

ON SEQUENTIAL ANALYTIC GROUPS

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ABSTRACT. We answer a question of S. Todorčević and C. Uzcátegui from their 2005 work by showing that the only possible sequential orders of sequential analytic groups are 1 and ω_1 . Other results on the structure of sequential analytic spaces and their relation to other classes of spaces are given as well. In particular, we provide a full topological classification of sequential analytic groups by showing that all such groups are either metrizable or k_ω -spaces, which, together with a result by Zelenyuk, implies that there are exactly ω_1 non-homeomorphic analytic sequential group topologies.

1. INTRODUCTION

Spaces with *definable* topologies are ubiquitous in mathematics. They often appear as examples when the topology construction does not use the axiom of choice and frequently show up inside function spaces (see [17] for references and further motivation). To make the notion of ‘definable’ more precise recall that a family of subsets of some countable set X viewed as a subset of 2^X in the natural product topology is called *analytic* (see [7]) if it is a continuous image of the irrationals \mathbb{N}^ω .

A variety of reasons to study *analytic spaces*, i.e., countable topological spaces whose topology is analytic is given in [17], [18], and [16]. The authors of [18] coined the term *effective topology* for the research involving such spaces and presented a number of questions whose answers depend on various set-theoretic assumptions in the realm of general topological spaces (such as the *Malykhin problem*; see [5]) that have effective counterparts that can be resolved in ZFC alone.

Recall that a space X is called *sequential* if whenever $\overline{A} \setminus A \neq \emptyset$ for some $A \subseteq X$ there exists a convergent sequence $S \subseteq A$ such that $S \rightarrow x \in \overline{A} \setminus A$. If X is sequential one naturally defines the *sequential closure* of a subset A of X as the set $[A]'$ of all the limits of all the convergent sequences in A . Recursively putting $[A]_0 = A$, $[A]_{\alpha+1} = [[A]_\alpha]'$, and taking unions at the limit stages, one arrives at the concept of an iterated sequential closure. It is well known that for any $A \subseteq X$ where X is sequential, there exists an $\alpha \leq \omega_1$ such that $[A]_\alpha = \overline{A}$. This observation naturally leads to the definition of the *sequential order* $\text{so}(X)$ of a sequential space X as the smallest ordinal $\gamma \leq \omega_1$ such that $\overline{A} = [A]_\gamma$ for every $A \subseteq X$. *Fréchet* spaces are defined as sequential spaces of sequential order ≤ 1 .

In [17] S. Todorčević and C. Uzcátegui show among other results that a countable topological group is metrizable if and only if it is analytic and Fréchet thus solving

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the effective version of Malykhin's question on the existence of a non-metrizable countable Fréchet group (the non-effective version of this question was answered in [5]). In the same paper they pose a question about the sequential orders of sequential analytic groups which can be considered an effective version of a question of Nyikos (see [10] and [13]).

In this paper we answer this question by showing that the only possible sequential orders of sequential analytic groups are 1 and ω_1 . In addition, we show that such groups have a very well-defined topological structure and their topologies are completely described by an ordinal invariant that measures the scatteredness of their compact subspaces (see below for a more precise discussion).

We assume that all topological spaces appearing below are regular and use the standard set-theoretic notation and terminology (see [8] and [7]). We proceed by defining some of the less common concepts.

A space X is called a k_ω -space if there exists a $\{K_n : n \in \omega\}$ where each K_n is a compact subset of X such that a set $U \subseteq X$ is open if and only if each $U \cap K_n$ is relatively open. The class of k_ω -spaces is stable under taking products, i.e., the product of two (or any finite number of) k_ω -spaces is again k_ω .

Countable k_ω -spaces are sequential and analytic (more precisely, their topology is $F_{\sigma\delta}$), and form a subclass of \aleph_0 -spaces (see [9]). Instead of the original definition we shall use the following characterization that describes \aleph_0 -spaces in the narrow case when X is sequential.

Lemma 1. *A sequential space X is an \aleph_0 -space if and only if there exists a countable collection $\{A_n : n \in \omega\}$ of subsets of X such that for any open $U \subseteq X$ and any converging sequence $S \subseteq U$ such that $S \rightarrow x \in U$ there is an $n \in \omega$ such that $A_n \subseteq U$ and $A_n \cap S$ is infinite.*

Sequential \aleph_0 -spaces are exactly the quotient images of separable metric spaces ([9]). Even in the case of a countable X , not every \aleph_0 -space is a k_ω -space, however, when X is a countable sequential non-Fréchet topological group, a corollary of a more general result in [1] implies that X is a k_ω -space if and only if X is an \aleph_0 -space. A result in [15] shows that for each such group $\text{so}(X) = \omega_1$. Perhaps the most surprising property of the class of all k_ω countable group topologies is that there are exactly ω_1 of them, moreover, the topological type of such group is uniquely described by the supremum of *Cantor-Bendixson ranks* of its compact subspaces (see [19] and [6]).

Recall that a collection of open subsets of a topological space is called a π -base if every open subset of the space contains a member of the collection. Furthermore, a collection of open subsets is called a *local π -base at $x \in X$* if every neighborhood of x contains a set in the collection. It is an easy observation that a collection of open subsets of X that is a local π -base at every point in some dense subset of X is a π -base of X . The following lemma is well known (the second statement is the famous Birkhoff-Kakutani metrization theorem).

Lemma 2. *Every topological group with a countable local π -base at any point is first countable and every first countable topological group is metrizable.*

The *countable sequential fan* $S(\omega)$ is defined as the set $\omega^2 \cup \{\omega\}$ equipped with the topology in which every $(n, i) \in \omega^2$ is isolated and the basic neighborhoods of ω are $U_f = \{(n, i) : i \geq f(n)\}$ where $f : \omega \rightarrow \omega$.

Definition 1. Let X be a topological space. Let $x \in X$ and $\langle D_n : n \in \omega \rangle$ be a collection of infinite countable closed discrete subsets of X such that for every open $U \subseteq X$ such that $x \in U$ there exists an $n \in \omega$ such that $U \cap D_n$ is infinite. Then $Y = \bigcup \{ D_n : n \in \omega \} \cup \{x\}$ is called a *wvD subspace* of X .

The utility of the previous definition is illustrated by the following lemma. A similar phenomenon was first noted by E. van Douwen in [3] (incidentally, wvD stands for *weak van Douwen space*).

Lemma 3. *If X has a wvD subspace, $X \times S(\omega)$ is not sequential.*

Proof. Let $Y = \bigcup \{ D_n : n \in \omega \} \cup \{x\}$ be a wvD subset of X where $x \in X$ and $D_n = \{ d_i^n : i \in \omega \}$ be as in Definition 1. Define

$$A = \bigcup \{ \bigcup \{ (d_i^n, (n, i)) : i \in \omega \} : n \in \omega \} \subseteq X \times S(\omega).$$

Now $(x, \omega) \in \overline{A} \setminus A$ but there is no infinite $S \subseteq A$ such that $S \rightarrow y$ for some $y \in X$. Indeed, otherwise the projection $\pi_2(S) \subseteq S(\omega)$ contains an infinite ‘diagonal’ convergent subsequence in $S(\omega)$ or one of the closed discrete subspaces $\{ (d_i^n, (n, i)) : i \in \omega \}$ contains an infinite convergent subsequence. \square

The following lemma is a corollary of Lemma 16 and Corollary 2 of [13].

Lemma 4. *Let τ be a sequential group topology on \mathbb{N} such that $\text{so}(\tau) < \omega_1$. If $\{ N_i : i \in \omega \}$ is a collection of nowhere dense subsets of \mathbb{N} there exists an $S \subseteq \mathbb{N}$ such that $S \rightarrow x$ for some $x \in \mathbb{N}$ and $S \cap N_i$ is finite for all $i \in \omega$.*

2. ANALYTIC AND OTHER CLASSES OF SPACES

In the arguments below, we shall assume that τ stands for some analytic topology on a countable set X . To simplify the notation we will assume that $X = \mathbb{N}$ whenever it is convenient. All the references to topological operations and properties such as convergence, etc., are relative to this topology.

We shall also fix a subtree T of $[\mathbb{N}]^{<\omega} \otimes [\mathbb{N}]^{<\omega}$ (see [16] for the definition of the tree order) that defines τ , i.e., such that $U = \pi_1(f)$ for some branch f of T whenever $U \in \tau$ is infinite. Given a $P \subseteq 2^{\mathbb{N}}$ and a $\sigma \in T$ we will use the notation

$$\sigma \uparrow P = \bigcap \{ \pi_1(f) : f \text{ is a branch of } T \text{ that extends } \sigma \text{ such that } \pi_1(f) \in P \}$$

if such f exist; otherwise we put $\sigma \uparrow P = \emptyset$.

Lemma 5. *Let $P \subseteq 2^{\mathbb{N}}$ and $S \rightarrow x$ for some infinite $S \subseteq \mathbb{N}$. Suppose there is an open $U \ni x$ such that $U \in P$. Then there exists a $\sigma \in T$ such that $S \subseteq^* \sigma \uparrow P$.*

Proof. Let f be a branch of T such that $\pi_1(f) = U$. Pick $\sigma_{-1} \in T$ such that f extends σ_{-1} and $x \in \pi_1(\sigma_{-1})$. Suppose no $\sigma \in T$ with the property stated in the lemma exists. Using this one can inductively construct a sequence $\langle \sigma_i : i \in \omega \rangle$ of elements of T and a strictly increasing sequence $\langle n_i : i \in \omega \rangle \subseteq S$ such that for every $i \in \omega$

- (1) σ_{i+1} extends σ_i ;
- (2) $n_i < \max \pi_1(\sigma_i)$ and $n_i \notin \pi_1(\sigma_i)$;
- (3) there exists a branch f_i of T that extends σ_i such that $\pi_1(f_i) \in P$.

Put $f_{-1} = f$. Let $i \in \omega \cup \{-1\}$ and note that (3) holds for $i = -1$. By the assumption and (3) $S \not\subseteq^* \sigma_i \upharpoonright P \neq \emptyset$ so one can pick a branch f_{i+1} of T that extends σ_i such that $n_{i+1} \notin \pi_1(f_i)$ for some $n_{i+1} \in S$, $n_{i+1} > n_i$. Let σ_{i+1} be such that f_{i+1} extends σ_{i+1} and $\max \sigma_{i+1} > \max \sigma_i$, $\max \sigma_{i+1} > n_{i+1}$.

Put $f' = \bigcup \langle \sigma_i : i \in \omega \rangle$. Then $S \not\subseteq^* \pi_1(f') \ni x$, a contradiction. \square

As usual, a set function $F : 2^X \rightarrow 2^X$ will be called *monotone* if $F(A) \subseteq F(B)$ whenever $A \subseteq B$ and $F(\emptyset) = \emptyset$.

Lemma 6. *Let $F : 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ be a monotone set function and let $\{Q_\alpha : \alpha \in \omega_1\}$ be such that each $Q_\alpha \subseteq 2^{\mathbb{N}}$ and $Q_\beta \subseteq Q_\alpha$ when $\beta \leq \alpha$. Define*

$$P_\alpha = \{B \subseteq \mathbb{N} : q \setminus F(B) \neq \emptyset \text{ for every } q \in Q_\alpha\}.$$

Then there exists a $\gamma \in \omega_1$ such that $q \not\subseteq F(\sigma \upharpoonright P_{\gamma'})$ for any $q \in \bigcup \{Q_\alpha : \alpha \in \omega_1\}$, any $\gamma' \geq \gamma$, and any $\sigma \in T$.

Proof. Since T is countable, it is enough to show that $\sigma \upharpoonright P_\alpha = \emptyset$ whenever there is a $q \in Q_\alpha$ such that $q \subseteq F(\sigma \upharpoonright P_\beta)$ for some $\beta < \omega_1$. Assuming such a q exists, suppose $\sigma \upharpoonright P_\alpha \neq \emptyset$. Then $\alpha > \beta$, otherwise $q \in Q_\alpha \subseteq Q_\beta$ and $q \setminus F(\sigma \upharpoonright P_\beta) \neq \emptyset$, since each P_β is closed under taking subsets, which contradicts $q \subseteq F(\sigma \upharpoonright P_\beta)$.

There exists a branch f of T that extends σ such that $\pi_1(f) \in P_\alpha \subseteq P_\beta$ so $q \setminus F(\pi_1(f)) \neq \emptyset$. Now $F(\pi_1(f)) \supseteq F(\sigma \upharpoonright P_\beta \cap \pi_1(f)) = F(\sigma \upharpoonright P_\beta) \supseteq q$, a contradiction. \square

In all applications of Lemma 6 below, the families P_α and Q_α are constructed recursively by starting with $Q_0 = \emptyset$ and $P_0 = 2^{\mathbb{N}}$, defining P_α as in the statement of the lemma above and defining Q_α in terms of P_β for $\beta < \alpha$.

Lemma 7. *Let τ be an analytic sequential topology on \mathbb{N} . Then there exists a countable family \mathcal{U} of open in τ sets and a countable family Ξ of nowhere dense subsets of \mathbb{N} such that at least one of the following alternatives holds for every $x \in \mathbb{N}$:*

- (1) *for any infinite sequence $S \subseteq \mathbb{N}$ such that $S \rightarrow x$ there is a $\xi \in \Xi$ such that $S \subseteq^* \xi$;*
- (2) *\mathcal{U} is a local π -base at x .*

Proof. Put $F(B) = \overline{B}$ and define $Q_\alpha = \{\text{Int}(\overline{\sigma \upharpoonright P_\beta}) : \sigma \in T, \beta < \alpha\}$ where P_β is defined as in Lemma 6. Find $\gamma \in \omega_1$ as in Lemma 6. It follows from the construction of Q_α that every $\sigma \upharpoonright P_\gamma$ is nowhere dense.

Put $P = P_\gamma$, $\Xi = \{\sigma \upharpoonright P_\gamma : \sigma \in T\}$, $\mathcal{U} = Q_\gamma$, and let $S \rightarrow x$ for some $x \in \mathbb{N}$. If \mathcal{U} is not a local π -base at x there exists an open $U \ni x$ such that $q \setminus \overline{U} \neq \emptyset$ for every $q \in Q_\gamma$. Thus $U \in P_\gamma$ and Lemma 5 implies that there is a $\sigma \in T$ such that $S \subseteq^* \sigma \upharpoonright P_\gamma \in \Xi$. \square

The following is an immediate corollary of Lemma 4, Lemma 2, and Lemma 7. It answers Question 7.1 from [17]. See also the remark immediately following Theorem 2 for an alternative proof of Corollary 1 that does not use Lemma 4.

Corollary 1. *A countable topological group is metrizable if and only if it has a sequential analytic topology with the sequential order less than ω_1 .*

The following result is mentioned (with proof) in [5] where it is attributed to [2] (and appears there as part of the proof of one of the results).

Lemma 8 ([2]). *Let X be a Fréchet space without isolated points, let $x \in X$ and let \mathcal{M} be a countable family of nowhere dense subsets of X . Then there is an infinite sequence C_x converging to x with only finite intersection with every element of \mathcal{M} .*

Lemma 8 and a simple argument result in the following corollary to Lemma 7.

Corollary 2. *Every analytic Fréchet space has a countable π -base.*

Proof. Let X be an analytic Fréchet space and put $U = X \setminus \overline{P}$ where P is the set of all isolated points of X . Observe that Lemma 8 shows that the first alternative of Lemma 7 does not hold in Fréchet spaces without isolated points so U has a countable π -base \mathcal{B} . Adding all the singletons from P to \mathcal{B} one obtains a countable π -base for X . \square

A similar proof shows that the conclusion of Lemma 7 can be sharpened for homogeneous spaces.

Corollary 3. *Let X be a homogeneous analytic sequential space. Then X has either a countable π -base or a countable collection Ξ of nowhere dense subsets such that for any $x \in X$ and any infinite sequence $S \subseteq X$ such that $S \rightarrow x$ there is a $\xi \in \Xi$ such that $S \subseteq^* \xi$.*

A quick observation reveals that a disjoint union $\mathbb{Q} \cup S_\omega$ of a copy of the rationals and the Arkhangel'skii-Franklin space S_ω (see [17] for a nice definition of S_ω and further references) does not satisfy the dichotomy of Corollary 3. Therefore the restrictions in the corollaries above cannot be removed.

Lemma 9. *Let X be an analytic sequential space. Then X is a k_ω -space or there exists a wvD subspace of X .*

Proof. Put $F(B) = \overline{B}$. As usual, we put $Q_0 = \emptyset$, $P_0 = 2^\mathbb{N}$, and define Q_α and P_α recursively as $Q_\alpha = \{\overline{\sigma \upharpoonright P_\beta} : \sigma \in T, \overline{\sigma \upharpoonright P_\beta} \text{ is not compact}, \beta < \alpha\}$ and $P_\alpha = \{B \subseteq \mathbb{N} : q \setminus F(B) \neq \emptyset \text{ for every } q \in Q_\alpha\}$. Let $\gamma \in \omega_1$ be as in Lemma 5. The construction of Q_α implies that every $\overline{\sigma \upharpoonright P_\gamma}$ is compact.

Suppose X has no wvD subspace. Put $P = P_\gamma$, define a countable family $K = \{\overline{\sigma \upharpoonright P_\gamma} : \sigma \in T, \overline{\sigma \upharpoonright P_\gamma} \text{ is compact}\}$, and let $S \rightarrow x$ for some $x \in X$. For each $q \in Q_\gamma$ pick a closed infinite discrete subset $D_q \subseteq q$. Call the collection just constructed \mathcal{D} . Suppose \mathcal{D} is infinite. Since $\bigcup \mathcal{D} \cup \{x\}$ is not a wvD subspace of \mathbb{N} there exists an open $U \ni x$ such that $D_q \setminus \overline{U} \subseteq q \setminus \overline{U} \neq \emptyset$ for every $q \in Q_\gamma$. If \mathcal{D} is finite (i.e. Q_γ is finite) the union $\bigcup \mathcal{D}$ is a closed discrete subset of X which implies the existence of an open U with the properties above. Thus $U \in P_\gamma$ and it follows from Lemma 5 that there exists a $\sigma \in T$ such that $S \subseteq^* \sigma \upharpoonright P_\gamma \subseteq \overline{\sigma \upharpoonright P_\gamma} \in K$. \square

The next corollary follows from Lemma 9 and Lemma 4.

Corollary 4. *Let X be an analytic space. Then $X \times S(\omega)$ is sequential if and only if X is a k_ω -space.*

Let C be a closed copy of $S(\omega)$ in X . If X is a topological group, it is convenient to assume that $\mathbf{1}_X$ is the limit point of C and write $C = \bigcup \{C_n : n \in \omega\} \cup \{\mathbf{1}_X\}$ where $C_n = \langle c_i^n : i \in \omega \rangle \rightarrow \mathbf{1}_X$ are disjoint subsets of X that do not contain $\mathbf{1}_X$, such that each $A \subseteq \bigcup C_n$ satisfying $|A \cap C_n| < \omega$ for every $n \in \omega$ is closed in X . We will refer to this representation of C as a *natural closed copy of $S(\omega)$ in X* and will use the notation above for the sake of brevity below.

Lemma 10. *Let X be an analytic group and let $\bigcup \{C_n : n \in \omega\}$ be a natural closed copy of $S(\omega)$ in X . There exists a countable family Ξ of subsets of X with the following properties:*

- (1) *for each $p \in \Xi$ there exists an $M_p \in \omega$ such that $|p \cap a \cdot C_n| = \omega$ implies $n \leq M_p$ for any $a \in X$;*
- (2) *for each infinite $S \subseteq X$ where $S \rightarrow x$ for some $x \in X$ there exists a $p \in \Xi$ such that $S \subseteq^* p$.*

Proof. Put $F(B) = (\overline{B})^{-1}\overline{B}$, define $Q_0 = \emptyset$, $P_0 = 2^{\mathbb{N}}$ and let

$$Q_\alpha = \{F(\sigma \uparrow P_\beta) : \sigma \in T, \beta < \alpha, |\{n \in \omega : F(\sigma \uparrow P_\beta) \cap C_n \neq \emptyset\}| = \omega\}$$

for $\alpha < \omega_1$, where $P_\alpha = \{B \subseteq \mathbb{N} : q \setminus F(B) \neq \emptyset \text{ for every } q \in Q_\alpha\}$ is as in Lemma 6.

Let $\gamma < \omega_1$ be the index provided by Lemma 6 and $\Xi = \{\sigma \uparrow P_\gamma : \sigma \in T\}$. Note that for $p = \sigma \uparrow P_\gamma \in \Xi$ the set $\{n \in \omega : F(p) \cap C_n \neq \emptyset\}$ is finite. Otherwise $q = F(p) \in Q_{\gamma+1}$ contrary to the choice of γ . Pick $M_p \in \omega$ so that $F(p) \cap C_n = \emptyset$ for $n \geq M_p$. Now if $|\overline{p} \cap a \cdot C_n| = \omega$ for some $n \in \omega$ and $a \in X$, then $a \in \overline{p}$ thus $F(p) \cap C_n = (\overline{p})^{-1}\overline{p} \cap C_n \supseteq a^{-1} \cdot \overline{p} \cap C_n \neq \emptyset$ so $n \leq M_p$.

Let $S \rightarrow x \in X$. Put $P = P_\gamma$. One can construct a set $D \subseteq \bigcup \{C_n : n \in \omega\}$ by induction such that $|D \cap C_n| \leq 1$ for each $n \in \omega$ and $D \cap q \neq \emptyset$ for each $q \in Q_\gamma$. Note that D is a closed discrete subset of X and $x^{-1}x = \mathbf{1}_X \notin D$. Therefore there exists an open neighborhood U of x such that $F(U) \cap D = (\overline{U})^{-1}\overline{U} \cap D = \emptyset$ and thus $U \in P$. Now Lemma 5 implies the existence of a $\sigma \in T$ such that $S \subseteq^* \sigma \uparrow P \in \Xi$. \square

Lemma 11. *Let X be an analytic non-Fréchet group. If X contains a wvD subspace, then X is not sequential.*

Proof. Suppose X is sequential. Since X is not Fréchet, X contains a closed copy of $S(\omega)$ (see [10]) so let $\bigcup \{C_n : n \in \omega\}$ be a natural closed copy of $S(\omega)$ in X . Let $\Xi = \{p_n : n \in \omega\}$ be the family provided by Lemma 10 and put $M^n = \max \{M_i : i \leq n\} + n + 1$ where M_i have the property of Lemma 10(2).

Let $D = \bigcup \{D_n : n \in \omega\} \cup \{\mathbf{1}_X\}$ be a wvD subspace of X where $D_n = \{d_i^n : i \in \omega\}$ are disjoint closed discrete subspaces of X . Now M^n is a strictly increasing sequence and by the construction the set $a \cdot C_{M^n} \cap (\bigcup \{p_i : i \leq n\})$ is finite for every $n \in \omega$ and $a \in X$. Pick a strictly increasing sequence $\langle r_i^n : i \in \omega \rangle \subseteq \omega$ such that $e_i^n = d_i^n \cdot c_{r_i^n}^{M^n} \notin \bigcup \{p_k : k \leq n\}$ and all $e_i^n \neq \mathbf{1}_X$ are distinct. Define $E = \{e_i^n : n, i \in \omega\}$ and suppose there is an infinite $S \subseteq E$ such that $S \rightarrow x$ for some $x \in X$.

Note that for every $n \in \omega$ the set $\{e_i^n : i \in \omega\} \subseteq D_n \cdot C_{M^n}$ is closed and discrete in X so we can assume that $S = \{e_{m_i}^{n_i} : i \in \omega\}$ for some strictly increasing $\langle n_i : i \in \omega \rangle \subseteq \omega$. Now $S \subseteq^* p_n$ for some $p_n \in \Xi$ by Lemma 10(2) but $e_{m_i}^{n_i} \notin p_n$ for $n_i > n$, a contradiction. Thus E is sequentially closed.

Let V be an open neighborhood of $\mathbf{1}_X$ in X and $U \ni \mathbf{1}_X$ be an open subset of X such that $U \cdot U \subseteq V$. Pick $n \in \omega$ such that $D_n \cap U$ is infinite and choose $k \in \omega$ large enough so that $c_{r_k^n}^{M^n} \in U$ and $d_k^n \in D_n \cap U$. Now $e_k^n = d_k^n \cdot c_{r_k^n}^{M^n} \in U \cdot U \subseteq V$. Thus $\mathbf{1}_X \in \overline{E} \setminus E$, a contradiction. \square

Lemmas 11 and 9 imply the following theorem.

Theorem 1. *Let X be a countable sequential group. Then the following are equivalent:*

- (1) *the topology of X is analytic;*
- (2) *the topology of X is $F_{\sigma\delta}$;*
- (3) *X is either first countable or k_ω .*

The following theorem was proved in [15].

Theorem 2 ([15]). *Let G be a Hausdorff sequential topological group with a point-countable k -network. If $\text{so}(G) < \omega_1$ then G is metrizable.*

Using the fact that every countable k_ω is an \aleph_0 -space and therefore has a countable k -network (see [9]), Corollary 1 immediately follows from Theorem 1 and Theorem 2.

The result of Zelenyuk (see [19] or [12]) mentioned in the introduction together with Theorem 1 imply

Corollary 5. *There are exactly ω_1 non-homeomorphic analytic sequential group topologies. Moreover, if X is an infinite analytic sequential group, then all finite powers X^n are such and are homeomorphic to each other.*

3. EXAMPLES AND QUESTIONS

It has been demonstrated by a number of authors that sequential \aleph_0 -spaces have a number of properties resembling those of separable metric spaces.

On the other hand, the properties that differentiate between the two classes of spaces are strikingly similar to those that separate analytic spaces and countable metrizable ones. As an example, it is easy to show that a sequential \aleph_0 -space with a *weak diagonal sequence property* is first countable (see [9] and [18]). The following example shows that analytic sequential spaces are not necessarily \aleph_0 -spaces, thus the statement of Theorem 1 is limited to topological groups.

Example 1. Consider the following basis for a topology on $[\mathbb{N}]^{<\omega}$, viewed as a tree with the usual order of end extension. Every point $x \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}$ is isolated and the basis of neighborhoods of \emptyset consists of complements of finite unions of branches together with \emptyset . It is shown in [18], Example 5.6 (see also [17], Remark 4.8) that the resulting topology is F_σ and Fréchet with the weak diagonal sequence property but not first countable. Thus the space constructed is not an \aleph_0 -space.

A partial result going in the opposite direction is possible. The next lemma is an easy corollary of the result that each quotient image of the rationals is *determined* (see [9]) by a countable family of metrizable subspaces.

Lemma 12. *If X is a quotient image of a countable metric space (equivalently, the rationals \mathbb{Q}), then X is analytic (more precisely, X has an $F_{\sigma\delta}$ topology).*

One might hope that due to their ‘tame’ convergence structure, \aleph_0 -spaces form a subfamily of all analytic spaces. The next simple example shows that this is not the case.

Example 2. There are 2^c countable Fréchet \aleph_0 -spaces with a single non-isolated point. In particular, there are non-analytic spaces of such kind.

Proof. Let $A \subseteq \mathbb{R}$ be an arbitrary subset of the real line. Put

$$M(A) = (\{0\} \times A) \cup \{(1/n, q) : n \in \mathbb{N}, q \in \mathbb{Q}\}.$$

Define a topology on $M(A)$ by making the Euclidean neighborhoods of points $(0, a)$, $a \in A$ the new basic neighborhoods and making all other points isolated. The space just constructed is a separable metrizable one. Consider the quotient map that sends $\{0\} \times A$ into a single point ∞ and is 1-1 on the rest of $M(A)$. Its image is a Fréchet \aleph_0 -space $P(A)$ with a single non-isolated point.

Let $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ be two different subsets of the real line, and let, say, $a \in A$ be such that $a \notin B$. Given any sequence of rationals $\langle q_n : n \in \omega \rangle \subseteq \mathbb{Q}$ that converges to a the set $\{(1/n, q_n) : n \in \omega\}$ is a convergent sequence in $P(A)$ and is a closed discrete subset of $P(B)$. Hence, the topologies of $P(A)$ and $P(B)$ differ, resulting in exactly $2^{\mathfrak{c}}$ different Fréchet \aleph_0 -topologies with a single non-isolated point. Noting that there are at most \mathfrak{c} possible homeomorphisms between topologies on a given countable set, one can pick $2^{\mathfrak{c}}$ pairwise non-homeomorphic spaces $P(A)$. Given that there are at most \mathfrak{c} analytic topologies on any countable set, most $P(A)$ are non-analytic. \square

Remark 1. The rather crude construction of the example above does not produce any ‘explicit’ $A \subseteq \mathbb{R}$ such that $P(A)$ is not analytic. A more precise proof is possible that shows that A is a projection of a Borel subset of the product of the irrationals and the topology of $P(A)$ viewed as the subset of the irrationals giving one more control over the complexity of $P(A)$.

The final example shows that the property established in Lemma 7 is not enough to show that the group is a k_ω -space.

Example 3 (CH). There exists a countable sequential group G and a countable collection Ξ of nowhere dense subsets of G such that G is not a k_ω -space and for every convergent sequence $S \subseteq G$ there is a $\xi \in \Xi$ such that $S \subseteq \xi$.

Proof. The full details of the construction are somewhat tedious and are of limited interest. We therefore present just a sketch of the proof. A number of similar arguments can be found in [14].

One starts with a non-discrete first countable topology τ_0 on \mathbb{Q} (any topologizable countable group would suffice; it is easy to see that a similar construction just as readily gives an example of a topological *field* with these properties). Pick a compact subset K of \mathbb{Q} such that $0 \in K$ has Cantor-Bendixson rank ω in K . Pick a countable collection of convergent sequences in τ_0 that witness the Cantor-Bendixson rank of each point of K . Let η_0 be the finest group topology on \mathbb{Q} in which each of these sequences converges. The existence of such a topology can be established by an easy argument (see, for example, [14]). Let $\{A_\alpha : \alpha \in \omega_1\} = 2^{\mathbb{Q}}$.

The construction proceeds by induction on $\alpha \in \omega_1$ where at stage α one defines a pair of topologies $\tau_\alpha \subseteq \eta_\alpha$ such that τ_α is first countable and η_α is determined by countably many compact subsets of finite Cantor-Bendixson rank. At limit stages the construction proceeds in a natural (and trivial) way.

At stage $\alpha + 1$ one picks $\tau_{\alpha+1} \supseteq \tau_\alpha$ such that $\tau_{\alpha+1}$ contains enough open in η_α subsets to show that a given $A_\alpha \subseteq \mathbb{Q}$ is closed in $\tau_{\alpha+1}$ provided it is closed in η_α and not compact in $\tau_{\alpha+1}$ provided it is not compact in η_α . Now $\eta_{\alpha+1} \subseteq \eta_\alpha$ is chosen as the finest topology coarser than η_α in which $S \rightarrow 0$ for some $S \subseteq K$ such

that $S \rightarrow 0$ in $\tau_{\alpha+1}$ and S is discrete in η_α . Such an S can be built inductively by first finding a compact (in $\tau_{\alpha+1}$) $K' \subseteq K$ of infinite Cantor-Bendixson rank.

Define

$$\tau = \bigcup \{ \tau_\alpha : \alpha \in \omega_1 \} = \bigcap \{ \eta_\alpha : \alpha \in \omega_1 \}$$

and put

$$\Xi = \{ d + (-1)^{\delta_0} K + \cdots + (-1)^{\delta_n} K : \delta_i \in \{0, 1\}, d \in [\mathbb{Q}]^{<\omega}, n \in \omega \}.$$

It is easy to see that the choice of τ_α ensures that τ is sequential and the choice of η_α and $\tau_{\alpha+1}$ prevents τ from being k_ω . Moreover, each compact in τ subset of \mathbb{Q} is compact in some η_α and therefore resides in some ‘monomial’ over K . Thus the family Ξ has the desired property. \square

Finally, it seems natural to ask whether Lemma 11 can be generalized to non-analytic groups.

Question 1. Do there exist (countable) sequential non-Fréchet groups that contain a (closed) wvD subspace?

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