has been proved in [2, Théorème A] for $\widetilde{N}^*_{\overline{0}}(-)$.

• The second and the fourth hypotheses are straightforward.

• The third one consists of the computation of the intersection cohomology of a cone. Let L be a compact connected pseudomanifold. It is well known that $H^0(\mathbb{R}^i\times\mathring{c}L)=R$ and $H^i(\mathbb{R}^i\times\mathring{c}L)=0$ if $i>0$. From [2, Propositions 6.1 and 7.1], we have

$$\bigoplus_{k\geq 0}H^k_{\mathrm{TW},\overline{0}}(\mathbb{R}^i\times\mathring{c}L)=H^0_{\mathrm{TW},\overline{0}}(L).$$

If $\omega\in\widetilde{N}^0_{\overline{0}}(L)$ is a cocycle, the cochain ω_σ is constant for any regular simplex $\sigma\colon\Delta\to L$. As σ has a non-empty intersection with the regular part, the connectedness of $L\backslash\Sigma$ (see [5, Lemma 2.6.3]) implies $H^0_{\mathrm{TW},\overline{0}}(L;R)=R$. ☐

Proposition 3.3 remains true for normal CS-sets, with the same proof. We do not introduce this notion here; see [12] for its definition.

Proof of Theorem A. Let $\overline{p}$ be a perversity. We consider the following diagram:

$$\begin{tikzcd}
H^*(X) \arrow[r, "-\cap[X]"] \arrow[d, "\mathcal{M}^*_{\overline{0}}"] \arrow[dd, bend right=60, "\mathcal{M}^*_{\overline{p}}"'] & H_{n-*}(X) \\
H^*_{\mathrm{TW},\overline{0}}(X) \arrow[d, "\alpha_{\overline{0},\overline{p}}"] & H^{\overline{t}}_{n-*}(X) \arrow[u, "\beta^{\overline{t}}"'] \\
H^*_{\mathrm{TW},\overline{p}}(X) \arrow[r, "-\cap[X]", "\cong"'] & H^{\overline{p}}_{n-*}(X), \arrow[u, "\beta^{\overline{p},\overline{t}}"']
\end{tikzcd} \tag{11}$$

where $\alpha_{\overline{0},\overline{p}}$, $\beta^{\overline{p},\overline{t}}$ and $\beta^{\overline{t}}$ are induced by the natural inclusions of complexes. The bottom isomorphism comes from [2, Théorème D]. It remains to check the commutativity of this diagram. Consider a cochain $c \in C^*(X)$ and a regular simplex $\sigma\colon \Delta_\sigma \to X$. (The chain $[X]$ being $\overline{0}$-allowable, each simplex in its decomposition is regular.) We have

$$\begin{aligned}
\mathcal{M}^*_{\overline{0}}(c) \cap \sigma \quad &=_{(1)} \quad \sigma_*\left(\mathcal{M}^*_{\overline{0}}(c)_\sigma \cap \widetilde{\Delta}_\sigma\right) =_{(2)} \sigma_*\left(\mu^*_{\Delta_\sigma}(\sigma^*(c)) \cap \widetilde{\Delta}_\sigma\right) \\
&=_{(3)} \quad \sigma_*\left(\sigma^*(c) \cap [\Delta_\sigma]\right) =_{(4)} c \cap \sigma,
\end{aligned} \tag{12}$$

where (1) is Definition 2.2, (2) comes from the definition of $\mathcal{M}^*_{\overline{0}}$, (3) comes from Proposition 3.2. and (4) is a property of the classical cap product. $\square$

Proof of Theorem B. By specifying the diagram (11) to the case $\overline{p} = \overline{0}$, we get the next commutative diagram:

$$\begin{tikzcd}
H^*(X) \arrow[r, "-\cap[X]"] \arrow[dd, "\mathcal{M}^*_{\overline{0}}"', "\cong"] & H_{n-*}(X) \\
 & H^{\overline{t}}_{n-*}(X) \arrow[u, "\cong", "\beta^{\overline{t}}"'] \\
H^*_{\mathrm{TW},\overline{0}}(X) \arrow[r, "-\cap[X]", "\cong"'] & H^{\overline{0}}_{n-*}(X). \arrow[u, "\beta^{\overline{0},\overline{t}}"']
\end{tikzcd}$$

From Proposition 3.3, [3, Proposition 5.5] and [2, Théorème D], we get that $\mathcal{M}^*_{\overline{0}}$, $\beta^{\overline{t}}$ and the bottom map are isomorphisms. This ends the proof. $\square$

4. Remarks and comments

Cup product and intersection product. In [1], [2], we define from the local structure on the euclidean simplices a cup product in intersection cohomology, induced by a chain map

$$\cup\colon \widetilde{N}^{k_1}_{\mathrm{TW},\overline{p}_1}(X) \otimes \widetilde{N}^{k_2}_{\mathrm{TW},\overline{p}_2}(X) \to \widetilde{N}^{k_1+k_2}_{\mathrm{TW},\overline{p}_1+\overline{p}_2}(X). \tag{13}$$

If X is a compact oriented n-dimensional pseudomanifold, the Poincaré duality induces (see [4, Section VIII.13]) an intersection product on the intersection homology defined by the commutativity of the next diagram:

$$\begin{tikzcd}
H^{k_1}_{\mathrm{TW},\overline{p}_1}(X)\otimes H^{k_2}_{\mathrm{TW},\overline{p}_2} \arrow[r, "\cup"] \arrow[d, "-\cap[X]\otimes-\cap[X]"', "\cong"] & H^{k_1+k_2}_{\mathrm{TW},\overline{p}_1+\overline{p}_2}(X) \arrow[d, "\cong"', "-\cap[X]"] \\
H^{\overline{p}_1}_{n-k_1}(X)\otimes H^{\overline{p}_2}_{n-k_2}(X) \arrow[r, "\pitchfork"] & H^{\overline{p}_1+\overline{p}_2}_{n-k_1-k_2}(X).
\end{tikzcd} \tag{14}$$

Let $[\omega],[\eta]\in H^*(X)$ and $\alpha^{\overline{p}}=(-\cap[X])\circ\mathcal{M}^*_{\overline{p}}$. From this definition, we deduce that

$$\alpha^{2\overline{p}}([\omega]\cup[\eta])=\alpha^{\overline{p}}([\omega])\pitchfork\alpha^{\overline{p}}([\eta]), \tag{15}$$

which is the analogue of the decomposition established in [7] in the PL case.

Lattice structure on the set of perversities. In diagram (2), the cap product $-\cap[X]\colon H^*(X)\to H_{n-*}(X)$ is factorized as $\beta^{\overline{p}}\circ\alpha^{\overline{p}}$ where $\beta^{\overline{p}}$ comes from the inclusion of complexes and the data $\overline{p}\mapsto\alpha^{\overline{p}}$ is compatible with the lattice structure on the set of perversities. More precisely, if $\overline{p}_1\leq\overline{p}_2$, we have a commutative diagram:

$$\begin{tikzcd}
 & & H^*(X) \arrow[ddll, dotted, "\mathcal{M}^*_{\overline{p}_2}"'] \arrow[d, dashed, "\mathcal{M}^*_{\overline{p}_1}"'] \arrow[ddr, dotted, "\alpha^{\overline{p}_2}"] \arrow[drr, dashed, "\alpha^{\overline{p}_1}"] & & \\
 & & H^*_{\mathrm{TW},\overline{p}_1}(X) \arrow[rr, dashed, "-\cap[X]"] \arrow[dll, "\alpha_{\overline{p}_1,\overline{p}_2}"] & & H^{\overline{p}_1}_{n-*}(X) \arrow[dl, "\beta^{\overline{p}_1,\overline{p}_2}"] \\
H^*_{\mathrm{TW},\overline{p}_2}(X) \arrow[rrr, dotted, "-\cap[X]"'] & & & H^{\overline{p}_2}_{n-*}(X), &
\end{tikzcd}$$

with $\alpha^{\overline{p}_2}=\beta^{\overline{p}_1,\overline{p}_2}\circ\alpha^{\overline{p}_1}$.

A second intersection cohomology. Alternately, as cohomology theory, we could choose (see [5], [6]) the dual complex of the intersection chains instead of $\widetilde{N}^*_{\overline{p}}(X)$. We denote

$$C^*_{\mathrm{GM},\overline{p}}(X;R)=\hom(C^{\overline{p}}_*(X;R),R) \text{ and } H^*_{\mathrm{GM},\overline{p}}(X;R) \text{ its cohomology.} \tag{16}$$

In [2, Théorème C], we have proved

$$H^*_{\mathrm{TW},\overline{p}}(X;R)\cong H^*_{\mathrm{GM},D\overline{p}}(X;R),$$

if R is a field, or more generally with a hypothesis on the torsion of the links, introduced in [8]. But, in the general case, these two cohomologies may differ. A natural question is the existence of a factorization of $\alpha^{\overline{p}}$ as above but in which the cohomology $H^*_{\mathrm{GM},D\overline{p}}(-)$ is substituted for $H^*_{\mathrm{TW},\overline{p}}(-)$. This question arises in [5, Section 8.1.6], together with a nice development on this point, that we are taking back partially in these lines. The next example shows that such factorization does not occur if we ask for the compatibility with the lattice structure.

Example 4.1. Recall that a cap product in the setting of $C^*_{\mathrm{GM},\overline{p}}(X;R)$ has been introduced by G. Friedman and J. McClure (see [6]) when R is a field. In the case of a commutative ring, its definition requires a condition on the torsion (see [5]) which is satisfied in the case of the top perversity. This justifies the existence of the horizontal isomorphism in the diagram below.

Consider the 4-dimensional compact oriented pseudomanifold $X = \Sigma\mathbb{RP}^3$ with $R = \mathbb{Z}$, $\overline{p}_1 = \overline{0}$, $\overline{p}_2 = \overline{1}$. The analogue of the previous diagram can be written as

$$\begin{tikzcd}
H^3(X) = \mathbb{Z}_2 \arrow[d, "\mathcal{M}^3_{\mathrm{GM},\overline{2}}"', "\cong"] \arrow[drr, "\alpha^{\overline{0}}_{\mathrm{GM}}"] & & \\
H^3_{\mathrm{GM},\overline{2}}(X) = \mathbb{Z}_2 \arrow[rr, "-\cap[X]", "\cong"'] \arrow[d] & & H^{\overline{0}}_1(X) = \mathbb{Z}_2 \arrow[d, "\cong"', "\beta^{\overline{0},\overline{1}}"] \\
H^3_{\mathrm{GM},\overline{1}}(X) = 0 & & H^{\overline{1}}_1(X) = \mathbb{Z}_2,
\end{tikzcd}$$

where $\mathcal{M}^3_{\mathrm{GM},\overline{2}}$ corresponds to the morphism $\omega^*_{\overline{2}}$ in [5, Section 8.1.6]. Let $\alpha^{\overline{0}}_{\mathrm{GM}} = (-\cap[X]) \circ \mathcal{M}^3_{\mathrm{GM},\overline{2}}$. Then the map $\alpha^{\overline{1}}_{\mathrm{GM}} = \beta^{\overline{0},\overline{1}} \circ \alpha^{\overline{0}}_{\mathrm{GM}}$ is an isomorphism which cannot factorize through $H^3_{\mathrm{GM},\overline{1}}(X)$.

Acknowledgement

The authors thank the referee for making suggestions which contributed to improving the writing.

References

[1] David Chataur, Martintxo Saralegi-Aranguren, and Daniel Tanré, *Intersection cohomology. Simplicial blow-up and rational homotopy*, arXiv mathematics e-prints 1205.7057v4 (2012). To appear in Mem. Amer. Math. Soc.

[2] ——, *Dualité de Poincaré et homologie d'intersection*, arXiv mathematics e-prints 1603.08773v1 (2016).

[3] ——, *Homologie d'intersection. Perversités générales et invariance topologique*, arXiv mathematics e-prints 1602.03009v1 (2016).

[4] Albrecht Dold, *Lectures on algebraic topology*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 200, Springer-Verlag, Berlin-New York, 1980. MR606196

[5] Greg Friedman, *Singular intersection homology*. Available at http://faculty.tcu.edu/gfriedman/index.html.

[6] Greg Friedman and James E. McClure, *Cup and cap products in intersection (co)homology*, Adv. Math. **240** (2013), 383–426, DOI 10.1016/j.aim.2013.02.017. MR3046315

[7] Mark Goresky and Robert MacPherson, *Intersection homology theory*, Topology **19** (1980), no. 2, 135–162, DOI 10.1016/0040-9383(80)90003-8. MR572580

[8] Mark Goresky and Paul Siegel, *Linking pairings on singular spaces*, Comment. Math. Helv. **58** (1983), no. 1, 96–110, DOI 10.1007/BF02564627. MR699009

[9] Henry C. King, *Topological invariance of intersection homology without sheaves*, Topology Appl. **20** (1985), no. 2, 149–160, DOI 10.1016/0166-8641(85)90075-6. MR800845

[10] Clinton Graydon McCrory, *Poincaré duality in spaces with singularities*, ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–Brandeis University, 1972. MR2622359

[11] Clint McCrory, *Zeeman's filtration of homology*, Trans. Amer. Math. Soc. **250** (1979), 147–166, DOI 10.2307/1998982. MR530047

[12] L. C. Siebenmann, *Deformation of homeomorphisms on stratified sets. I, II*, Comment. Math. Helv. **47** (1972), 123–136; ibid. **47** (1972), 137–163. MR0319207

[13] Andrei Verona, *Le théorème de de Rham pour les préstratifications abstraites* (French), C. R. Acad. Sci. Paris Sér. A-B **273** (1971), A886–A889. MR0290375

LAFMA, Université de Picardie Jules Verne, 33, rue Saint-Leu, 80039 Amiens Cedex 1, France

E-mail address: David.Chataur@u-picardie.fr

Laboratoire de Mathématiques de Lens, EA 2462, Université d'Artois, SP18, rue Jean Souvraz, 62307 Lens Cedex, France

E-mail address: martin.saraleguiaranguren@univ-artois.fr

Département de Mathématiques, UMR 8524, Université de Lille 1, 59655 Villeneuve d'Ascq Cedex, France

E-mail address: Daniel.Tanre@univ-lille1.fr