PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 145, Number 8, August 2017, Pages 3331–3342
http://dx.doi.org/10.1090/proc/13612
Article electronically published on April 6, 2017

NIELSEN EQUIVALENCE IN GUPTA-SIDKI GROUPS

AGLAIA MYROPOLSKA

(Communicated by Pham Huu Tiep)

ABSTRACT. For a group G generated by k elements, *the Nielsen equivalence classes* are defined as orbits of the action of $\mathrm{Aut}F_k$, the automorphism group of the free group of rank k, on the set of generating k-tuples of G.

Let $p \geqslant 3$ be prime and G_p the Gupta-Sidki p-group. We prove that there are infinitely many Nielsen equivalence classes on generating pairs of G_p.

1. INTRODUCTION

Let G be a finitely generated group. *The rank* $\mathrm{rank}(G)$ of G is the minimal number of generators of G. Fix $k \geqslant \mathrm{rank}(G)$ and let $\mathrm{Epi}(F_k, G)$ be the set of epimorphisms $\phi : F_k \to G$ from the free group F_k of rank k to G.

Consider the natural action of the group $\mathrm{Aut}\, F_k \times \mathrm{Aut}\, G$ on $\mathrm{Epi}(F_k, G)$: for $(\tau, \sigma) \in \mathrm{Aut}\, F_k \times \mathrm{Aut}\, G$ and for $\phi \in \mathrm{Epi}(F_k, G)$ define

$$\phi^{(\tau,\sigma)} = \sigma \cdot \phi \cdot \tau^{-1}.$$

Motivated by the study of presentations of finite groups, B. H. Neumann and H. Neumann defined in [NN51] *T_k-systems (systems of transitivity)* to be the orbits of this action. Later the action of $\mathrm{Aut}\, F_k$ on $\mathrm{Epi}(F_k, G)$ obtained much attention, especially in recent years: the motivation comes from the different areas, such as presentation of groups, computational group theory, product replacement algorithm, etc. The main question in this subject is whether the action $\mathrm{Aut}\, F_k \circlearrowright \mathrm{Epi}(F_k, G)$ is transitive, and if not, whether the number of orbits is finite or infinite. We will expand on $\mathrm{Aut}\, F_k$ to understand this action better.

It was proved by Nielsen [Nie18] that $\mathrm{Aut}\, F_k$ is generated by the following automorphisms, where $\{x_1, \dots, x_k\}$ is the basis of F_k:

$$R_{ij}^{\pm}(x_1, \dots, x_i, \dots, x_j, \dots, x_k) = (x_1, \dots, x_i x_j^{\pm 1}, \dots, x_j, \dots, x_k),$$
$$I_j(x_1, \dots, x_j, \dots, x_k) = (x_1, \dots, x_j^{-1}, \dots, x_k),$$

where $1 \leqslant i, j \leqslant k$, $i \neq j$. The transformations above are called *elementary Nielsen moves.*

There is an obvious one-to-one correspondence between $\mathrm{Epi}(F_k, G)$ and the set $\{(g_1, \dots, g_k) \mid \langle g_1, \dots, g_k \rangle = G\}$ of generating k-tuples of G. The action of $\mathrm{Aut}\, F_k$ on the generating k-tuple $(g_1, \dots, g_k)$ is determined by applying sequences of elementary Nielsen moves to $(g_1, \dots, g_k)$ by precomposition. For example, if $G = \mathbb{Z}^k$,

Received by the editors May 29, 2015 and, in revised form, September 20, 2016.

2010 *Mathematics Subject Classification.* Primary 20E05, 20E36, 20F05.

The author acknowledges the support of the Swiss National Science Foundation, grant 200021_144323.

then the set of generating k-tuples of $\mathbb{Z}^k$ coincides with $GL(k, \mathbb{Z})$ and the elementary Nielsen moves induce elementary row operations on these matrices. It follows that the action of $\operatorname{Aut} F_k$ on $\operatorname{Epi}(F_k, \mathbb{Z}^k)$ is transitive.

The orbits of the action $\operatorname{Aut} F_k \circlearrowright \operatorname{Epi}(F_k, G)$ are called *Nielsen (equivalence) classes* on generating k-tuples of G. In recent years the Nielsen equivalence classes became of particular interest as they correspond to the connected components of the Product Replacement Graph, whose set of vertices coincides with the set $\operatorname{Epi}(F_k, G)$ and whose edges correspond to elementary Nielsen moves (see [Eva07,Lub11,Pak01] and Section 3 for more on this topic).

In this article we study Nielsen equivalence in Gupta-Sidki p-groups $\{G_p\}_{p\geqslant 3}$ where p is odd prime. The Gupta-Sidki group G_p is an infinite p-group of rank 2, acting by automorphisms on the rooted p-ary tree T_p; moreover it is a just-infinite group, i.e., its every proper quotient is finite (Gupta-Sidki p-groups were defined in [GS83b]; the reader can refer to Section 2 for the definition and basic properties of G_p). It was shown by the author in [Myr16] that *all the generating k-tuples in the Gupta-Sidki p-group are Nielsen equivalent when $k \geqslant 3 = \operatorname{rank}(G_p) + 1$*. Here we show that there are infinitely many Nielsen classes when $k = \operatorname{rank}(G_p) = 2$. To the author's knowledge these are the first known examples of torsion groups with infinitely many Nielsen equivalence classes.

Theorem 1.1. *Let $p \geqslant 3$ be prime and G_p the Gupta-Sidki p-group. Then there are infinitely many Nielsen equivalence classes on generating pairs of G_p.*

It follows from [Myr16] that it is enough to allow an additional type of transformations, the Andrews-Curtis moves, to the elementary Nielsen moves so that the action on $\operatorname{Epi}(F_2, G_p)$ has a finite number of orbits (in particular, the action is transitive for $p = 3$).

More precisely, we say that elementary Nielsen moves together with the transformations

$$AC_{i,w}(x_1, ..., x_i, ..., x_k) = (x_1, ..., w^{-1}x_iw, ..., x_k)$$

where $1 \leqslant i \leqslant k$ and $w \in F_k$, form the set of *elementary Andrews-Curtis moves.* Elementary Andrews-Curtis moves transform *normally generating sets* (sets which generate F_k as a normal subgroup) into normally generating sets.

The Andrews-Curtis moves were introduced by Andrews and Curtis in [AC65] with the following conjecture, which remains open 50 years after it was formulated.

The Andrews-Curtis conjecture asserts that, for a free group F_k of rank $k \geqslant 2$ and a free basis $(x_1, ..., x_k)$ of F_k, any normally generating k-tuple $(y_1, ..., y_k)$ of F_k can be transformed into $(x_1, ..., x_k)$ by a finite sequence of elementary Andrews-Curtis moves.

We say that two normally generating k-tuples of F_k are *Andrews-Curtis equivalent* if one is obtained from the other by a finite chain of elementary Andrews-Curtis moves. The Andrews-Curtis equivalence corresponds to the actions of $\operatorname{Aut} F_k$ and of $(F_k)^k$ on normally generating k-tuples of F_k. More generally, for a finitely generated group G and $k \geqslant \operatorname{rank}(G)$, the above actions can be defined on the set $N_k(G)$ of normally generating k-tuples of G by precomposition. The orbits of this action are called *the Andrews-Curtis (equivalence) classes in G*. The analysis of Andrews-Curtis equivalence for arbitrary finitely generated groups is important to the analysis of potential counterexamples to the conjecture.

It follows from [Per05] and [Myr16] that the set of normally generating k-tuples $N_k(G_p)$ and the set of generating k-tuples $\mathrm{Epi}(F_k, G_p)$ coincide for $k \geqslant 2$. The main result of [Myr16] implies then that *for the Gupta-Sidki group G_p there are $\frac{p-1}{2}$ Andrews-Curtis classes on generating pairs of G_p*. Note however, that the question whether there are (in)finitely many T_2-systems on generating pairs of G_p remains open.

The Gupta-Sidki p-group, being a subgroup of $\mathrm{Aut}\, T_p$, has natural quotients by the level stabilizer subgroups $St_G(n)$. These quotients are finite nilpotent 2-generated groups with growing nilpotency class. The latter is true since the limit of these quotients in the space of marked 2-generated groups is the Gupta-Sidki p-group itself, which is not finitely presentable [Sid87]. In the last part of the paper we show a non-transitivity result for each finite quotient $G^{(n)} = G_3/St_{G_3}(n+3)$, $n \geqslant 1$, of the Gupta-Sidki 3-group. Note that there is only one Nielsen equivalence class in $(\mathbb{Z}/3\mathbb{Z})^2$, the abelianization of each $G^{(n)}$.

Proposition 1.2. *Let G_3 be the Gupta-Sidki 3-group and $St_{G_3}(n)$ the n-th level stabilizer subgroup of G_3. Set $G^{(n)} = G_3/St_{G_3}(n+3)$. Then the action $\mathrm{Aut}\, F_2 \circlearrowright \mathrm{Epi}(F_2, G^{(n)})$ is not transitive for any $n \geqslant 1$.*

It would be interesting to understand whether the number of Nielsen classes grows with n, but this remains an open question for the moment. An affirmative answer on this question would, in particular, imply that there were infinitely many Nielsen equivalence classes in G_3.

The paper is organized as follows. Section 2 contains preliminaries on Gupta-Sidki groups, in Section 3 we discuss relevant background on Nielsen equivalence, and Section 4 contains proofs of Theorem 1.1 and Proposition 1.2.

2. Preliminaries on groups acting on rooted trees

Let $X = \{1, 2, \ldots, d\}$ with $d \geqslant 2$ be a finite set. The vertex set of the rooted tree T_d is the set of finite sequences $\{x_1 x_2 \ldots x_k : x_i \in X\}$ over X; two sequences are connected by an edge when one can be obtained from the other by right-adjunction of a letter in X. The top node (the root) is the empty sequence $\varnothing$, and the children of σ are all the σs for $s \in X$. A map $f\colon T_d \to T_d$ is an *automorphism of the tree T_d* if it is bijective and it preserves the root and adjacency of the vertices. An example of an automorphism of T_d is the rooted automorphism a_π, defined as follows: for the permutation $\pi \in Sym(d)$, set $a_\pi(s\sigma) := \pi(s)\sigma$. Geometrically it can be viewed as the permutation of d subtrees just below the root $\varnothing$. Denote by $\mathrm{Aut}\, T_d$ *the group of automorphisms of the tree T_d*.

Let $G \leqslant \mathrm{Aut}\, T_d$. Denote by $St_G(\sigma)$ the subgroup of G consisting of the automorphisms that fix the sequence σ, i.e.,

$$St_G(\sigma) = \{g \in G \mid g(\sigma) = \sigma\}.$$

And denote by $St_G(n)$ the subgroup of G consisting of the automorphisms that fix all sequences of length n, i.e.,

$$St_G(n) = \bigcap_{\sigma \in X^n} St_G(\sigma).$$

Notice an obvious inclusion $St_G(n+1) \leqslant St_G(n)$. Moreover, observe that for any $n \geqslant 0$ the subgroups $St_G(n)$ are normal and of finite index in G. We therefore have a natural epimorphism between finite groups

$$G/St_G(n+1) \to G/St_G(n), \tag{1}$$

for any $n \geqslant 0$.

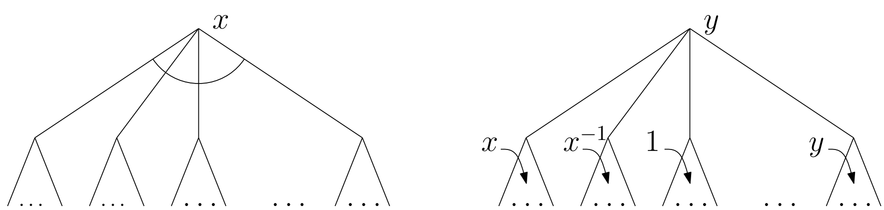
The examples of groups acting on rooted trees include remarkable examples of groups, such as the Grigorčuk group [Gri80] and the Gupta-Sidki p-groups [GS83a]. We define the latter family of groups below.

Fix $p \geqslant 3$ prime and $X = \{1, 2, \dots, p\}$. Let $\pi = (1, 2, \dots, p)$ be the cyclic permutation on X. Let s belong to X and σ belong to T_p. Denote by x the rooted automorphism of T_p defined by

$$x(s\sigma) = \pi(s)\sigma.$$

Denote by y the automorphism of T_p defined by

$$y(s\sigma) = \begin{cases} sx(\sigma) & \text{if } s = 1, \\ sx^{-1}(\sigma) & \text{if } s = 2, \\ sy(\sigma) & \text{if } s = p, \\ s\sigma & \text{otherwise.} \end{cases}$$



The *Gupta-Sidki* p-group is the group G_p of automorphisms of the tree T_p generated by x and y and we will write

$$G_p = \langle x, y \rangle.$$

To shorten the notation for the element y we will simply write

$$y = (x, x^{-1}, 1, \dots, 1, y).$$

Furthermore, for any element $g \in G_p$ we can write $g = x^i(g_1, \dots, g_p)$ for some $0 \leqslant i \leqslant p-1$ and $g_1, \dots, g_p \in G_p$. This writing is uniquely defined by the action of g on T_p. Namely, for any $g \in G_p$ the automorphism g acts as a permutation generated by x on the p subtrees just below the root $\varnothing$ and it induces the automorphisms $g_1, \dots, g_p$ on the respective subtrees.

For more details on the reach topic of groups acting on rooted trees we refer the reader to [Nek05].

We summarize here *some facts on the Gupta-Sidki p-group* which will be used in the following section:

(1) [GS83b] G_p is just-infinite, i.e., every proper quotient of G_p is finite.
(2) [Per05] All maximal subgroups of G_p are normal.
(3) [Per02] The abelianization $G_p^{ab} = G_p/[G_p, G_p]$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.

3. Nielsen equivalence

For a finitely generated group G and $k \geqslant \operatorname{rank}(G)$, we define the *Nielsen graph*[‡] $N_k(G)$ as follows:

- the set of vertices consists of generating k-tuples, i.e.,

$$V_{N_k}(G) = \{(g_1, ..., g_k) \in G^k \mid \langle g_1, ..., g_k \rangle = G\};$$

- two vertices are connected by an edge if one of them is obtained from the other by an elementary Nielsen move.

Observe that the graph $N_k(G)$ is connected if and only if the action of $\operatorname{Aut} F_k$ on $\operatorname{Epi}(F_k, G)$ is transitive.

Recall, that for a finitely generated group G the *Frattini subgroup* $\Phi(G) = \bigcap_{M <_{max} G} M$ is defined as the intersection of all maximal subgroups of G. Equivalently, the Frattini subgroup of G contains all the *non-generators*, i.e., the elements which can be removed from any generating set. The latter implies the following lemma.

Lemma 3.1 ([Eva93]). *Let G be a group generated by $\{x_1, ..., x_k\}$ and let $\varphi_1, ..., \varphi_k \in \Phi(G)$. Then $\langle x_1\varphi_1, ..., x_k\varphi_k \rangle = G$.*

Consider the class $\mathfrak{C}$ of finitely generated groups of which every maximal subgroup is normal in [Myr16]. The class $\mathcal{MN}$ includes finitely generated nilpotent groups; moreover all Grigorchuk groups and GGS groups, e.g., Gupta-Sidki p-groups G_p, belong to $\mathcal{MN}$ by [Per00, Per05]. In [AKT13] the result for GGS groups was generalized: the authors proved that all multi-edge spinal torsion groups acting on the regular p-ary rooted tree, with p odd prime, belong to $\mathcal{MN}$.

Observe that, for a group G in $\mathcal{MN}$, a normally generating set of G is, in fact, a generating set. Therefore, for groups in $\mathcal{MN}$ the partition (of the set of generating k-tuples) into Nielsen equivalence classes is a refinement of the partition into Andrews-Curtis classes.

We further describe what is known about Nielsen equivalence for some groups in the class $\mathcal{MN}$.

The most well-understood classification of Nielsen equivalence classes is known for finitely generated abelian groups (see [NN51, DG99, Oan11]). It also follows from the latter papers that for any finitely generated abelian group there is only one T_k-system for any $k \geqslant \operatorname{rank}(G)$.

Theorem 3.2 ([NN51, DG99, Oan11]).
Let A be a finitely generated abelian group with the primary decomposition $A \cong \mathbb{Z}^s \times \mathbb{Z}_{m_1} \times ... \times \mathbb{Z}_{m_r}$ with $r, s \geqslant 0$ and $m_r | m_{r-1} | ... | m_1$. Then $\operatorname{rank}(A) = r + s$ *and*

1. *$N_k(A)$ is connected if $k > r + s$;*
2. *if $r = 0$, i.e., $A \cong \mathbb{Z}^s$, then $N_s(G)$ is connected;*
3. *otherwise if $m_r = 2$ or $m_r = 3$, then $N_{r+s}(A)$ is connected and if $m_r > 3$, then $N_{r+s}(A)$ has $\varphi(m_r)/2$ connected components,*

where φ is the Euler function (the number of positive integers less than m_r which are coprime with m_r).

For a finitely generated nilpotent group the action of $\operatorname{Aut} F_k$ is transitive on $\operatorname{Epi}(F_k, G)$ when $k \geqslant \operatorname{rank}(G) + 1$ [Eva93]. However when $k = \operatorname{rank}(G)$ the

[‡] Also called the Extended Product Replacement Graph.

unicity of a Nielsen equivalence class generally breaks down. For instance, Dunwoody [Dun63] showed that to every pair of integers $n > 1$ and $N > 0$ there exists a finite nilpotent group of rank n and nilpotency class 2 for which there are at least N T_n-systems.

We consider the family of Gupta-Sidki p-groups $\{G_p\}_{p\geqslant 3}$, with p odd prime, as a generalization of finite nilpotent groups. It was shown by Pervova [Per05] that the groups G_p belong to the class $\mathcal{MN}$. This property was the main ingredient in [Myr16] for proving that there is only one Nielsen equivalence class for G_p for $k \geqslant 3 = \operatorname{rank}(G_p) + 1$. Moreover, for a group belonging to the class $\mathcal{MN}$, it is relevant to analyze Nielsen equivalence classes in the quotient $G/\Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G. Namely, if there are two generating k-tuples of $G/\Phi(G)$ which *are not* Nielsen equivalent, then their preimages in G are generating k-tuples of G which also *are not* Nielsen equivalent (see the section on the class $\mathcal{MN}$ in [Myr15] for details).

Therefore for groups in class $\mathcal{MN}$ the number of connected components of $N_k(G)$ is bounded below by the number of connected components of $N_k(G/\Phi(G))$. Since all maximal subgroups of G_p, the Gupta-Sidki p-group, are normal, it follows that the quotient $G_p/\Phi(G_p)$ is abelian. Moreover, any generating set of the quotient $G_p/\Phi(G_p)$ can be lifted up to the generating set of G_p [Myr16]. Therefore $G_p/\Phi(G_p)$ is a quotient of $(\mathbb{Z}/p\mathbb{Z})^2$ of rank 2; we deduce that $G_p/\Phi(G_p) \cong (\mathbb{Z}/p\mathbb{Z})^2$. Using Theorem 3.2, we find the number of connected components of the Nielsen graph $N_2((\mathbb{Z}/p\mathbb{Z})^2)$. Notice that for $p > 3$ the Nielsen graph $N_2(G_p)$ has at least $\frac{p-1}{2}$ connected components.

For $p = 3$ the question on the transitivity of the action of $\operatorname{Aut} F_2$ on $\operatorname{Epi}(F_2, G_3)$ is more subtle. In this paper we show in particular that, although there is only one Nielsen class on generating pairs of $G_3/\Phi(G_3) = (\mathbb{Z}/3\mathbb{Z})^2$, the action of $\operatorname{Aut} F_2$ is not transitive on $\operatorname{Epi}(F_2, G_3)$.

4. Nielsen equivalence in Gupta-Sidki groups

To prove Theorem 1.1 we use an observation by Nielsen (sometimes also attributed to Higman; see Lemma 4.1) as well as an analysis on conjugacy classes in the Gupta-Sidki p-group.

Lemma 4.1 (Nielsen)**.** *Let (u, v) and (u', v') be two Nielsen equivalent generating pairs of a group G. Then the commutator $[u, v]$ is conjugate either to $[u', v']$ or to $[u', v']^{-1}$.*

The proof of this lemma is a straightforward calculation of commutators of the pairs obtained from (u, v) by the elementary Nielsen moves.

In order to show that two elements are not conjugate in G_3, the Gupta-Sidki 3-group, sometimes we use the finite quotients $G_3/St_{G_3}(n)$ by the n-th level stabilizers. Consider a natural epimorphism

$$\pi\colon G_3 \to G_3/St_{G_3}(4).$$

The finite quotient $G_3/St_{G_3}(4)$ can be seen as a subgroup of $Sym(81)$ with

$$\pi(x) = (1, 28, 55)(2, 29, 56) \ldots (27, 54, 81)$$

and

$$\pi(y) = (1, 10, 19) \ldots (9, 18, 27)(28, 46, 37) \ldots (36, 54, 45)(55, 58, 61)$$
$$\cdot(56, 59, 62)(57, 60, 63)(64, 70, 67)(65, 71, 68)(66, 72, 69)(73, 74, 75)(76, 78, 77).$$

Recall that two elements are conjugate in the symmetric group if and only if their cycle types are the same. Therefore if for two elements $g, h \in G_3$ their images $\pi(g)$ and $\pi(h)$ have different cycle types in $Sym(81)$, then, in particular, they are not conjugate in G_3. Below all computations in $Sym(81)$ were done using GAP.

Example 4.2. The elements $yx^{-1}y^{-1}xy$ and y are not conjugate in G_3. Indeed,

$$\begin{aligned}\pi(yx^{-1}y^{-1}xy) &= (1,22,10,3,24,12,2,23,11)(4,25,13,5,26,14,6,27,15)\\ &\cdot(7,19,16)(8,20,17)(9,21,18)(55,64,79)(56,65,80)(57,66,81)\\ &\cdot(58,67,74,60,69,73,59,68,75)(61,70,78,62,71,76,63,72,77),\end{aligned}$$

and its cycle type differs from the one of $\pi(y)$.

Let G_3 be the Gupta-Sidki 3-group. Set $z_1 = [x, y] \in [G_3, G_3]$ and for all $n > 1$ set $z_n = (1, 1, z_{n-1})$. The fact that $z_n \in G_3$ follows from [GS84].

Proposition 4.3. *The elements $[x, yz_k]$, $[x, yz_j]^{\pm 1}$ and $z_1^{\pm 1}$ are not pairwise conjugate in G_3 for any $k, j > 2$ such that $k \neq j$.*

Proof. We prove the following two claims in order to conclude the proposition:

Claim 1. $[x, yz_n]$ is not conjugate to $z_1^{\pm 1}$ for any $n > 2$.

Claim 2. $[x, yz_k]$ and $[x, yz_j]^{\pm 1}$ are not conjugate for $k, j > 2$ and $k \neq j$.

The claims will be proved by contradiction. We compute that $z_1 = [x, y] = (y^{-1}x, x, xy)$ and $[x, yz_n] = (z_{n-1}^{-1}y^{-1}x, x, xyz_{n-1})$.

Proof of Claim 1. Assume that $[x, yz_n]$ and $z_1^{\pm 1}$ are conjugate; then there exists $g = x^i(g_1, g_2, g_3) \in G_3$ for some integer $i \in [0, 2]$, such that $[x, yz_n] = g^{-1}z_1^{\pm 1}g = (g_1^{-1}, g_2^{-1}, g_3^{-1})x^{-i}(y^{-1}x, x, xy)^{\pm 1}x^i(g_1, g_2, g_3)$. Observe that $i = 0$ because x is not conjugate to $(y^{-1}x)^{\pm 1}$ nor to $(xy)^{\pm 1}$. Moreover x is not conjugate to x^{-1} in G_3 therefore $[x, yz_n]$ can be conjugate only to z_1. We will prove that it is not the case. *For this it is enough to show that xyz_{n-1} and xy are not conjugate in G_3.* We will show it by induction assuming that

(*) yz_n and y are not conjugate in G_3 for any $n \geqslant 1$

and then will show that (*) is indeed the case.

Suppose that xyz_{n-1} and xy are conjugate in G; then there exists $g = x^i(g_1, g_2, g_3)$ for some integer $i \in [0, 2]$ such that

$$xyz_{n-1} = x(x, x^{-1}, yz_{n-2}) = (g_1, g_2, g_3)^{-1}x^{-i}(xy)x^i(g_1, g_2, g_3).$$

- If $i = 0$, then $(x, x^{-1}, yz_{n-2}) = (g_3^{-1}, g_1^{-1}, g_2^{-1})(x, x^{-1}, y)(g_1, g_2, g_3)$ and it follows that $g_2yz_{n-2}g_2^{-1} = y$.
- If $i = 1$, then $(x, x^{-1}, yz_{n-2}) = (g_3^{-1}, g_1^{-1}, g_2^{-1})(y, x, x^{-1})(g_1, g_2, g_3)$ and it follows that $xg_2yz_{n-2}g_2^{-1}x^{-1} = y$.
- If $i = 2$, then $(x, x^{-1}, yz_{n-2}) = (g_3^{-1}, g_1^{-1}, g_2^{-1})(x^{-1}, y, x)(g_1, g_2, g_3)$ and it follows that $g_2yz_{n-2}g_2^{-1} = y$.

By Assumption (*) elements yz_{n-2} and y are not conjugate in G_3 and we deduce that xyz_{n-1} is not conjugate to xy in G_3 modulo assumption (*). □

Proof of Assumption (*). yz_n and y are not conjugate in G_3 for any $n \geqslant 1$.

(1) The Assumption holds for $n = 1$. To see this, look at the action of yz_1 and y on the 4-th level of the tree; see Example 4.2.

(2) Suppose (*) is true for $n-1$.
(3) Consider $yz_n = (x, x^{-1}, yz_{n-1})$ and suppose it is conjugate to $y = (x, x^{-1}, y)$. Then there exists $g = x^i(g_1, g_2, g_3) \in G_3$ with $0 \leqslant i \leqslant 2$ such that $(g_1, g_2, g_3)^{-1}x^{-i}(x, x^{-1}, y)x^i(g_1, g_2, g_3) = (x, x^{-1}, yz_{n-1})$. Since x is not conjugate to x^{-1} nor to y in G_3, then $i = 0$. Therefore
$$(g_1^{-1}xg_1, g_2^{-1}x^{-1}g_2, g_3^{-1}yg_3) = (x, x^{-1}, yz_{n-1}).$$
We obtain the contradiction with the step of induction.

Proof of Claim 2. We will prove Claim 2 modulo Assumption (**) and (***) below and then in the end prove that both assumptions indeed hold.

Assumption (**). *For any $k, j \geqslant 1$ such that $k \neq j$ the elements yz_k and yz_j are not conjugate in G_3.*

Assumption (***). *For any $n \geqslant 2$ the element x is not conjugate to xyz_n or $z_n^{-1}y^{-1}x$ in G_3.*

We prove Claim 2 by contradiction. Suppose that there exists $g = x^i(g_1, g_2, g_3) \in G_3$ such that
$$[x, yz_k] = g^{-1}[x, yz_j]^{\pm 1}g$$
or equivalently
$$\begin{aligned}&(z_{k-1}^{-1}y^{-1}x, x, xyz_{k-1})\\ &= (g_1^{-1}, g_2^{-1}, g_3^{-1})x^{-i}(z_{j-1}^{-1}y^{-1}x, x, xyz_{j-1})^{\pm 1}x^i(g_1, g_2, g_3).\end{aligned} \tag{2}$$
Observe that x is not conjugate to x^{-1}, $z_{j-1}^{-1}y^{-1}x^{-1}$ and $x^{-1}yz_{j-1}$. To see this, look at the quotient $G_3/St_{G_3}(1) \cong \mathbb{Z}/3\mathbb{Z}$ and notice that the images of x and x^{-1} are not conjugate in $\mathbb{Z}/3\mathbb{Z}$. Therefore $[x, yz_k]$ cannot be conjugate to $[x, yz_j]^{-1}$. Moreover, it follows from Assumption (***) that $i = 0$ in equation (2).

To obtain the contradiction it is sufficient to show that xyz_{k-1} is not conjugate to xyz_{j-1}. Suppose they are conjugate; then there exists $g = x^i(g_1, g_2, g_3) \in G_3$ with $0 \leqslant i \leqslant 2$ such that
$$x(x, x^{-1}yz_{k-2}) = (g_1^{-1}, g_2^{-1}, g_3^{-1})x^{-i}x(x, x^{-1}, yz_{j-2})x^i(g_1, g_2, g_3).$$

- If $i = 0$, then $(x, x^{-1}, yz_{k-2}) = (g_3^{-1}xg_1, g_1^{-1}x^{-1}g_2, g_2^{-1}yz_{j-2}g_3)$ and it follows that $yz_{k-2} = g_2^{-1}yz_{j-2}g_2$.
- If $i = 1$, then $(x, x^{-1}, yz_{k-2}) = (g_3^{-1}yz_{j-2}g_1, g_1^{-1}xg_2, g_2^{-1}x^{-1}g_3)$ and it follows that $yz_{k-2} = g_2^{-1}x^{-1}yz_{j-2}xg_2$.
- If $i = 2$, then $(x, x^{-1}, yz_{k-2}) = (g_3^{-1}x^{-1}g_1, g_1^{-1}yz_{j-2}g_2, g_2^{-1}xg_3)$ and it follows that $yz_{k-2} = g_2^{-1}yz_{j-2}g_2$.

By Assumption (**), elements yz_{k-2} and yz_{j-2} are not conjugate in G_3 and we deduce that xyz_{k-1} and xyz_{j-1} are not conjugate in G_3 modulo Assumptions (**) and (***).

Proof of Assumption (**). Without loss of generality suppose that $j > k$. Suppose $yz_k = (x, x^{-1}yz_{k-1})$ and $yz_j = (x, x^{-1}, yz_{j-1})$ are conjugate. Then there exists $g = x^i(g_1, g_2, g_3) \in G_3$ with $0 \leqslant i \leqslant 2$ such that
$$(x, x^{-1}, yz_{k-1}) = (g_1, g_2, g_3)^{-1}x^{-i}(x, x^{-1}, yz_{j-1})x^i(g_1, g_2, g_3).$$
Since x is not conjugate to x^{-1} or to yz_{j-1} we conclude that $i = 0$ and hence yz_{k-1} and yz_{j-1} are conjugate. Continuing in the same way, we deduce that the elements

$yz_1 = (xy^{-1}x, 1, yxy)$ and $yz_{j-k+1} = (x, x^{-1}, yz_{j-k})$ are conjugate. We obtain a contradiction since x is not conjugate to $xy^{-1}x$ or to yxy (to see this it is enough to look at the action of these elements on the 4-th level of the tree) or to 1.

Proof of Assumption (***). To see that x is not conjugate to xyz_2 or $z_2^{-1}y^{-1}x$, it is enough to look at the action of these elements on the third level of the tree and to see that they have different cycle types, hence they are not conjugate in the quotient $G_3/St_{G_3}(3)$. And for $n \geqslant 3$, the action of z_n on the third level is trivial therefore it is enough to look at the action of x, xy and $y^{-1}x$ on the third level to see that they have different cycle types and therefore are not conjugate in $G_3/St_{G_3}(3)$. □

Let G_p be the Gupta-Sidki p-group for $p \geqslant 5$ prime. Set $z_1 = [x, y]$ and for $n > 1$ set $z_n = (1, \ldots, 1, z_{n-1})$. The fact that $z_n \in G_p$ follows from [GS84].

Proposition 4.4. *For any $k, j > 2$ and $k \neq j$ the elements $[x, yz_k]$ and $[x, yz_j]^{\pm 1}$ are not conjugate in G_p.*

Proof. By contradiction, suppose that there exists an element

$$g = x^i(g_1, \ldots, g_p) \in G_p$$

with $0 \leqslant i \leqslant p-1$ such that

$$[x, yz_k] = g^{-1}[x, yz_j]^{\pm 1}g$$

or, in other words,

$$\begin{aligned}&(z_{k-1}^{-1}y^{-1}x, x^{p-2}, x, 1, \ldots, 1, yz_{k-1})\\ &= (g_1^{-1}, \ldots, g_p^{-1})x^{-i}(z_{j-1}^{-1}y^{-1}x, x^{p-2}, x, 1, \ldots, 1, yz_{j-1})^{\pm 1}x^i(g_1, \ldots, g_p).\end{aligned} \tag{3}$$

Suppose $i \neq 0$. Observe that x is not conjugate to 1, x^{-1}, $x^{-1}yz_{j-1}$, x^{p-2}, x^2, and to $(yz_{j-1})^{\pm 1}$. To see this, look at the quotient $G_p/St_{G_p}(1) \cong \mathbb{Z}/pZ$, and notice that the image of x is not conjugate to the images of the elements above. Therefore x must be conjugate to $z_{j-1}^{-1}y^{-1}x$. In other words there exists $h = x^m(h_1, \ldots, h_p) \in G_p$ with $0 \leqslant m \leqslant p-1$ such that

$$x = (h_1, \ldots, h_p)^{-1}x^{-m} \cdot (a_1, \ldots, a_p)x \cdot x^m(h_1, \ldots, h_p),$$

where $a_1 = x^{-1}$, $a_2 = x$, $a_p = z_{j-2}^{-1}y^{-1}$ and $a_k = 1$ otherwise.

It follows that the following system of equations holds:

$$\begin{cases} h_p^{-1}a_{\pi^{m+1}(1)}h_1 & = 1,\\ h_1^{-1}a_{\pi^{m+1}(2)}h_2 & = 1,\\ \ldots & \\ h_{p-1}^{-1}a_{\pi^{m+1}(p)}h_p & = 1,\end{cases}$$

where π^{m+1} is the m-th power of the permutation $(1, 2, \ldots, p)$ and, for each $1 \leqslant r \leqslant p$, $\pi^{m+1}(r)$ denotes the image of r under π^{m+1}.

After solving the system one obtains that

$$h_p^{-1}a_{\pi^{m+1}(1)}a_{\pi^{m+1}(2)} \ldots a_{\pi^{m+1}(p)}h_p = 1,$$

which gives us a contradiction to $i \neq 0$.

In view of equation (3) and that $i = 0$, in order to obtain a contradiction to the initial assumption that $[x, yz_k]$ is conjugate to $[x, yz_j]^{\pm 1}$, it is enough to show that yz_{k-1} is not conjugate to yz_{j-1}. Without loss of generality suppose that $k > j$.

Suppose by contradiction that yz_{k-1} is conjugate to yz_{j-1}, i.e., there exists $h = (h_1, \dots, h_p)x^l \in G_p$ with $0 \leqslant l \leqslant p-1$ such that

$$\begin{aligned}&(x, x^{-1}, 1, \dots, 1, yz_{k-2})\\ &= (h_1, \dots, h_p)^{-1}x^{-l}(x, x^{-1}, 1, \dots, 1, yz_{j-2})x^l(h_1, \dots, h_p).\end{aligned}$$

Observe that x is not conjugate to 1, x^{-1} and yz_{j-2}. Hence $l = 0$ and therefore yz_{k-2} is conjugate to yz_{j-2}. We repeat the same arguments $j-2$ times to conclude that $yz_{k-j+1} = (x, x^{-1}, 1, \dots, 1, yz_{k-j})$ and $yz_1 = (xy^{-1}x, x^{p-3}, x, 1, \dots, 1, y^2)$ are conjugate. Observe that x^{-1} is not conjugate to 1, $xy^{-1}x$, x^{p-3}, x and y^2. The contradiction then follows and we deduce that yz_{k-1} is not conjugate to yz_{j-1} which concludes the proof. $\square$

We are now able to deduce that there are infinitely many Nielsen equivalence classes on generating pairs of the Gupta-Sidki p-group for any $p \geqslant 3$ prime.

Proof of Theorem 1.1. Fix $p \geqslant 3$ prime. Let $z_1 = [x, y] \in [G_p, G_p]$ and for all $n > 1$ let $z_n = (1, \dots, 1, z_{n-1}) \in G_p$. It follows from Theorem 4.1.1 in [GS84] that $z_n \in [G_p, G_p]$. Since $[G_p, G_p] = \Phi(G_p)$ and $\langle x, y\rangle = G_p$, then by Lemma 3.1 we deduce that $\langle x, yz_n\rangle = G_p$. We conclude by Lemma 4.1, Proposition 4.3 and Proposition 4.4 that there are infinitely many orbits of the action $\operatorname{Aut} F_2 \circlearrowright \operatorname{Epi}(F_2, G_p)$. $\square$

Proof of Proposition 1.2. First, we show that the graph $N_2(G_3/St_{G_3}(4))$ is not connected. Consider two pairs $(u, v) = (x, y)$ and $(u', v') = (x^{-1}y^{-1}xy \cdot x, y)$ in G_3. Since $\langle x, y\rangle = G_3$ and $[G_3, G_3] = \Phi(G_3)$, it follows that (u', v') is also a generating pair of G_3 by Lemma 3.1.

Denote the images of (u, v) and (u', v') in the finite quotient $G_3/St_{G_3}(4)$ by $(\overline{u}, \overline{v})$ and $(\overline{u'}, \overline{v'})$. Clearly the pairs $(\overline{u}, \overline{v})$ and $(\overline{u'}, \overline{v'})$ are generating. If they are Nielsen equivalent, then by Nielsen criterion (Lemma 4.1) their commutators $[\overline{u}, \overline{v}]$ and $[\overline{u'}, \overline{v'}]^{\pm 1}$ must be conjugate in $Sym(81)$ and, in particular, their cycle types must be the same. We will obtain the contradiction with the latter.

We calculate the commutators respectively:

$[\overline{u}, \overline{v}] = (1,16,19,3,18,21,2,17,20)(4,10,22,5,11,23,6,12,24)(7,13,25)(8,14,26)(9,15,27)(28,37,46)(29,38,47)(30,39,48)(31,40,49)(32,41,50)(33,42,51)(34,43,52)(35,44,53)(36,45,54)(55,70,79)(56,71,80)(57,72,81)(58,64,74,59,65,75,60,66,73)(61,67,78,63,69,77,62,68,76)$,

$[\overline{u'}, \overline{v'}] = (1,10,25,2,11,26,3,12,27)(4,15,21)(5,13,19)(6,14,20)(7,17,23,9,16,22,8,18,24)(28,41,53,29,42,54,30,40,52)(31,45,47,32,43,48,33,44,46)(34,37,50,35,38,51,36,39,49)(55,70,79)(56,71,80)(57,72,81)(58,64,74,59,65,75,60,66,73)(61,67,78,63,69,77,62,68,76)$.

The cycle types of $[\overline{u}, \overline{v}]$ and $[\overline{u'}, \overline{v'}]^{\pm 1}$ are different therefore $(\overline{u}, \overline{v})$ and $(\overline{u'}, \overline{v'})$ are not Nielsen equivalent.

For any $l \geqslant 4$, there exists an epimorphism from $G_3/St_{G_3}(l)$ to $G_3/St_{G_3}(4)$. We will show that the Nielsen graph $N_2(G_3/St_{G_3}(l))$ is not connected using the Gaschütz lemma [Gas55]. The Gaschütz lemma asserts that if there exists an epimorphism between finite groups $f\colon G \to H$ and $m \geqslant \operatorname{rank}(G)$, then for any generating m-tuple $(h_1, \dots, h_m)$ of H there exists a generating m-tuple $(g_1, \dots, g_m)$ of G with $f(g_i) = h_i$ for $i = 1, \dots, m$. Hence the generating pairs $(\overline{u}, \overline{v})$ and $(\overline{u'}, \overline{v'})$ of $G_3/St_{G_3}(4)$ have preimages, generating pairs in $G_3/St_{G_3}(l)$, which are not Nielsen equivalent. The proof is completed. $\square$

Acknowledgments

The author would like to thank Pierre de la Harpe, Tatiana Nagnibeda and Said Sidki for stimulating discussions on this work, and Laurent Bartholdi for valuable suggestions during the conference "Growth in Groups" in Le Louverain.

References

[AC65] J. J. Andrews and M. L. Curtis, *Free groups and handlebodies*, Proc. Amer. Math. Soc. **16** (1965), 192–195, DOI 10.2307/2033843. MR0173241

[AKT13] Theofanis Alexoudas, Benjamin Klopsch, and Anitha Thillaisundaram, *Maximal subgroups of multi-edge spinal groups*, Groups Geom. Dyn. **10** (2016), no. 2, 619–648, DOI 10.4171/GGD/359. MR3513111

[DG99] Persi Diaconis and Ronald Graham, *The graph of generating sets of an abelian group*, Colloq. Math. **80** (1999), no. 1, 31–38. MR1684568

[Dun63] M. J. Dunwoody, *On T-systems of groups*, J. Austral. Math. Soc. **3** (1963), 172–179. MR0153745

[Eva93] Martin J. Evans, *Presentations of groups involving more generators than are necessary*, Proc. London Math. Soc. (3) **67** (1993), no. 1, 106–126, DOI 10.1112/plms/s3-67.1.106. MR1218122

[Eva07] Martin J. Evans, *Nielsen equivalence classes and stability graphs of finitely generated groups*, Ischia group theory 2006, World Sci. Publ., Hackensack, NJ, 2007, pp. 103–119. MR2405933 (2009g:20062)

[Gas55] Wolfgang Gaschütz, *Zu einem von B. H. und H. Neumann gestellten Problem* (German), Math. Nachr. **14** (1955), 249–252 (1956), DOI 10.1002/mana.19550140406. MR0083993

[Gri80] R. I. Grigorčuk, *On Burnside's problem on periodic groups* (Russian), Funktsional. Anal. i Prilozhen. **14** (1980), no. 1, 53–54. MR565099

[GS83a] Narain Gupta and Saïd Sidki, *On the Burnside problem for periodic groups*, Math. Z. **182** (1983), no. 3, 385–388, DOI 10.1007/BF01179757. MR696534

[GS83b] N. Gupta and Said Sidki, *Some infinite p-groups* (English, with Russian summary), Algebra i Logika **22** (1983), no. 5, 584–589. MR759409

[GS84] N. Gupta and Said Sidki, *Extension of groups by tree automorphisms*, Contributions to group theory, Contemp. Math., vol. 33, Amer. Math. Soc., Providence, RI, 1984, pp. 232–246. MR767112 (86e:20047)

[Lub11] Alexander Lubotzky, *Dynamics of* $\mathrm{Aut}(F_N)$ *actions on group presentations and representations*, Geometry, rigidity, and group actions, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 2011, pp. 609–643. MR2807845

[Myr15] A. Myropolska, *The class $\mathcal{MN}$ of groups in which all maximal subgroups are normal*, Preprint arXiv:1509.08090.

[Myr16] Aglaia Myropolska, *Andrews-Curtis and Nielsen equivalence relations on some infinite groups*, J. Group Theory **19** (2016), no. 1, 161–178, DOI 10.1515/jgth-2015-0031. MR3441132

[Nek05] Volodymyr Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs, vol. 117, American Mathematical Society, Providence, RI, 2005. MR2162164

[Nie18] J. Nielsen, *Über die Isomorphismen unendlicher Gruppen ohne Relation* (German), Math. Ann. **79** (1918), no. 3, 269–272, DOI 10.1007/BF01458209. MR1511927

[NN51] Bernhard H. Neumann and Hanna Neumann, *Zwei Klassen charakteristischer Untergruppen und ihre Faktorgruppen* (German), Math. Nachr. **4** (1951), 106–125, DOI 10.1002/mana.19500040112. MR0040297

[Oan11] Daniel Oancea, *A note on Nielsen equivalence in finitely generated abelian groups*, Bull. Aust. Math. Soc. **84** (2011), no. 1, 127–136, DOI 10.1017/S0004972711002279. MR2817667

[Pak01] Igor Pak, *What do we know about the product replacement algorithm?*, Groups and computation, III (Columbus, OH, 1999), Ohio State Univ. Math. Res. Inst. Publ., vol. 8, de Gruyter, Berlin, 2001, pp. 301–347. MR1829489

[Per00] E. L. Pervova, *Everywhere dense subgroups of a group of tree automorphisms* (Russian, with Russian summary), Tr. Mat. Inst. Steklova **231** (2000), no. Din. Sist., Avtom. i Beskon. Gruppy, 356–367; English transl., Proc. Steklov Inst. Math. **4(231)** (2000), 339–350. MR1841763

[Per02] E. L. Pervova, *The congruence property of AT-groups* (Russian, with Russian summary), Algebra Logika **41** (2002), no. 5, 553–567, 634, DOI 10.1023/A:1020979720331; English transl., Algebra Logic **41** (2002), no. 5, 306–313. MR1953179

[Per05] E. L. Pervova, *Maximal subgroups of some non locally finite p-groups*, Internat. J. Algebra Comput. **15** (2005), no. 5-6, 1129–1150, DOI 10.1142/S0218196705002803. MR2197824

[Sid87] Said Sidki, *On a* 2*-generated infinite* 3*-group: the presentation problem*, J. Algebra **110** (1987), no. 1, 13–23, DOI 10.1016/0021-8693(87)90034-2. MR904179

Financial Engineering, École Polytechnique Fédérale de Lausanne, CH 1015 Lausanne, Switzerland