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CONTINUITY OF CONVEX FUNCTIONS AT THE BOUNDARY OF THEIR DOMAINS: AN INFINITE DIMENSIONAL GALE-KLEE-ROCKAFELLAR THEOREM

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Abstract. Given C a closed convex set spanning the real Banach space X and x_0 a boundary point of C, this article proves that the two following statements are equivalent: (i) any lower semi-continuous convex function $f : C \to \mathbb{R}$ is continuous at x_0, and (ii) at x_0, C is Maserick polyhedral; that is, C is locally the intersection of a finite family of closed half-spaces.

1. Introduction

Given C, a closed convex subset of a real Banach space, and $f : C \to \mathbb{R}$, a convex lower semi-continuous function, a basic functional analysis result ensures us that f is continuous at every interior point of C. The proof of this requires a delicate balance among the completeness of the underlying space, the lower semi-continuity property of the function, and the convexity of the function. However, at boundary of points of C, this balance no longer operates.

In the finite dimensional setting, the celebrated Gale-Klee-Rockafellar theorem gives a detailed account of the continuity properties of convex lsc functions at boundary points of their domains by using the notion of point of polyhedrality of a set: C is said to be *polyhedral* at x_0 if some neighborhood of x_0 relative to C is a polytope (that is, the convex hull of a finite number of points). In this notation, it holds that any convex lsc function $f : C \to \mathbb{R}$ is continuous at some point x_0 iff C is polyhedral at x_0 (the *if part* is the initial Gale-Klee-Rockafellar theorem [7, Theorem 2]; the *only if* part has been completely solved in [5, Theorem 2.4]).

On these grounds, several so-called "automatic continuity" results emerged: one can prove that any convex lsc function defined on a polytope or on a polyhedral cone (a cone with a polytope for a base) is always continuous, whence the well-known and highly used mathematical economics and game theory lemma (see for instance [1, Theorem 4.2]) which says that each lsc convex function defined on P^n_+, the cone of the vectors from $\mathbb{R}^n$ with positive coordinates, is continuous.

The present study addresses the question of the continuity of a convex lsc function at a boundary point of its domain in the general setting of the infinite dimensional Banach spaces.

Our main result, Theorem 3.1, states that, given C a closed convex set spanning the real Banach space X, any convex lsc function $f : C \to \mathbb{R}$ is continuous at $x_0 \in C$

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iff, at x_0, C is *Maserick polyhedral*[1](as defined in [12]), that is, iff C is locally at x_0 the intersection of a finite number of closed half-spaces.

Corollary 4.1, an obvious consequence of our main result, captures the contrast between the case of a finite dimensional space, as depicted by the classical Gale-Klee-Rockafellar theorem, and the infinite dimensional setting. Namely, this result proves that, given x_0, an extreme point of an infinite dimensional subset C of a Banach space X, there is at least a convex lsc function $f : C \to \mathbb{R}$ which is discontinuous at x_0.

Accordingly, on every infinite dimensional line-free closed convex cone (for instance on ℓ^2_+, the positive cone of the space of square summable sequences ℓ^2), as well as on every weakly compact convex set, and in particular on every bounded closed convex subset of a reflexive Banach space, one can construct a discontinuous yet lsc convex function.

It remains possible however to achieve automatic continuity results even in the infinite dimensional setting. A generic example of such a result is provided by Corollary 4.2, which proves that any convex lsc function defined on the closed unit ball of c_0 is continuous.

A special mention must be made of Theorem 2.3, a key step in proving Theorem 3.1. This apparently simple result establishes that any infinite dimensional and line-free closed convex cone can always be expressed as the union of a strictly increasing sequence of closed convex cones; its surprisingly difficult proof covers the largest part of Section 2.

Our article ends with a section dedicated to conclusions and open questions.

2. Conical covers

Suppose K is a closed convex cone of a real Banach space X. A conical cover of K consists of an increasing family $(K_n)_{n\in\mathbb{N}}$ of closed convex cones satisfying

$$K_i \neq K \quad \forall i \in \mathbb{N} \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} K_n = K.$$

The following result establishes that the class of closed convex cones admitting a conical cover plays an important role in studying the continuity of a convex lsc function at a point of the boundary of its domain.

Proposition 2.1. *Let $K \subset X$ be a closed convex cone possessing a conical cover. Then, there is $f : K \to \mathbb{R}$ a convex lsc function discontinuous at θ_X, the null vector of X.*

Proof. Let $(K_n)_n$ be a conical cover of K; since none of the cones $(K_n)_n$ coincides with K, it is possible to pick a sequence $(x_n)_{n\in\mathbb{N}}$ converging to θ_X such that

$$x_n \in (K \setminus K_n) \quad \forall n \in \mathbb{N}.$$

The researched function f is defined by the following formula:

$$f(x) := \sup_{n\in\mathbb{N}} \frac{\operatorname{dist}(x, K_n)}{\operatorname{dist}(x_n, K_n)}, \quad \forall x \in K.$$

As K is the union of the increasing family of cones $(K_n)_n$, it follows that, at a given $x \in K$, only finitely many among the numbers $\left\{ \frac{\operatorname{dist}(x,K_n)}{\operatorname{dist}(x_n,K_n)} : n \in \mathbb{N} \right\}$ have

[1]For details on the eight forms of infinite dimensional polyhedrality, the reader is referred to the influential articles of Durier and Papini [4], and Fonf and Veselý [6].

positive values, the rest amounting to zero. Thus, $f(x)$ is well-defined for any vector $x \in K$.

Moreover, for any $n \in \mathbb{N}$, the function

$$\phi_n : X \to \mathbb{R}_+, \quad \phi_n(x) := \frac{\operatorname{dist}(x, K_n)}{\operatorname{dist}(x_n, K_n)}$$

is convex and continuous, $\phi_n(\theta_X) = 0$ and $\phi_n(x_n) = 1$. Accordingly, the function

$$\phi : X \to \mathbb{R}_+ \cup \{+\infty\}, \quad \phi := \sup_{n\in\mathbb{N}} \phi_n(x),$$

is convex and lsc, $\phi(\theta_X) = 0$, and $\phi(x_n) \geq 1$ for any $n \in \mathbb{N}$.

But f is nothing other than the restriction on the closed convex set K of the convex lsc function ϕ, so f is itself a convex lsc function. Finally, since $f(\theta_X) = 0$, while

$$\limsup_{x\to\theta_X} f(x) \geq \limsup_{n\to+\infty} f(x_n) \geq 1,$$

it yields that the function f is discontinuous at θ_X. □

In light of the previous result, it appears that characterizing the cones possessing at least a conical cover is an important step in studying the continuity properties of an lsc convex function.

Let us denote by X^* the topological dual of X, and by $\langle\cdot,\cdot\rangle : X^* \times X \to \mathbb{R}$ the duality pairing between X^* and X. The following lemma provides us with a standard manner to construct a conical cover for a closed convex cone K.

Lemma 2.2. *Let $K \subset X$ be a closed convex cone, and assume that there is a sequence $(x_n^*)_{n\in\mathbb{N}} \subset X^*$ such that*

$$\forall n \in \mathbb{N},\ \exists x_n \in K \ \text{ s.t. } \ \langle x_n^*, x_n\rangle > 0, \tag{2.1}$$

and

$$\forall x \in K,\ \ \exists n_x \in \mathbb{N} \ \text{ s.t. } \ \langle x_n^*, x\rangle \leq 0 \ \ \forall n \geq n_x. \tag{2.2}$$

Then, K admits a conical cover.

Proof. Let us set

$$K_n := \{x \in K : \ \langle x_m, x\rangle \leq 0 \ \ \forall m \geq n\};$$

obviously, $(K_n)_{n\in\mathbb{N}}$ is an increasing family of closed convex cones. From relation (2.2) it follows that

$$x \in K_{n_x} \qquad \forall x \in K,$$

so

$$\bigcup_{n\in\mathbb{N}} K_n = K,$$

while from relation (2.1) we deduce that

$$x_n \notin K_n \qquad \forall n \in \mathbb{N},$$

so

$$K_n \neq K \qquad \forall n \in \mathbb{N}.$$

The family $(K_n)_n$ is accordingly the researched conical cover of K. □

2.1. **Conical covers for line-free convex cones.** A cone K of the Banach space X is called *line-free* (the name *pointed* is also in use) if it does not contain any line of form $\mathbb{R}\, v$, with v a non-null vector from X or, equivalently, if $K \cap (-K) = \{\theta_X\}$.

Theorem 2.3. *Let $K \subset X$ be a line-free closed convex cone. The following two statements are equivalent:*
i) there is no conical cover for K,
ii) K is the convex hull of a finite number of rays.

Proof. *ii)* $\Rightarrow$ *i)* Let us consider K, the convex hull of the finite family of rays $(\mathbb{R}_+\, x_i)_{i=\overline{1,m}}$, and let us assume, to the end of achieving a contradiction, that there exists $(K_n)_n$, a conical cover of K.

The union of $(K_n)_n$ amounts to K; hence, for every $i = \overline{1, m}$, there is an index $s_i \in \mathbb{N}$ such that $x_i \in K_{s_i}$. Set p for the largest of the indices $\{s_i :\ i = \overline{1, m}\}$; then the set K_p contains the set $\{x_i \in X : i = \overline{1, m}\}$, and, since K_p is a cone and K is the convex hull of the rays spanned by the elements $(x_i)_{i=\overline{1,m}}$, we deduce that $K \subset K_p$, a contradiction.

i) $\Rightarrow$ *ii)* Let K be a closed convex cone which cannot be expressed as the convex hull of a finite set of rays. In constructing a conical cover for K, we distinguish three cases:

Case 1. K has a weakly compact base.

Case 2. K has a bounded base which is not weakly compact.

Case 3. K does not have a bounded base.

Case 1. Since K has a weakly compact base, there exists

$$x^* \in K^- := \{y^* \in X^* :\ \langle y^*, x\rangle \leq 0\ \forall x \in K\}, \quad \|x^*\| = 1,$$

such that the set

$$C := \{x \in K :\ \langle x^*,\ x\rangle = -1\} \tag{2.3}$$

is weakly compact. A major role in constructing the desired conical cover of K will be played by the denting points of K, that is, points which are contained in open slices

$$S(y^*, \alpha) = \{x \in C :\ \langle y^*, x\rangle < \alpha\}, \quad y^* \in X^*,\ \alpha \in \mathbb{R},$$

of arbitrary small diameter. The Lindenstrauss-Troyanski theorem [2, page 200] (or, to be more specific, as noted by Phelps [13, page 89], "Troyanski's renorming theorem [14] combined with a result of Lindenstrauss [11]") shows that every weakly compact convex subset of a Banach space is the closed convex hull of its denting points.

Accordingly, the set $D(C)$ of all the denting points of C is infinite (as the cone K cannot be expressed as the convex hull of a finite number of rays, its base C cannot be the convex hull of a finite family of points). On the other hand, it is obvious that, given an infinite subset A of a metric space, it is always possible to find an open ball centered at one of the points of A leaving outside an infinite number of elements from A (simply pick x an element of A, and if any ball centered at x contains all but a finite number of points from A, then any ball which does not contain x being however centered at a point of A does the job). By repeatedly applying this reasoning to the subset $D(C)$ of X, we can recursively define a sequence $(x_n)_n$ of denting points of C and $(\varepsilon_n)_n$, a sequence of positive numbers, such that the balls of center x_n and radius ε_n are mutually disjoint.

As $(x_n)_n$ are denting points of C, there are vectors $(y_n^*)_n \subset X^*$ and real constants $(\alpha_n)_n$ such that the slices $\{S(y_n^*, \alpha_n) : n \in \mathbb{N}\}$ are contained in the balls of radius ε_n centered at x_n. Accordingly, the slices $S(y_n^*, \alpha_n)$ are mutually disjoint, which means that the visibly increasing sequence of closed convex subsets of C given by the relation

$$C_n := C \setminus \left(\bigcup_{i \geq n} S(y_i^*, \alpha_i) \right)$$

fulfills the following relations:

$$C_n \neq C \quad \forall n \in \mathbb{N}, \qquad \bigcup_{n \in \mathbb{N}} C_n = C.$$

Let us set $K_n := \mathbb{R}_+ C_n$; clearly, the increasing family of cones $(K_n)_n$ is a conical cover of K.

Case 2. In this case, there is $x^* \in K^-$, $\|x^*\| = 1$, such that the set C defined by relation (2.3) is bounded but not weakly compact (of course, this can happen only when X is not a reflexive Banach space).

To solve this case, we need to construct a sequence $(x_n^*)_{n \in \mathbb{N}} \subset X^*$ satisfying relations (2.1) and (2.2). In this respect, we use James' celebrated theorem ([9, Theorem 4]) in order to deduce that there exists $y^* \in X^*$ such that

$$\sup_{x \in C} \langle y^*, x \rangle = 1 \tag{2.4}$$

and

$$\langle y^*, x \rangle < 1 \qquad \forall x \in C. \tag{2.5}$$

We set

$$x_n^* := \frac{n\, x^*}{n+1} + y^*;$$

for every $x \in C$ it follows that $\langle x_n^*, x \rangle = \langle y^*, x \rangle - \frac{n}{n+1}$. In order to prove that the sequence $(x_n^*)_n$ satisfies relation (2.1), it suffices to pick $x_n \in C$ such that $\langle y^*, x_n \rangle > \frac{n}{n+1}$, which is possible thanks to relation (2.4). It is moreover clear that relation (2.2) needs to be proved only for $x \in C$ and that, in this case, it is enough to pick $n_x \in \mathbb{N}$ such that $\frac{n_x}{n_x+1} > \langle y^*, x \rangle$ (again possible as a consequence of relation (2.5)).

Case 3. Let us first establish some notation. For any non-null element $x \in K$, we define the real number

$$\beta(x) := \inf_{x^* \in K^-,\, x^* \neq \theta_{X^*}} \frac{\langle x^*, x \rangle}{\|x\|\, \|x^*\|};$$

let us remark that the real numbers $\beta(x)$ are always negative.

Indeed, by construction, $\beta(x)$ is always a non-positive real number, and $\beta(x) = 0$ means that $\langle x, x^* \rangle = 0$ for any $x^* \in K^-$, which means that the negative dual cone K^- is completely contained in the hyperplane $H_x = \{x^* \in X^* : \langle x^*, x \rangle = 0\}$, which, as a consequence of the bi-dual theorem, means that K contains the line $\mathbb{R}\,x$, a contradiction. Accordingly, the sets

$$L_n := \{\theta_X\} \cup \left\{ x \in K \setminus \{\theta_X\} : \ \beta(x) < -\frac{1}{n} \right\}$$

form an increasing family of closed cones (not necessarily convex) whose union amounts to K. Finally, let K_n be the closed convex hull of L_n. Obviously, $(K_n)_n$ is an increasing family of closed convex cones, and their union equals K.

Since when neither of the cones $(K_n)_n$ coincides with K, $(K_n)_n$ is a conical cover for K and there is nothing to prove, we shall only address the case when there is an index $m \in \mathbb{N}$ such that $K_m = K$.

In solving this case, our objective is to construct a sequence $(x_n^*)_{n\in\mathbb{N}} \subset X^*$ satisfying relations (2.1) and (2.2). Let us first consider D, the closed convex hull of the set $\{x \in L_m : \ \|x\| = 1\}$, and let us assume, to the end of achieving a contradiction, that θ_X is disjoint from the closed convex set D. Accordingly, the point θ_X and the set D can be strictly separated by the means of a closed hyperplane: in other words, there is $x^* \in X^*$ and $\gamma > 0$ such that

$$\langle x^*, \theta_X \rangle = 0 < \gamma < \inf_{x\in D} \langle x^*,\ x\rangle \leq \inf_{x\in L_m,\, \|x\|=1} \langle x^*,\ x\rangle\,.$$

Hence,

$$\langle x^*,\ x\rangle \geq \gamma \|x\| \quad \forall x \in L_m,$$

and, since K is nothing but the closed convex hull of L_m, it follows that

$$\langle x^*,\ x\rangle \geq \gamma \|x\| \quad \forall x \in K.$$

The previous inequality clearly implies that the base $C := \{x \in K : \ \langle x^*,\ x\rangle = -1\}$ of K is bounded, a contradiction.

We have thus proved that θ_X belongs to the closed convex hull of the set of norm one elements from the cone L_m. Accordingly, for any $n \in \mathbb{N}$, there is $\overline{x}_n$, a vector belonging to the convex hull of the set $\{x \in L_m : \ \|x\| = 1\}$, and thus to the convex hull of some finite subset J_n of $\{x \in L_m : \ \|x\| = 1\}$, whose norm is smaller than $\frac{1}{n}$.

Let us gather together the elements of the sets $(J_n)_n$ in a sequence $(y_n)_n$ (this is always possible, since $(J_n)_n$ is a countable family of finite sets). By definition, for every element y_n of this sequence it holds that $y_n \in L_m$ and $\|y_n\| = 1$, so there are elements $(y_n^*)_n \subset K^-$, $\|y_n^*\| = 1$, such that

$$\langle y_n^*,\ y_n\rangle \leq -\frac{1}{m} \qquad \forall n \in \mathbb{N}. \tag{2.6}$$

Let us set

$$z^* = \sum_{n\in\mathbb{N}} \frac{y_n^*}{2^n};$$

obviously, $z^* \in K^-$.

Given $p \in \mathbb{N}$, we remark that it is impossible to have $\langle z^*, x\rangle \leq -\frac{2\,\|z^*\|}{p}$ for all the vectors $x \in J_p$, since this would also imply that $\langle z^*, \overline{x}_p\rangle \leq -\frac{2\,\|z^*\|}{p}$, a relation which contradicts the fact that $\|\overline{x}_p\| \leq \frac{1}{p}$. Accordingly, it results that there is at least an element, say y_{k_p}, among the elements of J_p, and consequently among the elements of the newly defined sequence $(y_n)_n$, such that

$$\left\langle z^*,\ y_{k_p}\right\rangle > -\frac{2\,\|z^*\|}{p}. \tag{2.7}$$

On the other hand,

$$\left\langle z^*,\ y_{k_p}\right\rangle = \sum_{n\in\mathbb{N}} \frac{\left\langle y_n^*,\ y_{k_p}\right\rangle}{2^n}; \tag{2.8}$$

since $y_n^* \in K^-$ for any $n \in \mathbb{N}$ and $y_{k_p} \in K$, it holds that

$$\left\langle y_n^*,\ y_{k_p}\right\rangle \leq 0 \quad \forall n \in \mathbb{N}, \tag{2.9}$$

while relation (2.6) applied to the index k_p implies that

$$\left\langle y_{k_p}^*,\ y_{k_p}\right\rangle \leq -\frac{1}{m}. \tag{2.10}$$

By combining relations (2.7), (2.8), (2.9) and (2.10) we deduce that

$$-\frac{2\,\|z^*\|}{p} < \left\langle z^*,\ y_{k_p}\right\rangle \leq -\frac{1}{m\,2^{k_p}} \quad \forall p \in \mathbb{N}. \tag{2.11}$$

We are now in a position to define the desired sequence:

$$x_n^* := z^* - 2\frac{\langle z^*,\ y_{k_n}\rangle}{\left\langle y_{k_n}^*,\ y_{k_n}\right\rangle}\, y_{k_n}^* \quad \forall n \in \mathbb{N};$$

all that we need to prove is that relations (2.1) and (2.2) hold true.

A direct calculus shows that, for any $n \in \mathbb{N}$,

$$\langle x_n^*,\ y_{k_n}\rangle = -\langle z^*,\ y_{k_n}\rangle,$$

and from relation (2.11) it yields that

$$\langle x_n^*,\ y_{k_n}\rangle \geq \frac{1}{m\,2^{k_n}} > 0 \quad \forall n \in \mathbb{N},$$

so relation (2.1) is verified.

Another obvious consequence of relation (2.11) is that

$$\lim_{n\to+\infty} \langle z^*,\ y_{k_n}\rangle = 0;$$

since $\|y_{kp}^*\| = 1$ and (see relation (2.10))

$$-1 = -\|y_{k_p}^*\|\,\|y_{k_p}\| \leq \left\langle y_{k_p}^*,\ y_{k_p}\right\rangle \leq -\frac{1}{m},$$

we may conclude that

$$\lim_{n\to+\infty} x_n^* = z^*. \tag{2.12}$$

Let us pick a non-null vector $x \in K$; since $z^* \in K^-$, there are only two possibilities: either $\langle z^*, x\rangle = 0$ or $\langle z^*, x\rangle < 0$.

Let us first assume that $\langle z^*, x\rangle = 0$; in this case

$$\sum_{n\in\mathbb{N}} \frac{\langle y_n^*,\ x\rangle}{2^n} = 0,$$

and since $\langle y_n^*, x\rangle \leq 0$ for any $n \in \mathbb{N}$, we infer that

$$\langle y_n^*,\ x\rangle = 0 \qquad \forall n \in \mathbb{N}.$$

The definition of x_n^* implies now that

$$\langle x_n^*,\ x\rangle = 0 \qquad \forall n \in \mathbb{N},$$

so relation (2.2) holds true.

Let us turn now to the case when $\langle z^*,\, x\rangle < 0$. By virtue of relation (2.12), we deduce that the sequence $\langle x_n^*,\ x\rangle$ converges to a negative real number, so its values must be non-positives for n large enough; relation (2.2) is therefore verified. □

3. Boundary points of automatic continuity

Following Klee ([10, p. 86]), we call the set C *Maserick polyhedral* (for short *M-polyhedral*) at $x_0 \in C$ if there are $\varepsilon > 0$, a real constant, and K, a Maserick polyhedral cone (for short an M-polyhedral cone), that is, a cone which can be expressed as the intersection of a finite family of closed half-spaces of X, such that

$$C \cap (x_0 + \varepsilon\, B_X) = (x_0 + K) \cap (x_0 + \varepsilon\, B_X)\,, \tag{3.1}$$

where B_X is the closed unit ball of X.

Similarly, if relation (3.1) holds with K being a closed convex (but not necessarily M-polyhedral) cone, the set C is simply called conical at x_0; in other words (Howe, [8, p. 1198]), "near x_0, the set C looks like a [. . .] cone." Of course, if a set is M-polyhedral at some point, it is also conical at the same point, but the converse does not generally hold.

The following statement proves a criterion of automatic continuity for convex lsc functions at the boundary of their domain.

Theorem 3.1. *Let $C \subset X$ be a closed convex set spanning X, and x_0 a boundary point of C. The following two statements are equivalent:*

i) every convex lsc function $f : C \to \mathbb{R}$ is continuous at x_0,

ii) C is M-polyhedral at x_0.

Proof. ii) $\Rightarrow$ i) Let us consider x_0, a boundary point of C at which C is polyhedral in the sense of Maserick, that is, a point x_0 such that relation (3.1) holds true for some real constant $\varepsilon > 0$, and some M-polyhedral cone K. Given $f : C \to \mathbb{R}$, an lsc convex function, and $(x_n)_n$, a sequence of points from C converging to x_0, our aim is to prove that

$$f(x_0) \geq \limsup_{n\to+\infty}\ f(x_n). \tag{3.2}$$

Let us first remark that the set $H_1 := K \cap (-K)$ is a closed subspace of X of finite co-dimension; accordingly, H_1 is complemented, so there is H_2, a finite dimensional subspace of X such that $H_1 \cap H_2 = \{\theta_X\}$ and $H_1 + H_2 = X$. Any vector $x \in X$ may thus be written as the sum of two vectors from H_1 and H_2,

$$x = r_1(x) + r_2(x), \qquad r_1(x) \in H_1,\ r_2(x) \in H_2,$$

in exactly one way. Moreover, the functions $r_1 : X \to H_1$ and $r_2 : X \to H_2$ are linear and continuous, so there exists $M > 0$ such that

$$\|r_1(x)\| \leq M\ \|x\| \ \text{ and } \ \|r_2(x)\| \leq M\ \|x\| \qquad \forall x \in X. \tag{3.3}$$

Another useful relation establishes that, for any $x \in K$, the vectors $r_1(x)$ and $r_2(x)$ necessarily belong to K (indeed, $H_1 \subset K$, so $r_1(x) \in K$, while $r_2(x) = x - r_1(x)$, and our claim yields from the fact that both the vectors x and $-r_1(x)$ belong to the convex cone K).

Obviously, for every vector $x \in X$ it holds that

$$x = x_0 + (x - x_0) = x_0 + r_1(x - x_0) + r_2(x - x_0),$$

so every $x \in X$ may be expressed as follows:

$$x = \frac{(x_0 + 2\,r_1(x - x_0)) + (x_0 + 2\,r_2(x - x_0))}{2}.$$

As already remarked, the elements $x_0 + 2\,r_1(x - x_0)$ and $x_0 + 2\,r_2(x - x_0)$ belong to $x_0 + K$, provided that $x \in x_0 + K$. If moreover $x \in x_0 + K$ and $\|x_0 - x\| \leq \frac{\varepsilon}{2\,M}$, then both the real numbers $\|2\,r_1(x - x_0)\|$ and $\|2\,r_2(x - x_0)\|$ are smaller than ε, so the vectors $x_0 + 2\,r_1(x - x_0)$ and $x_0 + 2\,r_2(x - x_0)$ lie in $x_0 + (K \cap \varepsilon\, B_X)$, which, by virtue of relation (3.1), means that they actually lie in C.

Since x_n converges to x_0, it holds that $\|x_0 - x_n\| \leq \frac{\varepsilon}{2\,M}$ for n large enough; hence, for such n, x_n is the mid-point of a segment whose end-points, $x_0 + 2\,r_1(x_n - x_0)$ and $x_0 + 2\,r_2(x_n - x_0)$, are two vectors from the convex set C, and the Jensen inequality proves that

$$f(x_n) \leq \frac{f\left(x_0 + 2\,r_1(x_n - x_0)\right) + f\left(x_0 + 2\,r_2(x_n - x_0)\right)}{2}. \tag{3.4}$$

A basic result of convex analysis says that the restriction of an lsc convex function on a closed linear manifold of a Banach space is always continuous; when specified to the restriction of f on $x_0 + K$, this result says that

$$\lim_{n \to +\infty} f\left(x_0 + 2\,r_1(x_n - x_0)\right) = f\left(\lim_{n \to +\infty} x_0 + 2\,r_1(x_n - x_0)\right) = f(x_0). \tag{3.5}$$

An important step in our reasoning is to remark that the intersection between the cone K and the closed linear space H_2 is a polyhedral cone. Indeed, the intersection between a closed linear subspace and an M-polyhedron is again an M-polyhedron (this easy-to-prove fact is an obvious consequence of [6, Theorem 2.1]). Thus $K \cap H_2$ is an M-polyhedral cone of finite dimension, and our claim is proved by using [12, Theorem 2.2], a result implying, *inter alia*, that any finite dimensional M-polyhedral cone is the convex hull of a finite number of rays. Let us now invoke the Gale-Klee-Rockafellar theorem [7, Theorem 2], a statement which affirms that, in the finite dimensional setting, an lsc convex function is continuous at every point at which its domain is polyhedral, in order to deduce that the restriction of f on $K \cap H_2$ is continuous at x_0, so

$$\lim_{n \to +\infty} f\left(x_0 + 2\,r_2(x_n - x_0)\right) = f\left(\lim_{n \to +\infty} x_0 + 2\,r_2(x_n - x_0)\right) = f(x_0). \tag{3.6}$$

The desired relation (3.2) is a consequence of relations (3.4), (3.5) and (3.6).

i) $\Rightarrow$ *ii)* Let us consider x_0, a boundary point at which C is not M-polyhedral. Our objective is to construct an lsc convex function $f : C \to \mathbb{R}$ which is not continuous at x_0. Let us distinguish two cases:

Case 1. C is not a conical set at x_0.

Case 2. At x_0, C is conical without being M-polyhedral (for instance C is a circular cone and x_0 is its apex).

Case 1. As remarked by Howe (see [8, Proposition 1]), the Minkowski gauge of C at x_0,

$$\mu_{C,x_0} : C \to \mathbb{R}, \quad \mu_{C,x_0}(x) = \inf\left\{\gamma > 0 : \;\; x_0 + \frac{1}{\gamma}\,x \in C\right\}$$

provides us with an example of an lsc convex function defined on C which is not continuous at x_0.

Case 2. With the same notation $H_1 := K \cap (-K)$ as above (however, in this case, H_1 is no longer necessarily of finite co-dimension; in point of fact, it may even be reduced to the singleton $\{\theta_X\}$), let us denote by $\hat{X}$ the quotient space of X by H_1, and by $\hat{K}$, the quotient cone of K by H_1.

As a first step in constructing the function f, let us prove that $\hat{K}$ cannot be expressed as the conical hull of a finite number of rays from $\hat{X}$. Let us assume the contrary; as C spans X, it follows that K also spans X, hence that $\hat{K}$ spans $\hat{X}$. Accordingly, $\hat{H}$ is a Banach space of finite dimension, and we can apply to $\hat{K} \subset \hat{X}$ the Minkowski-Weyl theorem for cones ([3, Theorem 3.11]), a statement which proves that $\hat{K}$ is the intersection of a finite family of closed half-spaces of $\hat{X}$. This fact obviously entails that K itself is the intersection of a finite set of closed half-spaces of X, a contradiction which proves our claim.

We are now in a position to apply to $\hat{K}$ the conclusions of Theorem 2.3 and deduce that the cone $\hat{K}$ possesses a conical cover. Clearly, the same statement holds true for the cone K, and the existence of the lsc convex function defined on C and discontinuous at x_0 follows as a simple consequence of Proposition 2.1. ☐

4. Conclusions and open problems

The following statement is an obvious consequence of Theorem 3.1.

Corollary 4.1. *Let K be a finite dimensional face of the infinite dimensional subset C of the Banach space X, and let x_0 be a point in K. There is at least a convex lsc function $f : C \to \mathbb{R}$ which is discontinuous at x_0.*

In particular, no automatic continuity result may be achieved on infinite dimensional closed convex sets with extreme points, and in particular on line-free cones, or on bounded subsets of reflexive Banach spaces. It is however easy to combine the well-known fact that the closed unit ball of c_0 is Maserick-polyhedral with the conclusions of Theorem 3.1 in order to imply the following infinite dimensional theorem on continuity of convex functions.

Corollary 4.2. *Let B be the closed unit ball of c_0, the sequence space of real null sequences, and let $f : B \to \mathbb{R}$ be a convex function. If f is lsc, then f is continuous.*

Let us also mention the following statement, which implies Theorem 2.3, but for which, to our best knowledge, no proof or counterexample is available.

Conjecture 4.3. *Let X be a real Banach space and $K \subset X$ a line-free closed convex cone. The following two statements are equivalent:*

i) $\Pi(K)$ is closed for any linear and continuous application $\Pi : X \to \mathbb{R}^N$, $N \in \mathbb{N}$,

ii) K is the convex hull of a finite number of rays.

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