PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 145, Number 10, October 2017, Pages 4465–4471
http://dx.doi.org/10.1090/proc/13541
Article electronically published on April 12, 2017

REVERSE ISOPERIMETRIC INEQUALITY IN TWO-DIMENSIONAL ALEXANDROV SPACES

ALEXANDER A. BORISENKO

(Communicated by Lei Ni)

ABSTRACT. We prove a reverse isoperimetric inequality for domains homeomorphic to a disc with the boundary of curvature bounded below lying in two-dimensional Alexandrov spaces of curvature $\geqslant c$. We also study the equality case.

The well-known isoperimetric inequality for the Euclidean plane states that the area F and the length L of the boundary of any plane domain with a rectifiable boundary satisfy the inequality

$$L^2 - 4\pi F \geqslant 0,$$

and equality is attained only for a circle [1].

On the planes of constant curvature there is a similar theorem, and the following inequality holds [2]:

$$L^2 - 4\pi F + cF^2 \geqslant 0,$$

where c is the curvature of the plane.

In two-dimensional manifolds of bounded curvature for domains homeomorphic to a disc A. D. Alexandrov proved [3] the inequality

$$F \leqslant \frac{L^2}{2(2\pi - \omega^+)},$$

where ω^+ is a positive curvature of a domain. In the inequality above equality holds only if the domain is isometric to a lateral surface of a right circular cone with curvature $\omega^+ < 2\pi$ at the vertex.

The isoperimetric inequality for domains with a compact closure and bounded by a finite number of rectifiable curves in two-dimensional manifolds of bounded curvature was proved in [4].

But if we don't put any conditions on the boundary curves, then the areas of the enclosed domains can be arbitrarily close to zero, and the perimeters of these curves can be arbitrarily large. If we assume that the boundary curve is λ-convex, then, given the perimeter of it, the area of the domain is bounded from below by some constant. We prove the following main theorem.

Theorem. *Let G be a domain homeomorphic to a disc and lying in a 2-dimensional Alexandrov space of curvature (in the sense of Alexandrov) $\geqslant c$. If the boundary*

Received by the editors July 11, 2016 and, in revised form, September 22, 2016 and October 19, 2016.

2010 *Mathematics Subject Classification.* Primary 53C45, 52A40.

Key words and phrases. Alexandrov metric spaces, isoperimetric inequality, λ-convex curve.

curve γ of G is λ-convex and the perimeter of γ is equal to L, then the area F of the domain G satisfies:

(1) *for $c = 0$,*

$$F \geqslant \frac{L}{2\lambda} - \frac{1}{\lambda^2}\sin\frac{L\lambda}{2}; \tag{0.1}$$

(2) *for $c = k^2$,*

$$F \geqslant \frac{4}{k^2}\arctan\left(\frac{\lambda}{\sqrt{\lambda^2+k^2}}\tan\left(\frac{\sqrt{\lambda^2+k^2}}{4}L\right)\right) - \frac{L\lambda}{k^2}; \tag{0.2}$$

(3) (a) *for $c = -k^2$ and $\lambda > k$,*

$$F \geqslant \frac{L\lambda}{k^2} - \frac{4}{k^2}\arctan\left(\frac{\lambda}{\sqrt{\lambda^2-k^2}}\tan\left(\frac{\sqrt{\lambda^2-k^2}}{4}L\right)\right); \tag{0.3}$$

(b) *for $c = -k^2$ and $\lambda = k$,*

$$F \geqslant \frac{L}{k} - \frac{4}{k^2}\arctan\frac{kL}{4}; \tag{0.4}$$

(c) *for $c = -k^2$ and $\lambda < k$,*

$$F \geqslant \frac{L\lambda}{k^2} - \frac{4}{k^2}\arctan\left(\frac{\lambda}{\sqrt{k^2-\lambda^2}}\tanh\left(\frac{\sqrt{k^2-\lambda^2}}{4}L\right)\right). \tag{0.5}$$

In all inequalities (0.1) – (0.5) *equality is attained if and only if the domain G is a λ-convex lune of length L lying on the plane of constant curvature equal to c.*

By λ-convex lune we understand a convex domain bounded by two arcs of constant geodesic curvature and of length $L/2$. In particular:

1) For $c = 0$ the domain is bounded by two circular arcs of radius $1/\lambda$. In this case the perimeter satisfies $L \leqslant 2\pi/\lambda$.
2) For $c = k^2 > 0$ the domain is also bounded by two circular arcs of curvature equal to λ, and the perimeter satisfies $L \leqslant 2\pi/\sqrt{\lambda^2+k^2}$.
3) (a) For $c = -k^2 < 0$ and $\lambda > k$ the lune is also bounded by two arcs of a circle of curvature equal to λ, and the perimeter satisfies $L \leqslant 2\pi/\sqrt{\lambda^2-k^2}$;
   (b) for $c = -k^2 < 0$, $\lambda = k$ the domain is bounded by arcs of horocycles, and the perimeter of the domain can be arbitrary;
   (c) for $c = -k^2 < 0$, $0 < \lambda < k$ the domain is bounded by two arcs of equidistants, the perimeter can also be arbitrary.

For domains in two-dimensional simply-connected spaces of constant curvature equal to c the main theorem was proved when $c = 0$ in [5], when $c = k^2$ in [6], and when $c = -k^2$ in [7].

For J-holomorphic curves some variant of a reverse isoperimetric inequality was proved in [8]. In particular, it's been shown that the length of the boundary of a J-holomorphic curve with Lagrangian boundary conditions is dominated by a constant times its area.

Let R be an Alexandrov space of curvature $\geqslant c$ (see [9, p. 308] and [10, p. 103]) homeomorphic to a disc. Suppose G is a domain in R bounded by a rectifiable curve γ. Denote by $\varphi(\gamma_1)$ the *integral geodesic curvature* (the swerve) of an arc γ_1, $\gamma_1 \subset \gamma$ [9, p. 309].

A curve γ is called λ-*convex* with $\lambda > 0$ if for each sub-arc γ_1 of γ,

$$\frac{\varphi(\gamma_1)}{s(\gamma_1)} \geqslant \lambda,$$

where $s(\gamma_1)$ is the length of an arc γ_1. For regular curves in a two-dimensional Riemannian manifold this condition is equivalent to the assumption that the geodesic curvature at each point $\geqslant \lambda > 0$. In the general case such a condition allows a curve to have corner points.

Theorem A (A. D. Alexandrov, [9, p. 269])**.** *A metric space with inner metric of curvature $\geqslant c$ homeomorphic to a sphere is isometric to a closed convex surface in a simply-connected space of constant curvature equal to c.*

Theorem B (A. V. Pogorelov, [11, pp. 119-167, 267, 320-321]; Yu. Volkov [12, pp. 463-493])**.** *Closed isometric convex surfaces in the three-dimensional Euclidean and spherical spaces are equal up to a rigid motion.*

Theorem C (A. D. Milka, [13])**.** *Closed isometric convex surfaces in the three-dimensional Lobachevsky space are equal up to a rigid motion.*

Theorem D (W. Meeks and S. T. Yau, [14])**.** *Let M be a convex three-dimensional manifold in the spherical space S^3, and $\gamma \subset \partial M$ be a closed Jordan curve. Then the curve γ bounds an embedded surface which is a solution to the Plateau problem. Moreover, this surface either entirely lies on the boundary ∂M or the interior of the surface lies inside M.*

Theorem E. *Let γ be a closed embedded λ-convex curve, with $\lambda > 0$, lying in a two-dimensional model space of constant curvature equal to c. If L and F are, respectively, the length of γ and the area of the domain, enclosed by the curve, then*

(1) (A. Borisenko, K. Drach, [5]) *for the Euclidean plane, i.e. for $c = 0$,*

$$F \geqslant \frac{L}{2\lambda} - \frac{1}{\lambda^2} \sin \frac{L\lambda}{2};$$

(2) (A. Borisenko, K. Drach, [6]) *for the spherical space, i.e. for $c = k^2$,*

$$F \geqslant \frac{4}{k^2} \arctan\left(\frac{\lambda}{\sqrt{\lambda^2+k^2}} \tan\left(\frac{\sqrt{\lambda^2+k^2}}{4} L\right)\right) - \frac{L\lambda}{k^2};$$

(3) (K. Drach, [7]) *for the hyperbolic space, i.e. for $c = -k^2$, when*

(a) $\lambda > k$,

$$F \geqslant \frac{L\lambda}{k^2} - \frac{4}{k^2} \arctan\left(\frac{\lambda}{\sqrt{\lambda^2-k^2}} \tan\left(\frac{\sqrt{\lambda^2-k^2}}{4} L\right)\right);$$

(b) $\lambda = k$,

$$A \geqslant \frac{L}{k} - \frac{4}{k^2} \arctan \frac{kL}{4};$$

(c) $\lambda < k$,

$$F \geqslant \frac{L\lambda}{k^2} - \frac{4}{k^2} \arctan\left(\frac{\lambda}{\sqrt{k^2-\lambda^2}} \tanh\left(\frac{\sqrt{k^2-\lambda^2}}{4} L\right)\right).$$

Proof of the main Theorem

Let G_1 and G_2 be two copies of the domain G. Let us glue the domains G_1 and G_2 along their boundary curves γ_1 and γ_2 by isometry between these curves. We obtain a manifold F homeomorphic to the two-dimensional sphere with intrinsic metric. Since the sum of the integral geodesic curvatures of any two identified arcs of the boundary curves is non-negative, from the Alexandrov gluing theorem [9, p. 318] it follows that F is Alexandrov space of curvature $\geqslant c$. By Theorem A, this manifold can be isometrically embedded as a closed convex surface F_1 in the simply-connected space $M^3(c)$ of constant curvature equal to c. From Theorems B and C it follows that up to a rigid motion this surface is unique.

By *plane domains* we will understand domains on totally geodesic two-dimensional surfaces in spaces of constant curvature; similarly, we will call *lines* geodesic lines in these spaces.

Perform the reflection of the surface F_1 with respect to a plane π passing through 3 points on γ that do not lie on a line. We will get the mirrored surface F_2. The domains G_1 and G_2 are mapped to domains $\tilde{G}_1$ and $\tilde{G}_2$ on F_2; the curve γ is mapped to $\tilde{\gamma}$. But G_1 is isometric to G_2; $\tilde{G}_2$ is isometric to $\tilde{G}_1$. Let us reverse the orientation of the domains $\tilde{G}_1$, $\tilde{G}_2$. Then the surface F_2 will be isometric to F_1 and they will have the same orientation. By Theorems B and C the surface F_1 can be mapped onto the surface F_2 by rigid motion of the ambient space. But the 3 points of the curve γ are fixed under this rigid motion. Thus it follows that this motion is the identity mapping and, moreover, the curve γ coincides with the curve $\tilde{\gamma}$. Such situation is possible only when the curve γ is a plane curve and is the boundary of a convex cup isometric to the domain G. Recall that a *convex cap* is a convex surface with a plane boundary γ such that the surface is a graph over a plane domain $\bar{G}$ enclosed by γ. Note that since γ is a convex curve on the plane, the integral geodesic curvature of any arc of the curve γ is non-negative viewed both as a curve on the cap and as a curve on a plane.

In the cases $c = 0$ or $c = -k^2$ with $k > 0$, by a direct computation of the area of a graph over a plane domain, we obtain that the area of G is not less than the area of the plane domain $\bar{G}$. And the equality holds if and only if the cap G coincides with $\bar{G}$.

In the spherical space $S^3(k^2)$ of curvature $c = k^2$ let us consider a domain M bounded by the two-dimensional totally geodesic sphere containing γ; M is a closed hemisphere. This hemisphere is a convex three-dimensional manifold in the sense of [14]; thus we can apply Theorem D. By this theorem, the minimal surface M^2, which is a solution of the Plateau problem, either coincides with the domain $\bar{G}$ on the boundary of M or its interior lies inside the hemisphere.

Let us show that the integral geodesic curvature of any arc of γ calculated on $\bar{G}$ is not less than the corresponding integral geodesic curvature of this calculated on the cap G. This means that γ as a boundary curve of $\bar{G}$ is also λ-convex.

Recall that the intrinsic curvature $\omega(D)$ of a Borel set D on a convex surface in a space of constant curvature c is

$$\omega(D) = \psi(D) + cF(D),$$

where $\psi(D)$ is extrinsic curvature, $F(D)$ – area of D [9, p. 397]. Consider a closed convex surface M^2 bounded by G and the plane domain $\bar{G}$, and a surface $\bar{M}^2$ made up from the double-covered domain $\bar{G}$.

The intrinsic curvature concentrated on γ equals

$$\omega(\gamma) = \tau_\gamma(G) + \tau_\gamma(\bar{G}),$$

where $\tau_\gamma(G)$, $\tau_\gamma(\bar{G})$ are the integral geodesic curvatures of γ computed in G and $\bar{G}$, respectively.

Since $F(\gamma) = 0$, we have

$$\begin{aligned}\psi_M(\gamma) &= \tau_\gamma(G) + \tau_\gamma(\bar{G}),\\ \psi_{\bar{M}}(\gamma) &= 2\tau_\gamma(\bar{G}).\end{aligned}$$

From the definition of the extrinsic curvature [9, p. 397] it follows that $\psi_{\bar{M}}(\gamma) \geqslant \psi_M(\gamma)$, since each plane supporting to M^2 at a point on γ is also supporting to $\bar{M}^2$. Thus we obtain $\tau_\gamma(\bar{G}) \geqslant \tau_\gamma(G)$. Moreover, this inequality holds for any sub-arc of γ as well (for the Euclidean space this inequality was proved by V. A. Zalgaller [15, p. 65]).

Since the curve γ on the sphere is λ-convex, we have that the length of γ is less then 2π and it lies inside a two-dimensional open totally geodesic hemisphere of the boundary ∂M. Assume that the minimal surface lies inside the domain M. This means that this surface lies in some open hemisphere of the spherical space $S^3(k^2)$.

Without loss of generality it can be assumed that $k = 1$. Denote by $S^3(1)$ a sphere of radius 1 in the Euclidean space $\mathbb{E}^4$ with the orthogonal Cartesian coordinates x^0, x^1, x^2, x^3. By S^3_+ denote the hemisphere of $S^3(1)$ with $x^0 > 0$. By definition, the *geodesic map* of S^3_+ onto the Euclidean space $\mathbb{E}^3$ (given by $x^0 = 1$) with the orthogonal Cartesian coordinates y^1, y^2, y^3 is the mapping

$$y^i = \frac{x^i}{x^0}, \quad i = 1, 2, 3.$$

It is clear from the definition that lines and planes of S^3_+ are mapped into lines and planes of the Euclidean space $\mathbb{E}^3$. Let $\big(x^0(u_1, u_2), x^1(u_1, u_2), x^2(u_1, u_2), x^3(u_1, u_2)\big)$ be the position vector of the minimal surface $M_2 \subset S^3_+$ with the boundary curve γ. Suppose $\tilde{M}^2$ is the image of M^2 under the geodesic map; then the position vector of $\tilde{M}^2$ is

$$\tilde{r} = \frac{r}{x^0},$$

where $r = r(u_1, u_2) = \big(x^1(u_1, u_2), x^2(u_1, u_2), x^3(u_1, u_2)\big)$.

The second fundamental forms of M^2 and $\tilde{M}^2$ are proportional (see [16, Lemma 8]). From minimality of M^2 it follows that the Gaussian curvature of the surface $\tilde{M}^2$ is non-positive.

Since the curve γ as a curve of the sphere S^2 (with $x^0 = 0$, $\big(x^1\big)^2 + \big(x^2\big)^2 + \big(x^3\big)^2 = 1$) is λ-convex, γ belongs to an open hemisphere of S^2. We choose in $\mathbb{E}^4$ the Cartesian coordinates such that for the points on the curve γ we have $x^3 > c_0 > 0$ for some constant c_0 less than 1.

If a sequence of points $Q_i \in M^2$ converges to a point on the curve γ, then the sequence $\tilde{Q}_i \in \tilde{M}^2$ converges to infinity. The directions of the position vectors $\tilde{r}\left(\tilde{Q}_i\right)$ converge to the vector

$$a = \left(\frac{x^1(u_1, u_2)}{x^3(u_1, u_2)}, \frac{x^2(u_1, u_2)}{x^3(u_1, u_2)}, 1\right),$$

where $(0, r(u_1, u_2))$, defined above, is the position vector of the points on the curve γ.

If φ is the angle between the axis y^3 and the vector a, then

$$\cos\varphi = x^3(u_1, u_2) > c_0.$$

Therefore, in the infinity the surface $\tilde{M}^2$ lies inside the cone

$$y^3 = \frac{c_0}{\sqrt{1-c_0^2}}\sqrt{(y^1)^2 + (y^2)^2},$$

and $y^3 \to +\infty$ if a sequence of points $Q_i \in M^2$ converges to the point on the curve γ.

Let F_0 be the graph of the function

$$y^3 = \sqrt{1 + \epsilon^2\left((y^1)^2 + (y^2)^2\right)} - h$$

with $0 < \epsilon \leqslant c_0/\sqrt{1-c_0^2}$ and sufficiently big $h > 1$. The surface $\tilde{M}^2$ lies on the one side with respect to the surface F_0.

On $\tilde{M}^2$ consider the function

$$f(u_1, u_2) := y^3(u_1, u_2) - \sqrt{1 + \epsilon^2(y^1(u_1, u_2)^2 + y^2(u_1, u_2)^2)} + h,$$

where $y^i(u_1, u_2)$ $(i = 1, 2, 3)$ are coordinates of the points in $\tilde{M}^2$. We have $f > 0$, and if points $\tilde{Q}_i \in \tilde{M}^2$ converge to infinity, the function f converges to $+\infty$. Thus it follows that the minimum of the function f is attained at some point $\tilde{Q}_0 \in \tilde{M}^2$. It is easy to compute that at this point the Gaussian curvature of $\tilde{M}^2$ is positive. This contradicts that $\tilde{M}^2$ has a non-positive Gaussian curvature.

Therefore, the solution to the Plateau problem is the plane domain $\bar{G}$. From this we get that either the area of G is strictly larger than the area of $\bar{G}$ or the domains G and $\bar{G}$ coincide.

Consider the domain $\bar{G}$ with the boundary curve γ. This domain lies in a two-dimensional space of constant curvature c and satisfies the conditions of Theorem E, which yields the desired conclusion of the main Theorem.

References

[1] Wilhelm Blaschke, *Kreis und Kugel* (German), Walter de Gruyter & Co., Berlin, 1956. 2te Aufl. MR0077958

[2] Felix Bernstein, *Über die isoperimetrische Eigenschaft des Kreises auf der Kugeloberfläche und in der Ebene* (German), Math. Ann. **60** (1905), no. 1, 117–136, DOI 10.1007/BF01447496. MR1511289

[3] A. D. Aleksandrov, *An isoperimetric problem* (Russian), Doklady Akad. Nauk SSSR (N.S.) **50** (1945), 31–34. MR0052136

[4] V. K. Ionin, *Isoperimetric and certain other inequalities for a manifold of bounded curvature* (Russian), Sibirsk. Mat. Ž. **10** (1969), 329–342. MR0240753

[5] A. A. Borisenko and K. D. Drach, *Isoperimetric inequality for curves with curvature bounded below*, Math. Notes **95** (2014), no. 5-6, 590–598, DOI 10.1134/S0001434614050034. Translation of Mat. Zametki **95** (2014), no. 5, 656–665. MR3306202

[6] Alexander Borisenko and Kostiantyn Drach, *Extreme properties of curves with bounded curvature on a sphere*, J. Dyn. Control Syst. **21** (2015), no. 3, 311–327, DOI 10.1007/s10883-014-9221-z. MR3348570

[7] K. Drach, *About the isoperimetric property of λ-convex lunes on the Lobachevsky plane*, Dopov. Nats. Akad. Nauk Ukr., Mat. Pryr. Tekh. Nauky 2014, **11** (2014), 11–15 (in Russian; English version arXiv:1402.2688 [math.DG]).

[8] Yoel Groman and Jake P. Solomon, *A reverse isoperimetric inequality for J-holomorphic curves*, Geom. Funct. Anal. **24** (2014), no. 5, 1448–1515, DOI 10.1007/s00039-014-0295-2. MR3261632

[9] A. D. Alexandrov, *A. D. Alexandrov selected works. Part II*, *Intrinsic geometry of convex surfaces*, edited by S. S. Kutateladze, translated from the Russian by S. Vakhrameyev. Chapman & Hall/CRC, Boca Raton, FL, 2006. MR2193913

[10] Dmitri Burago, Yuri Burago, and Sergei Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001. MR1835418

[11] A. V. Pogorelov, *Extrinsic geometry of convex surfaces*, translated from the Russian by Israel Program for Scientific Translations, Translations of Mathematical Monographs, Vol. 35. American Mathematical Society, Providence, R.I., 1973. MR0346714

[12] A. D. Alexandrov, *Convex polyhedra*, translated from the 1950 Russian edition by N. S. Dairbekov, S. S. Kutateladze and A. B. Sossinsky, with comments and bibliography by V. A. Zalgaller and appendices by L. A. Shor and Yu. A. Volkov. Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005. MR2127379

[13] A. D. Milka, *Unique determinacy of general closed convex surfaces in Lobačevskiĭ space* (Russian), Ukrain. Geom. Sb. **23** (1980), 99–107, iii. MR614279

[14] William H. Meeks III and Shing Tung Yau, *The classical Plateau problem and the topology of three-dimensional manifolds. The embedding of the solution given by Douglas-Morrey and an analytic proof of Dehn's lemma*, Topology **21** (1982), no. 4, 409–442, DOI 10.1016/0040-9383(82)90021-0. MR670745

[15] V. A. Zalgaller, *On a class of curves with curvature of bounded variation on a convex surface* (Russian), Mat. Sbornik N.S. **30(72)** (1952), 59–72. MR0048833

[16] A. A. Borisenko, *Complete l-dimensional surfaces of non-positive extrinsic curvature in a Riemannian space* (Russian), Mat. Sb. (N.S.) **104(146)** (1977), no. 4, 559–576, 662–663. MR0470905

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Avenue, Kharkiv, 61103, Ukraine

E-mail address: aborisenk@gmail.com