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# MAASS FORM TWISTED SHINTANI $\mathcal{L}$-FUNCTIONS

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ABSTRACT. The Maass form twisted Shintani $\mathcal{L}$-functions are introduced, and some of their analytic properties are studied. These functions contain data regarding the distribution of shapes of cubic rings.

## 1. INTRODUCTION

The space of binary cubic forms over a commutative ring $\mathscr{R}$

$$V_{\mathscr{R}} = \left\{ f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 : a,b,c,d \in \mathscr{R} \right\} \tag{1.1}$$

has a rich algebraic structure. $\mathrm{GL}_2(\mathscr{R})$ acts by changing coordinates:

$$g \cdot f(x,y) = f((x,y)g^t). \tag{1.2}$$

Over $\mathbb{C}$, this makes $V_{\mathbb{C}}$ an example of a prehomogeneous vector space. Over $\mathbb{R}$, $V_{\mathbb{R}}$ splits into a pair of open $\mathrm{GL}_2(\mathbb{R})$ orbits, having positive and negative discriminant, and a singular set having discriminant zero. The non-singular forms have finite stabilizer, so that these are naturally identified with finite quotients of $\mathrm{GL}_2(\mathbb{R})$. Over $\mathbb{Z}$, one considers in addition to the lattice $L = V_{\mathbb{Z}}$, the dual lattice

$$\hat{L} = \{ f \in L : 3 | b, c \} . \tag{1.3}$$

For a fixed non-zero $m \in \mathbb{Z}$ those integral forms from $L$ and $\hat{L}$ of discriminant $m$ each split into finitely many orbits, the number of which is the class number, denoted $h(m)$ and $\hat{h}(m)$, respectively. The space of integral binary cubic forms taken modulo $\mathrm{GL}_2(\mathbb{Z})$-equivalence has extra significance, as it is in discriminant-preserving bijection with cubic rings taken up to isomorphism [3], [4], [2].

Shintani [12] introduced zeta functions enumerating the class numbers $h(m)$, $\hat{h}(m)$. These Dirichlet series, initially defined only in the half-plane $\{s \in \mathbb{C} : \Re(s) > 1\}$, have meromorphic continuation to all of $\mathbb{C}$ and satisfy a functional equation relating $s$ to $1-s$. Shintani determined the poles and residues, and hence obtained strong results on the average behavior of $h(m)$.

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Having fixed a base point, a class of integral forms $\tilde{f}$ of non-zero discriminant is identified with a point in $\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{GL}_2(\mathbb{R})$, and it is natural to ask for the distribution of these points on average. We call the point $g_f \in \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{GL}_2(\mathbb{R})$ the 'shape' of the form $f$, a name which becomes more natural in the case that $f$ is associated to an order of a cubic field, in which case $g_f$ describes the shape of the corresponding lattice in its natural embedding. The distribution of these shapes was studied by Terr [14], who proved the asymptotic uniform distribution of the shape of cubic orders and fields when ordered by discriminant; see also [5]. In related work, the author proved the quantitative equidistribution of 3-torsion ideal classes in imaginary quadratic fields [6] and the corresponding equidistribution statements for quartic and quintic fields have been demonstrated by Bhargava and Harron [1].

The purpose of this note is to give a strong estimate for the equidistribution of binary cubic forms with respect to the cuspidal spectrum of $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$, by modifying the method of Shintani. Let $\phi$ be a non-constant automorphic cusp form on

$$\mathscr{X} = \mathrm{SO}_2(\mathbb{R})\backslash \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z}), \tag{1.4}$$

which is an eigenfunction of the Hecke algebra, and extend $\phi$ to $\mathrm{GL}_2(\mathbb{R})$ by projecting by a diagonal matrix. Fix base forms $x^0_\pm$ of discriminant $\pm 1$, and for each $m \neq 0$ choose representatives $\{g_{i,m}\}_{i=1}^{h(m)}, \{\hat{g}_{i,m}\}_{i=1}^{\hat{h}(m)}$ such that $\left\{g_{i,m} \cdot x^0_{\mathrm{sgn}(m)}\right\}_{i=1}^{h(m)}$, $\left\{\hat{g}_{i,m} \cdot x^0_{\mathrm{sgn}(m)}\right\}_{i=1}^{\hat{h}(m)}$ are representatives for the classes of integral forms of discriminant $m$. Denote $\Gamma(i,m)$, $\hat{\Gamma}(i,m)$ the stability groups of $g_{i,m} \cdot x^0_{\mathrm{sgn}(m)}$, resp. $\hat{g}_{i,m} \cdot x^0_{\mathrm{sgn}(m)}$ in $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Introduce '$\phi$-twisted Shintani $\mathscr{L}$-functions' defined for $\Re(s) > 4$ by absolutely convergent Dirichlet series

$$\begin{aligned} \mathscr{L}_\pm(L,s;\phi) &= \sum_{\pm m \geqslant 1} \frac{1}{|m|^s} \sum_{i=1}^{h(m)} \frac{\phi\left(g_{i,m}^{-1}\right)}{|\Gamma(i,m)|}, \\ \mathscr{L}_\pm(\hat{L},s;\phi) &= \sum_{\pm m \geqslant 1} \frac{1}{|m|^s} \sum_{i=1}^{\hat{h}(m)} \frac{\phi\left(\hat{g}_{i,m}^{-1}\right)}{\left|\hat{\Gamma}(i,m)\right|}. \end{aligned} \tag{1.5}$$

It is shown that these series may be factored from an orbital integral as in [12]. The trick which permits introducing $\phi$ is due to Selberg [11], exploiting the mean-value property of harmonic functions.

The twisted $\mathscr{L}$-functions appear less natural than the case $\phi = 1$ of [12]. For instance, we are not aware that a functional equation is satisfied, and suspect that none exists. We are, however, able to demonstrate the holomorphic continuation past the region of absolute convergence, which is sufficient to prove equidistribution statements.

**Theorem 1.1.** *Let $\phi$ be a Maass Hecke-eigen cusp form on $\mathscr{X}$. The $\phi$-twisted Shintani $\mathscr{L}$-functions extend to holomorphic functions in the half-plane $\Re(s) > \frac{1}{8}$.*

*Remark* 1.2. Theorem 1.1 exhibits substantial orthogonality of the shapes of binary cubic forms to the Maass spectrum. In particular, for $\psi \in C_c^\infty(\mathbb{R}^+)$, the proof of

Theorem 1.1 permits the estimate

$$\sum_{\pm m \geqslant 1} \psi\left(\frac{|m|}{X}\right) \sum_{i=1}^{h(m)} \frac{\phi\left(g_{i,m}^{-1}\right)}{|\Gamma(i,m)|} \ll_{\epsilon,\psi} X^{\frac{1}{8}+\epsilon} \tag{1.6}$$

with the same estimate for dual forms. By comparison, the number of forms counted is order $X$. The best estimate in (1.6) obtainable from [14] is of order $X^{\frac{15}{16}}$, while [1] proves the qualitative statement $o(X)$.

*Remark* 1.3. Recall that a cusp form $\phi$ of $\mathrm{SO}_2(\mathbb{R})\backslash \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ satisfies an exponential decay condition in the cusp. Our argument applies with appropriate modifications also to the Eisenstein spectrum, and to automorphic forms that transform on the left by a fixed character of $\mathrm{SO}_2(\mathbb{R})$. See [8] for a general description of automorphic forms on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$. We omit the details here, but intend to give detailed equidistribution statements in a future paper treating cubic fields.

**Related work.** We discovered the twisted $\mathscr{L}$-functions during work on the AIM Square on alternative proofs of the Davenport-Heilbronn theorems. See work of Sato [9], [10] for some related objects.

## 2. Background

Set $G = \mathrm{GL}_2(\mathbb{R})$, $G^1 = \mathrm{SL}_2(\mathbb{R})$, $G^+ = \{g \in G : \det g > 0\}$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $\Gamma_\infty = \Gamma \cap \begin{pmatrix} \pm 1 & 0 \\ * & \pm 1 \end{pmatrix}$ and standard subgroups[1]

$$\begin{aligned} K &= \left\{ k_\theta = \begin{pmatrix} c(\theta) & s(\theta) \\ -s(\theta) & c(\theta) \end{pmatrix} : \theta \in \mathbb{R}/\mathbb{Z} \right\}, \\ A &= \left\{ a_t = \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix} : t \in \mathbb{R}_{>0} \right\}, \\ N &= \left\{ n_u = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} : u \in \mathbb{R} \right\}. \end{aligned} \tag{2.1}$$

Haar measure is normalized on $G^1$ by setting, for $f \in L^1(G^1)$,

$$\int_{G^1} f(g) dg = \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}^+} \int_{\mathbb{R}} f(k_\theta a_t n_u) d\theta \frac{dt}{t^3} du \tag{2.2}$$

and, for $f \in L^1(G)$,

$$\int_{G^+} f(g) dg = \int_{\mathbb{R}^+} \int_{G^1} f\left( \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix} g \right) dg \frac{d\ell}{\ell}. \tag{2.3}$$

2.1. **Automorphic forms.** For consistency with Shintani we work on $L^2(K\backslash G^1/\Gamma)$, with the lattice quotient on the right. This differs from many modern authors. See [15] for a summary of the results discussed here, and note the normalization $y = t^2$.

A convenient basis for $L^2(K\backslash G^1/\Gamma)$ consists in joint eigenfunctions of the Laplacian and the Hecke operators. These automorphic forms split into discrete and continuous spectrum. The discrete spectrum has an $L^2$ basis of Hecke-eigen Maass forms while the continuous spectrum is spanned by the real analytic Eisenstein series.

[1] $c(\theta) = \cos(2\pi\theta)$, $s(\theta) = \sin(2\pi\theta)$.

Let $\phi(g)$ be a Hecke-eigen Maass form with Laplace eigenvalue $\lambda = s(1-s)$, $s = \frac{1}{2} + it_\phi$. The Maass forms split into even and odd forms. An even Maass form $\phi$ has a Fourier development in the parabolic direction

$$\phi(g) = 2t \sum_{n=1}^{\infty} \rho_\phi(n) K_{s-\frac{1}{2}}\left(2\pi n t^2\right) \cos(2\pi n u) \tag{2.4}$$

whereas an odd form replaces $\cos(\cdot)$ with $\sin(\cdot)$ in the Fourier expansion. We use the Mellin transforms

$$\begin{aligned} &\int_0^\infty K_\nu(x) x^{s-1} dx = 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right), \qquad \Re s > |\Re \nu|, \\ &\int_0^\infty \cos(x) x^{s-1} dx = \Gamma(s)\cos\left(\frac{\pi s}{2}\right), \qquad 0 < \Re s < 1. \end{aligned} \tag{2.5}$$

We assume the Maass forms considered are even, although the argument applies to odd forms without change. Let the Maass forms be Hecke-normalized, that is, $\rho_\phi(1) = 1$. This means that the Fourier coefficients satisfy the Hecke relations

$$\rho_\phi(m)\rho_\phi(n) = \sum_{d|\,\mathrm{GCD}(m,n)} \rho_\phi\left(\frac{mn}{d^2}\right), \tag{2.6}$$

from which it follows that there exists constant $C > 1$ such that for all primes $p$ and $n \geqslant 1$,

$$|\rho_\phi(p^n)| \leqslant \left(C(1 + |\rho_\phi(p)|)\right)^n. \tag{2.7}$$

The sup bound

$$|\rho_\phi(n)| \ll n^{\frac{7}{64}+\epsilon} \tag{2.8}$$

was proven in [7] while the $L^2$-bound

$$\sum_{n \leqslant X} |\rho_\phi(n)|^2 \ll X \tag{2.9}$$

follows from the Rankin-Selberg theory.

We follow Shintani's convention regarding the real analytic Eisenstein series, which puts the symmetry line for these forms at $\Re(z) = 0$. Define

$$E(z,g) = \sum_{\gamma \in \Gamma/\Gamma_\infty} t(g\gamma)^{z+1} \tag{2.10}$$

the real analytic Eisenstein series with complex parameter $z$. This satisfies a functional equation

$$\xi(z+1)E(z,g) = \xi(1-z)E(-z,g); \qquad \xi(z) = \pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z), \tag{2.11}$$

and has a Fourier development in $z \neq 0$ given by

$$\begin{aligned} E(z,g) &= t^{z+1} + t^{1-z}\frac{\xi(z)}{\xi(z+1)} \\ &\quad + \frac{4t}{\xi(z+1)} \sum_{m=1}^{\infty} \eta_{\frac{z}{2}}(m) K_{\frac{z}{2}}(2\pi m t^2)\cos(2\pi m u), \\ \eta_{\frac{z}{2}}(m) &= \sum_{ab=m} \left(\frac{a}{b}\right)^{\frac{z}{2}}. \end{aligned} \tag{2.12}$$

Say that $f \in C(K\backslash G^1/\Gamma)$ is of polynomial growth if $f$ is bounded by a polynomial in $\theta, u, t$, similarly, is Schwarz class if it decays when multiplied by any polynomial in $\theta, u, t$. The Maass forms are Schwarz class, while the Eisenstein series has polynomial growth. After subtracting the constant term in the Fourier expansion, the resulting modified Eisenstein series again is Schwarz class.

Due to convergence issues resulting from the constant term it is convenient to work with a truncated Eisenstein series. Let $\Psi$ denote the space of entire functions such that for all $\psi \in \Psi$, for all $-\infty < C_1 < C_2 < \infty$, for all $N > 0$,

$$\sup_{C_1 < \Re(w) < C_2} \left(1 + (\Im w)^2\right)^N |\psi(w)| < \infty. \tag{2.13}$$

For $\psi \in \Psi$ and $\Re(w) > 1$ define the incomplete Eisenstein series at $\psi$ by choosing $1 < c < \Re(w)$ and setting

$$\mathscr{E}(\psi, w; g) = \oint_{\Re(z)=c} \psi(z) \frac{E(z,g)}{w-z} dz. \tag{2.14}$$

2.2. **Binary cubic forms.** $G$ acts naturally on the space

$$V_{\mathbb{R}} = \left\{ax^3 + bx^2y + cxy^2 + dy^3 : (a,b,c,d) \in \mathbb{R}^4\right\} \tag{2.15}$$

of binary cubic forms via, for $f \in V_{\mathbb{R}}$ and $g \in G$,

$$g \cdot f(x,y) = f((x,y) \cdot g^t). \tag{2.16}$$

The discriminant $D$, which is a homogeneous polynomial of degree four on $V_{\mathbb{R}}$, is a relative invariant: $D(g \cdot f) = \chi(g) D(f)$ where $\chi(g) = \det(g)^6$. One identifies the dual space of $V_{\mathbb{R}}$ with $\mathbb{R}^4$ via alternating pairing

$$\langle x, y\rangle = x_4 y_1 - \frac{1}{3} x_3 y_2 + \frac{1}{3} x_2 y_3 - x_1 y_4. \tag{2.17}$$

Let $\tau$ be the map $V_{\mathbb{R}} \to V_{\mathbb{R}}$ carrying each basis vector to its dual basis vector; the discriminant $\hat{D}$ on the dual space is normalized such that $\tau$ is discriminant-preserving. There is an involution $\iota$ on $G$ given by

$$g^\iota = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (g^{-1})^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.18}$$

This satisfies, for all $g \in G$, $x \in V_{\mathbb{R}}$, $y \in \hat{V}_{\mathbb{R}}$,

$$\langle x, y\rangle = \langle g \cdot x, g^\iota \cdot y\rangle. \tag{2.19}$$

The set of forms of zero discriminant are called the singular set, $S$. The non-singular forms split into spaces $V_+$ and $V_-$ of positive and negative discriminant. The space $V_+$ is a single $G^+$ orbit with representative $x_+ = (0,1,-1,0)$ and stability group

$$I_{x_+} = \left\{ I, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \right\}. \tag{2.20}$$

Set $x_+^0 = \lambda_+ x_+$, rescaled to have discriminant 1. $V_-$ is also a single $G^+$ orbit with representative $x_- = (0,1,0,1)$ with trivial stabilizer. $x_-^0 = \lambda_- x_-$ is also rescaled to have discriminant 1.

Set $w_1 = (0,0,1,0), w_2 = (0,0,0,1)$. The singular set is the disjoint union

$$S = \{0\} \sqcup G^1 \cdot w_1 \sqcup G^1 \cdot w_2. \tag{2.21}$$

The stability group for the action of $G^1$ on $w_1$ is trivial $I_{w_1} = \{1\}$, while on $w_2$ it is $I_{w_2} = N$.

Over $\mathbb{Z}$ write $L$ and $\hat{L}$ for the lattices of integral forms and their dual, and write $L_0$ and $\hat{L}_0$ for those integral forms, resp. dual forms, of discriminant zero. $L_0$ and $\hat{L}_0$ are the disjoint unions

$$L_0 = \{0\} \sqcup L_0(I) \sqcup L_0(II), \qquad \hat{L}_0 = \{0\} \sqcup L_0(I) \sqcup \hat{L}_0(II), \tag{2.22}$$

with

$$\begin{aligned} L_0(I) &= \bigsqcup_{m=1}^{\infty} \bigsqcup_{\Gamma/\Gamma\cap N} \gamma \cdot (0,0,0,m), \\ L_0(II) &= \bigsqcup_{m=1}^{\infty} \bigsqcup_{n=0}^{m-1} \bigsqcup_{\gamma\in\Gamma} \gamma \cdot (0,0,m,n), \\ \hat{L}_0(II) &= \bigsqcup_{m=1}^{\infty} \bigsqcup_{n=0}^{3m-1} \bigsqcup_{\gamma\in\Gamma} \gamma \cdot (0,0,3m,n). \end{aligned} \tag{2.23}$$

Given Schwarz function $f \in \mathscr{S}(V_{\mathbb{R}})$ one has the Fourier transforms

$$\hat{f}(x) = \int_{V_{\mathbb{R}}} f(y) e(\langle x, y\rangle) dy, \qquad f(x) = \frac{1}{9} \int_{V_{\mathbb{R}}} \hat{f}(y) e(\langle x, y\rangle) dy. \tag{2.24}$$

For $\ell \in \mathbb{R}_{>0}$ write $f_\ell(x) = f(\ell x)$. Say that $f$ is left-$K$-invariant if, for all $x \in V_{\mathbb{R}}$, for all $k \in K$, $f(k \cdot x) = f(x)$. One easily checks that $f$ and $\hat{f}$ are simultaneously left-$K$-invariant. Say that $f$ is right-$K$-invariant if, for both choices of $\pm$, for all $g \in G^+$, for all $k \in K$, $f(gk \cdot x_\pm) = f(g \cdot x_\pm)$. Let $\phi$ be a Maass form and let $f$ be right-$K$-invariant. Identify $f_\ell\left(g \cdot x_\pm^0\right)$ as functions $f_{\ell,\pm}$ on $G^1/I_{x_\pm}$. Interpret, for $h \in G^1$,

$$\int_{G^1} f_\ell\left(g \cdot x^0_{\operatorname{sgn} m}\right) \phi\left(gh\right) dg \tag{2.25}$$

as group convolution on $G^1$, written $\check{f}_{\ell,\pm} * \phi(h)$. The result obtained is left-$K$-invariant. Since the Laplacian and Hecke operators commute with translation, it follows by multiplicity 1 that $\check{f}_{\ell,\pm} * \phi = \lambda(f_{\ell,\pm}, \phi)\phi$ is a scalar multiple times $\phi$.

## 3. Dirichlet series

Let $\phi \in C(K\backslash G^1/\Gamma)$ be a Hecke-eigen Maass form and extend $\phi$ to $G$ by projecting onto $G^1$. Note that this means that $\phi(g) = \phi(g^\iota)$ since $g$ and $g^\iota$ differ by a scalar. Let $f \in \mathscr{S}(V_{\mathbb{R}})$. Adapting Shintani's construction, introduce orbital integrals

$$\begin{aligned} Z(f, L; s, \phi) &= \int_{G_+/\Gamma} \chi(g)^s \phi\left(g\right) \sum_{x\in L\backslash L_0} f(g \cdot x) dg, \\ Z(f, \hat{L}; s, \phi) &= \int_{G_+/\Gamma} \chi(g)^s \phi\left(g\right) \sum_{x\in \hat{L}\backslash \hat{L}_0} f(g \cdot x) dg. \end{aligned} \tag{3.1}$$

**Lemma 3.1.** *Let $f \in \mathscr{S}(V_{\mathbb{R}})$ be right-$K$-invariant. Let $\phi$ be a Maass form satisfying, for $\ell > 0$,*

$$\check{f}_{\ell,\pm} * \phi = \lambda(f_{\ell,\pm}, \phi)\phi. \tag{3.2}$$

*For $\Re(s)$ sufficiently large, the orbital integrals $Z(f,L;s,\phi), Z(f,\hat{L};s,\phi)$ satisfy*

$$\begin{aligned} Z(f,L;s,\phi) =& \mathscr{L}_+(L,s;\phi)\int_0^\infty \lambda(f_{\ell,+},\phi)\ell^{12s-1}d\ell \\ &+ \mathscr{L}_-(L,s;\phi)\int_0^\infty \lambda(f_{\ell,-},\phi)\ell^{12s-1}d\ell \\ Z(f,\hat{L};s,\phi) =& \mathscr{L}_+(\hat{L},s;\phi)\int_0^\infty \lambda(f_{\ell,+},\phi)\ell^{12s-1}d\ell \\ &+ \mathscr{L}_-(\hat{L},s;\phi)\int_0^\infty \lambda(f_{\ell,-},\phi)\ell^{12s-1}d\ell. \end{aligned} \tag{3.3}$$

*Proof.* One finds for $\Re(s)$ sufficiently large

$$\begin{aligned} Z(f,L;s,\phi) &= \int_{G_+/\Gamma}\sum_{m\neq 0}\sum_{i=1}^{h(m)}\frac{\chi(g)^s\phi(g)}{|\Gamma(i,m)|}\sum_{\gamma\in\Gamma} f(g\gamma g_{i,m}\cdot x^0_{\operatorname{sgn} m})dg \\ &= \sum_{m\neq 0}\frac{1}{|m|^s}\sum_{i=1}^{h(m)}\frac{\ell^{12s-1}}{|\Gamma(i,m)|}\int_0^\infty\int_{G^1} f_\ell\left(g\cdot x^0_{\operatorname{sgn} m}\right)\phi\left(gg^{-1}_{i,m}\right)dgd\ell \\ &= \sum_{m\neq 0}\frac{1}{|m|^s}\sum_{i=1}^{h(m)}\frac{\phi\left(g^{-1}_{i,m}\right)}{|\Gamma(i,m)|}\int_0^\infty \lambda(f_{\ell,\operatorname{sgn} m},\phi)\ell^{12s}\frac{d\ell}{\ell} \\ &= \mathscr{L}_+(L,s;\phi)\int_0^\infty \lambda(f_{\ell,+},\phi)\ell^{12s}\frac{d\ell}{\ell} + \mathscr{L}_-(L,s;\phi)\int_0^\infty \lambda(f_{\ell,-},\phi)\ell^{12s}\frac{d\ell}{\ell}. \end{aligned} \tag{3.4}$$

The proof for the dual $\mathscr{L}$-functions is the same. □

Following Shintani, introduce

$$\begin{aligned} Z^+(f,L;s,\phi) &= \int_{G_+/\Gamma,\chi(g)\geqslant 1}\chi(g)^s\phi(g)\sum_{x\in L\backslash L_0} f(g\cdot x)dg, \\ Z^+(f,\hat{L};s,\phi) &= \int_{G_+/\Gamma,\chi(g)\geqslant 1}\chi(g)^s\phi(g)\sum_{x\in \hat{L}\backslash \hat{L}_0} f(g\cdot x)dg. \end{aligned} \tag{3.5}$$

These functions converge absolutely and are entire.

The following proposition is the analogue of [12, Proposition 2.14].

**Proposition 3.2.** *For $\Re(s) > 4$,*

$$\begin{aligned} Z(f,L;s,\phi) &= Z^+(f,L;s,\phi) + Z^+(\hat{f},\hat{L};1-s,\phi) \\ &- \int_{\substack{G_+/\Gamma,\\ \chi(g)<1}}\chi(g)^s\phi(g)\left\{\sum_{x\in L_0} f(g\cdot x) - \chi(g)^{-1}\sum_{x\in\hat{L}_0}\hat{f}(g^\iota\cdot x)\right\}dg, \end{aligned} \tag{3.6}$$

$$\begin{aligned} Z(f,\hat{L};s,\phi) &= Z^+(f,\hat{L};s,\phi) + \frac{1}{9}Z^+(\hat{f},L;1-s,\phi) \\ &- \int_{\substack{G_+/\Gamma,\\ \chi(g)<1}}\chi(g)^s\phi(g)\left\{\sum_{x\in \hat{L}_0} f(g\cdot x) - \frac{1}{9}\chi(g)^{-1}\sum_{x\in L_0}\hat{f}(g^\iota\cdot x)\right\}dg. \end{aligned} \tag{3.7}$$

*Proof.* Write

$$Z(f,L;s,\phi) = \int_{G_+/\Gamma} \chi(g)^s \phi(g) \sum_{x\in L} f(g\cdot x)dg - \int_{G_+/\Gamma} \chi(g)^s \phi(g) \sum_{x\in L_0} f(g\cdot x)dg. \tag{3.8}$$

Split the first integral at $\det(g) \geqslant 1$. In the integral with $\det(g) < 1$ perform Poisson summation in the sum over $L$, using that $F_g(x) = f(g\cdot x)$ has $\hat{F}_g(y) = \frac{1}{\chi(g)}\hat{f}(g^\iota \cdot y)$. The proof for $\hat{L}$ is similar. □

The objective now is to give the holomorphic continuation of (3.6) and (3.7). This closely follows the evaluation of Shintani leading up to the Corollary to Proposition 2.16 of [12].

Given $f \in \mathscr{S}(V_{\mathbb{R}})$ which is left-$K$-invariant, introduce distributions, for $z, z_1, z_2 \in \mathbb{C}$ and $u \in \mathbb{R}$,

$$\begin{aligned} \Sigma_1(f,z_1,z_2) &= \int_0^\infty \int_0^\infty (f(0,0,t,u) + f(0,0,t,-u)) t^{z_1-1} u^{z_2-1} dt du, \\ \Sigma_2(f,z) &= \int_0^\infty f(0,0,0,u) u^{z-1} du, \\ \Sigma_3(f,z,u) &= \int_0^\infty f(0,0,t,u) t^{z-1} dt. \end{aligned} \tag{3.9}$$

Following Shintani, for $g \in G^1/\Gamma$ define

$$J_L(f)(g) = \sum_{x\in L_0} f(g\cdot x), \qquad J_{\hat{L}}(f)(g) = \sum_{x\in \hat{L}_0} f(g\cdot x). \tag{3.10}$$

It follows from [12, Lemma 2.10] that for $f \in \mathscr{S}(V_{\mathbb{R}})$, for $\phi$ of at most polynomial growth, $\phi(g)J_L(f)(g)$ and $\phi(g)J_{\hat{L}}(\hat{f})(g)$ have at most polynomial growth, while

$$\phi(g)\left(J_L(f)(g) - J_{\hat{L}}(\hat{f})(g)\right) \tag{3.11}$$

is a Schwarz class function on $G^1/\Gamma$.

The starting point is the formula (see e.g. [12, p. 174])

$$\begin{aligned} &\frac{\psi(1)}{\xi(2)} \int_{G^1/\Gamma} \left(J_L(f)(g) - J_{\hat{L}}(\hat{f})(g)\right) \phi(g) dg \\ &= \lim_{w\downarrow 1} (w-1) \int_{G^1/\Gamma} \left(J_L(f)(g) - J_{\hat{L}}(\hat{f})(g)\right) \phi(g) \mathscr{E}(\psi,w;g) dg. \end{aligned} \tag{3.12}$$

We have the following evaluation of integrals.

**Lemma 3.3.** *Let $\phi$ be a Maass form. Then*

$$\int_{G^1/\Gamma} \mathscr{E}(\psi,w;g)\phi(g)dg = 0. \tag{3.13}$$

*Proof.* Let $1 < c < \Re(w)$. Opening $\mathscr{E}(\psi, w; g)$ as a contour integral, then unfolding the Eisenstein series, one obtains

$$\int_{G^1/\Gamma} \mathscr{E}(\psi, w; g)\phi(g)dg = \frac{1}{2}\int_{G^1/\Gamma\cap N}\left(\oint_{\Re(z)=c} \frac{t(g)^{1+z}}{w-z}\psi(z)dz\right)\phi(g)dg. \tag{3.14}$$

This integral now vanishes by integrating in the parabolic direction, since the Maass form has no constant term. □

Let $\psi_1, \psi_2$ be two holomorphic functions in the half-plane $\Re(w) > 4$. Say these functions are equivalent $\psi_1 \sim \psi_2$ if $(\psi_1 - \psi_2)$ may be meromorphically continued to $\Re(w) > 0$ and is holomorphic in a neighborhood of $w = 1$. Equivalent functions are interchangeable in the integrand of (3.12).

Let $\phi \in C(K\backslash G^1/\Gamma)$. Set

$$\begin{aligned}
\Theta_\psi^{(1)}(w;\phi) &= \int_{G^1/\Gamma} \mathscr{E}(\psi, w; g)\phi(g) \sum_{x\in L_0(I)} f(g\cdot x)dg,\\
\Theta_\psi^{(2)}(w;\phi) &= \int_{G^1/\Gamma} \mathscr{E}(\psi, w; g)\phi(g) \sum_{x\in L_0(II)} f(g\cdot x)dg,\\
\hat{\Theta}_\psi^{(2)}(w;\phi) &= \int_{G^1/\Gamma} \mathscr{E}(\psi, w; g)\phi(g) \sum_{x\in \hat{L}_0(II)} f(g\cdot x)dg.
\end{aligned} \tag{3.15}$$

Also, write $\phi_c(t)$ for its constant term, found by integrating away the parabolic direction.

**Lemma 3.4.** *Let $f \in \mathscr{S}(V_\mathbb{R})$ be left-$K$-invariant. Given Maass form $\phi$,*

$$\Theta_\psi^{(1)}(w;\phi) \sim 0. \tag{3.16}$$

*Proof.* Let $\Re(w) > 2$. Write

$$\Theta_\psi^{(1)}(w;\phi) = \int_{G^1/\Gamma} \sum_{m=1}^{\infty} \sum_{\Gamma/\Gamma\cap N} f(g\gamma\cdot(0,0,0,m))\mathscr{E}(\psi, w; g)\phi(g)dg. \tag{3.17}$$

Introduce the Dirichlet series

$$F_\phi(u;z) = \sum_{n\geqslant 1} \frac{\eta_{\frac{z}{2}}(n)\rho_\phi(n)}{n^u}, \tag{3.18}$$

which converges absolutely in $\Re(u) - \frac{|\Re(z)|}{2} > 1$. After unfolding the sum over $\Gamma/\Gamma \cap N$ and integrating in the compact and parabolic directions, this becomes (see [12, p. 178], middle display, for the first evaluation)

$$\begin{aligned}
(3.19)\quad \Theta_\psi^{(1)}(w;\phi) &= \sum_{m=1}^{\infty} \int_0^\infty f(0,0,0,t^{-3}m) \left( \oint_{\Re(z)=2} \frac{(E(z;\cdot)\phi)_c(t)}{w-z} \psi(z) dz \right) \frac{dt}{t^3} \\
&= 4 \sum_{m=1}^{\infty} \int_0^\infty f(0,0,0,t^{-3}m) \\
&\times \left( \oint_{\Re(z)=2} \frac{\left(\sum_{n\geqslant 1} \eta_{\frac{z}{2}}(n)\rho_\phi(n) K_{\frac{z}{2}}(2\pi n t^2) K_{s-\frac{1}{2}}(2\pi n t^2)\right)}{\xi(z+1)(w-z)} \psi(z) dz \right) \frac{dt}{t} \\
&= 4 \oint\!\!\oint_{\substack{\Re(u,z)\\=(3,2)}} \Sigma_2(f,u) \int_0^\infty \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\eta_{\frac{z}{2}}(n)\rho_\phi(n)}{\xi(z+1)(w-z)} \\
&\quad \times \left(\frac{t^3}{m}\right)^u K_{\frac{z}{2}}(2\pi n t^2) K_{s-\frac{1}{2}}(2\pi n t^2) \psi(z) \frac{dt}{t} dz du \\
&= 4 \oint\!\!\oint_{\substack{\Re(u,z)\\=(3,2)}} \int_0^\infty \frac{\Sigma_2(f,u)\zeta(u) F_\phi\left(\frac{3u}{2};z\right)}{\xi(z+1)(w-z)} \psi(z) t^{3u} K_{\frac{z}{2}}(2\pi t^2) K_{s-\frac{1}{2}}(2\pi t^2) \frac{dt}{t} dz du.
\end{aligned}$$

Shift the $z$ contour to $\Re(z) = 0$ to verify that $\Theta_\psi^{(1)}(w;\phi)$ is holomorphic in $\Re(w) > 0$. □

Introduce

$$(3.20)\qquad G_\phi(x) = \sum_{\ell,m=1}^{\infty} \frac{\rho_\phi(\ell m)}{\ell^{1+x} m^{1+3x}}.$$

Form $\hat{G}_\phi$ by dilating the sum over $m$ by 3.

**Lemma 3.5.** *Given Maass form $\phi$, $G_\phi(x)$ is holomorphic in the half-plane $\Re(x) > \frac{-1}{4}$.*

*Proof.* Let $L_p(s,\phi) = \sum_{n\geqslant 0} \frac{\rho_\phi(p^n)}{p^{ns}}$ be the local factor in the $L$-function $L(s,\phi) = \prod_p L_p(s,\phi)$ in $\Re(s) > 1$. For $\Re(x) > -\frac{1}{4}$, write the local factor at prime $p$ in $G_\phi(x)$ as

$$\begin{aligned}
(3.21)\qquad G_{\phi,p}(x) =& L_p(1+x,\phi) L_p(1+3x,\phi) \\
&\times \left(1 + \frac{\rho_\phi(p^2) - \rho_\phi(p)^2}{p^{2+4x}} + O\left(\frac{(1+|\rho_\phi(p)|)^3}{p^{3+7x}}\right)\right).
\end{aligned}$$

It follows that $G_\phi(x) = L(1+x,\phi)L(1+3x,\phi)H_\phi(x)$ where $H_\phi$ is given by an absolutely convergent Euler product in $x > -\frac{1}{4}$. □

The Archimedean counterpart to $G_\phi$ is

$$W_\phi(w_1,w_2) = \frac{2^{w_2-3}}{\pi^{\frac{1+w_1+w_2}{2}}}\Gamma(1-w_2)\cos\left(\frac{\pi}{2}(1-w_2)\right) \times \Gamma\left(\frac{-1+w_1+3w_2+2it_\phi}{4}\right)\Gamma\left(\frac{-1+w_1+3w_2-2it_\phi}{4}\right) \tag{3.22}$$

which is holomorphic in $\{w_1,w_2 : \Re(w_1+3w_2)>1, \Re(w_2)<1\}$.

**Lemma 3.6.** *Let $f \in \mathscr{S}(V_\mathbb{R})$ be left-$K$-invariant. Given Maass form $\phi$,*

$$\Theta_\psi^{(2)}(w;\phi) \sim \frac{\psi(1)}{\xi(2)(w-1)} \oint\!\!\!\oint_{\Re(w_1,w_2)=(1,\frac{1}{2})} \Sigma_1(f,w_1,w_2) \times W_\phi(w_1,w_2)G_\phi\left(\frac{w_1+w_2-1}{2}\right)dw_1dw_2. \tag{3.23}$$

*To obtain the corresponding terms for $\hat{\Theta}_\psi^{(2)}(w;\cdot)$ replace $G$ with $\hat{G}$.*

*Proof.* Calculate (see [12, p. 179], next to last display)

(3.24)

$$\begin{aligned}
&\Theta_\psi^{(2)}(w;\phi)\\
&= \int_{G^1/\Gamma} \mathscr{E}(\psi,w;g)\phi(g)\sum_{m=1}^\infty\sum_{n=-\infty}^\infty\sum_{\gamma\in\Gamma/\Gamma\cap N} f(g\gamma\cdot(0,0,m,n))dg\\
&= \int_0^\infty\int_0^1\sum_{m=1}^\infty\sum_{n=-\infty}^\infty f(a_t\cdot(0,0,m,n+mu)) \oint_{\Re(z)=5}\frac{E(z,g)\phi(g)}{w-z}\psi(z)dzdu\frac{dt}{t^3}.
\end{aligned}$$

In the Eisenstein series, separate the constant term, writing $\tilde{E}(z,g) = E(z,g) - E(z,g)_c$. The contribution of the non-constant part of $\tilde{E}(z,g)\phi(g)$ is holomorphic in $\Re(w)>0$ by tracing [12, p. 180], top.

The contribution of $(\tilde{E}(z,g)\phi(g))_c$ is given by

$$\begin{aligned}
&\frac{4}{\xi(z+1)}\int_0^\infty\int_{-\infty}^\infty \oint_{\Re(z)=5}\sum_{m=1}^\infty f(0,0,t^{-1}m,u)\frac{\psi(z)}{w-z}\\
&\quad\times\left(\sum_{n=1}^\infty \rho_\phi(n)\eta_{\frac{z}{2}}(n)K_{\frac{z}{2}}(2\pi nt^2)K_{s-\frac{1}{2}}(2\pi nt^2)\right)dzdut^2dt\\
&= \frac{2}{\xi(z+1)}\int_0^\infty\int_{-\infty}^\infty \oint\!\!\!\oint_{\substack{\Re(z,z')\\=(5,5)}} \Sigma_3(f,z',u)\frac{\zeta(z')F_\phi(\frac{3+z'}{2};z)}{(2\pi)^{\frac{3+z'}{2}}}\frac{\psi(z)}{w-z}\\
&\quad\times K_{\frac{z}{2}}(t)K_{s-\frac{1}{2}}(t)t^{\frac{3+z'}{2}}dz'dzdu\frac{dt}{t}.
\end{aligned} \tag{3.25}$$

Shift the $z$ contour to $\Re(z)=0$ to verify that this is holomorphic in $\Re(w)>0$.

From the constant term of $E(z,g)$, only the term $\frac{\xi(z)}{\xi(z+1)}t^{1-z}$ contributes, and from this term one picks up a pole at $z=1$. Following Shintani, this yields

$$\Theta_\psi^{(2)}(w;\phi) \sim \frac{2\psi(1)}{\xi(2)(w-1)}\int_0^\infty\int_{-\infty}^\infty \sum_{\ell,m=1}^\infty \rho_\phi(\ell m) \times K_{s-\frac{1}{2}}(2\pi\ell m t^2) f(0,0,t^{-1}m,u)\cos(2\pi\ell t^3 u)\,du\,dt\,dt. \tag{3.26}$$

Split the integral over $u$ by writing $f(0,0,*,u) = f_+(0,0,*,u)$ for $u>0$ and $f(0,0,*,u) = f_-(0,0,*,-u)$ for $u<0$. Now open $f$ by taking Mellin transforms in both variables,

$$\Theta_\psi^{(2)}(w;\phi) \sim \frac{2\psi(1)}{\xi(2)(w-1)}\int_0^\infty\int_0^\infty \oint\!\!\oint_{\substack{\Re(w_1,w_2)\\=(1,\frac{1}{2})}} \Sigma_1(f,w_1,w_2)\sum_{\ell,m=1}^\infty \frac{\rho_\phi(\ell m)}{m^{w_1}} \times K_{s-\frac{1}{2}}(2\pi\ell m t^2)\cos(2\pi\ell t^3 u)u^{-w_2}t^{1+w_1}\,dw_1\,dw_2\,du\,dt. \tag{3.27}$$

Replace $u := 2\pi\ell t^3 u$, then $t := 2\pi\ell m t^2$ to obtain

$$\begin{aligned}\Theta_\psi^{(2)}(w;\phi) &\sim \frac{\psi(1)}{\xi(2)(w-1)}\oint\!\!\oint_{\substack{\Re(w_1,w_2)\\=(1,\frac{1}{2})}} \Sigma_1(f,w_1,w_2)\frac{G_\phi\left(\frac{w_1+w_2-1}{2}\right)}{(2\pi)^{\frac{1+w_1+w_2}{2}}} \\ &\times \int_0^\infty\int_0^\infty K_{s-\frac{1}{2}}(t)\cos(u)\frac{du}{u^{w_2}}\frac{dt}{t^{\frac{3-w_1-3w_2}{2}}}\,dw_1\,dw_2 \\ &\sim \frac{\psi(1)}{\xi(2)(w-1)} \\ &\times \oint\!\!\oint_{\substack{\Re(w_1,w_2)\\=(1,\frac{1}{2})}} \Sigma_1(f,w_1,w_2)W_\phi(w_1,w_2)G_\phi\left(\frac{w_1+w_2-1}{2}\right)dw_1\,dw_2.\end{aligned} \tag{3.28}$$

□

Putting together the above lemmas we conclude

$$\begin{aligned}&\int_{G^1/\Gamma}\left(J_L(f) - J_{\hat{L}}(\hat{f})\right)\phi(g)\,dg \\ &= \oint\!\!\oint_{\substack{\Re(w_1,w_2)\\=(1,\frac{1}{2})}} \Sigma_1(f,w_1,w_2)W_\phi(w_1,w_2)G_\phi\left(\frac{w_1+w_2-1}{2}\right)dw_1\,dw_2 \\ &- \text{ terms replacing } f,G \text{ with } \hat{f},\hat{G}.\end{aligned} \tag{3.29}$$

We now holomorphically extend the orbital integrals. Note that

$$\Sigma_1(f_t,w_1,w_2) = t^{-w_1-w_2}\Sigma_1(f,w_1,w_2). \tag{3.30}$$

*Proof of Theorem* 1.1. Choose $f$ which is bi-$K$-invariant, and arrange $f$ such that the integrals $\int_0^\infty \lambda(f_{\ell,\pm},\phi)\ell^{12s}\frac{d\ell}{\ell}$ are entire. Since $Z^+$ is entire, it suffices to consider

the integral

$$-\int_{G^+/\Gamma,\chi(g)\leqslant 1}\chi(g)^s\phi(g)\left\{\sum_{x\in L_0}f(g\cdot x)-\chi^{-1}(g)\sum_{x\in\hat{L}_0}\hat{f}(g^\iota x)\right\}dg \tag{3.31}$$
$$=-\int_0^1 t^{12s}\int_{G^1/\Gamma}\phi(g_1)\left\{\sum_{x\in L_0}f_{t^3}(g_1\cdot x)-\sum_{x\in\hat{L}_0}\widehat{f_{t^3}}(g_1\cdot x)\right\}dg_1\frac{dt}{t}.$$

The contribution from $f$ may be expressed

$$-\int_0^1\oint\!\!\oint_{\Re(w_1,w_2)=(1,\frac{1}{2})} t^{12s-3w_1-3w_2}\Sigma_1(f,w_1,w_2) \tag{3.32}$$
$$\times W_\phi(w_1,w_2)G_\phi\left(\frac{w_1+w_2-1}{2}\right)dw_1dw_2\frac{dt}{t}.$$

Shift the $w_1$ contour left to $\Re(w_1)=\epsilon$. This expression is holomorphic in $\Re(s)>\frac{1}{8}+\epsilon$. The contribution from $\hat{f}$ may be expressed (see [12, p. 182])

$$\int_0^1\oint\!\!\oint_{\Re(w_1,w_2)=(1,\frac{1}{2})} t^{12s-12+3w_1+3w_2}\Sigma_1(\hat{f},w_1,w_2) \tag{3.33}$$
$$\times W_\phi(w_1,w_2)\hat{G}_\phi\left(\frac{w_1+w_2-1}{2}\right)dw_1dw_2\frac{dt}{t}.$$

In this integral, integration with respect to $w_1$ may be pushed right as far as we like, so that the integral itself is holomorphic. □

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