

INCLUSION RELATIONS BETWEEN MODULATION AND TRIEBEL-LIZORKIN SPACES

WEICHAO GUO, HUOXIONG WU, AND GUOPING ZHAO

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ABSTRACT. In this paper, we obtain the sharp conditions of the inclusion relations between modulation spaces $M_{p,q}^s$ and Triebel-Lizorkin spaces $F_{p,r}$ for $p \leq 1$, which greatly improve and extend the results for the embedding relations between local Hardy spaces and modulation spaces obtained by Kobayashi, Miyachi and Tomita in [Studia Math. **192** (2009), 79–96].

1. INTRODUCTION

The modulation spaces $M_{p,q}^s$ were introduced by Feichtinger [2] in 1983 by the short-time Fourier transform. One can find some motivations and basic properties in [3]. Modulation spaces have a close relationship with the topics of time-frequency analysis (see [6]), and they have been regarded as appropriate spaces for the study of PDE (see [16]). As function spaces associated with the uniform decomposition, modulation spaces have many beautiful properties, for instance, the properties of product and convolution on modulation spaces (see [4, 9, 10]), and the boundedness of unimodular multipliers on modulation spaces (see [1, 14, 28]). One can also see [8, 24, 26, 27] for the study of nonlinear evolution equations on modulation spaces.

A basic but important problem is to find the relations between modulation spaces and classical function spaces. Many authors have paid attention to this topic, for example, one can see Gröbner [5], Okoudjou [15], Toft-Wahlberg [19], Toft [20], Sugimoto-Tomita [18], Kobayashi-Sugimoto [13] and Wang-Huang [25]. In particular, Kobayashi-Miyachi-Tomita [12] studied the inclusion relations between local Hardy spaces h_p and modulation spaces $M_{p,q}^s$ and established the following results.

Theorem A (cf. [12]). *Let $0 < p \leq 1$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. Then, $M_{p,q}^s \subset h_p$ if and only if one of the following conditions holds:*

- (1) $1/p \leq 1/q, s \geq 0$;
- (2) $1/p > 1/q, s > n(1/p - 1/q)$.

Theorem B (cf. [12]). *Let $0 < p \leq 1$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. Then, $h_p \subset M_{p,q}^s$ if and only if one of the following conditions holds:*

- (1) $1/q \leq 1/p, s \leq n(1 - 1/p - 1/q)$;
- (2) $1/q > 1/p, s < n(1 - 1/p - 1/q)$.

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We recall that the local Hardy spaces h_p ($0 < p < \infty$), which were introduced by Goldberg in [7], is equivalent to the inhomogeneous Triebel-Lizorkin space $F_{p,r}$ with $r = 2$. Note that the Triebel-Lizorkin spaces contain many important function spaces, such as Lebesgue spaces, Sobolev spaces, and Hardy spaces. It is natural to ask whether the inclusion relations above in Theorems A and B still hold if replaced $F_{p,2}$ by the general $F_{p,r}$ for $r > 0$. This question will be addressed by the following theorems.

Theorem 1.1. *Let $0 < p \leq 1$, $0 < q, r \leq \infty$ and $s \in \mathbb{R}$. Then, $M_{p,q}^s \subset F_{p,r}$ if and only if one of the following conditions holds:*

- (1) $1/p \leq 1/q, s \geq 0, 1/r \leq 1/q$;
- (2) $1/p > 1/q, s > n(1/p - 1/q)$.

Theorem 1.2. *Let $0 < p \leq 1$, $0 < q, r \leq \infty$ and $s \in \mathbb{R}$. Then, $F_{p,r} \subset M_{p,q}^s$ if and only if one of the following conditions holds:*

- (1) $1/q \leq 1/p, s \leq n(1 - 1/p - 1/q)$;
- (2) $1/q > 1/p, s < n(1 - 1/p - 1/q)$.

Remark 1.3. To make the reader more clearly understand our work in this paper, we would like to make a comparison between our theorems and the main results in [12], i.e., Theorems A and B.

Clearly, Theorems 1.1 and 1.2 in this paper are a great improvement and extension of Theorems A and B. Also, at the technical level, we have to work under the framework of Triebel-Lizorkin spaces, which is a lack of such a simple form of atoms as in h_p , and then the most important problem we face is how to estimate the more complicated atoms of Triebel-Lizorkin spaces in modulation spaces. In addition, we drop the method used in [12], which deeply depends on an equivalent norm of modulation space (see Lemma 2.2 in [12]), and use the “standard” norm of modulation spaces. We would like to give an independent proof (quite different from [12]) on both sufficiency and necessity parts, which seems more efficient and is clearer than those in [12].

This paper is organized as follows. In Section 2, we will recall some basic notation and definitions, and present some preliminary lemmas, which will be used in our proofs. The proofs of our main results will be given in Section 3. Also, we will end this paper by presenting an open problem and its difficulties.

2. PRELIMINARIES

In this section, we first recall some notation. Let C be a positive constant that may depend on n, p, q, r, s . The notation $X \lesssim Y$ denotes the statement that $X \leq CY$, the notation $X \sim Y$ means the statement $X \lesssim Y \lesssim X$, and the notation $X \simeq Y$ denotes the statement $X = CY$. For a multi-index $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$, we denote $|k|_\infty := \sup_{i=1,2,\dots,n} |k_i|$, and $\langle k \rangle := (1 + |k|^2)^{1/2}$.

Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ be the Schwartz space and $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^n)$ be the space of tempered distributions. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi ix \cdot \xi} dx, \quad \mathcal{F}^{-1}f(x) = \hat{f}(-x) = \int_{\mathbb{R}^n} f(\xi)e^{2\pi ix \cdot \xi} d\xi.$$

We recall some definitions of the function spaces treated in this paper.

Definition 2.1. Let $s \in \mathbb{R}, 0 < p, q \leq \infty$. The weighted Lebesgue space $L_{x,p}^s$ consists of all measurable functions f such that

$$(2.1) \quad \|f\|_{L_{x,p}^s} = \begin{cases} \left(\int_{\mathbb{R}^n} |f(x)|^p \langle x \rangle^{ps} dx \right)^{1/p}, & p < \infty, \\ ess \sup_{x \in \mathbb{R}^n} |f(x) \langle x \rangle^s|, & p = \infty, \end{cases}$$

is finite. If f is defined on \mathbb{Z}^n , we denote

$$(2.3) \quad \|f\|_{l_{k,p}^{s,0}} = \begin{cases} \left(\sum_{k \in \mathbb{Z}^n} |f(k)|^p \langle k \rangle^{ps} \right)^{1/p}, & p < \infty, \\ \sup_{k \in \mathbb{Z}^n} |f(k) \langle k \rangle^s|, & p = \infty, \end{cases}$$

and $l_{k,p}^{s,0}$ as the (quasi) Banach space of functions $f : \mathbb{Z}^n \rightarrow \mathbb{C}$ whose $l_{k,p}^{s,0}$ norm is finite. If f is defined on $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$, we denote

$$(2.5) \quad \|f\|_{l_{j,p}^{s,1}} = \begin{cases} \left(\sum_{j \in \mathbb{N}} 2^{jsp} |f(j)|^p \right)^{1/p}, & p < \infty, \\ \sup_{j \in \mathbb{N}} |2^{js} f(j)|, & p = \infty, \end{cases}$$

and $l_{j,p}^{s,1}$ as the (quasi-) Banach space of functions $f : \mathbb{N} \rightarrow \mathbb{C}$ whose $l_{j,p}^{s,1}$ norm is finite. We write $L_p^s, l_p^{s,0}, l_p^{s,1}$ for short, respectively, if there is no confusion.

The translation operator is defined as $T_{x_0} f(x) = f(x - x_0)$ and the modulation operator is defined as $M_\xi f(x) = e^{2\pi i \xi \cdot x} f(x)$, for $x, x_0, \xi \in \mathbb{R}^n$. Fix a nonzero function $\phi \in \mathcal{S}$. The short-time Fourier transform of $f \in \mathcal{S}'$ with respect to the window ϕ is given by

$$(2.7) \quad V_\phi f(x, \xi) = \langle f, M_\xi T_x \phi \rangle,$$

and that can be written as

$$(2.8) \quad V_\phi f(x, \xi) = \int_{\mathbb{R}^n} f(y) \overline{\phi(y - x)} e^{-2\pi i y \cdot \xi} dy$$

if $f \in \mathcal{S}$. We give the (continuous) definition of modulation space $\mathcal{M}_{p,q}^s$ as follows.

Definition 2.2. Let $s, t \in \mathbb{R}, 0 < p, q \leq \infty$. The (weighted) modulation space $\mathcal{M}_{p,q}^s$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the (weighted) modulation space norm

$$(2.9) \quad \begin{aligned} \|f\|_{\mathcal{M}_{p,q}^s} &= \|\|V_\phi f(x, \xi)\|_{L_{x,p}}\|_{L_{\xi,q}^s} \\ &= \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_\phi f(x, \xi)|^p dx \right)^{q/p} \langle \xi \rangle^{sq} d\xi \right)^{1/q} \end{aligned}$$

is finite, with the usual modifications when $p = \infty$ or $q = \infty$. This definition is independent of the choice of the window $\phi \in \mathcal{S}$.

Applying the frequency-uniform localization techniques, one can give an alternative definition of modulation spaces (see [21, 27] for details).

We denote by Q_k the unit cube with the center at k . Then the family $\{Q_k\}_{k \in \mathbb{Z}^n}$ constitutes a decomposition of \mathbb{R}^n . Let $\rho \in \mathcal{S}(\mathbb{R}^n)$, $\rho : \mathbb{R}^n \rightarrow [0, 1]$ be a smooth function satisfying that $\rho(\xi) = 1$ for $|\xi|_\infty \leq 1/2$ and $\rho(\xi) = 0$ for $|\xi| \geq 3/4$. Let

$$(2.10) \quad \rho_k(\xi) = \rho(\xi - k), k \in \mathbb{Z}^n$$

be a translation of ρ . Since $\rho_k(\xi) = 1$ in Q_k , we have that $\sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \geq 1$ for all $\xi \in \mathbb{R}^n$. Denote

$$(2.11) \quad \sigma_k(\xi) = \rho_k(\xi) \left(\sum_{l \in \mathbb{Z}^n} \rho_l(\xi) \right)^{-1}, \quad k \in \mathbb{Z}^n.$$

It is easy to know that $\{\sigma_k(\xi)\}_{k \in \mathbb{Z}^n}$ constitutes a smooth decomposition of \mathbb{R}^n , and $\sigma_k(\xi) = \sigma(\xi - k)$. The frequency-uniform decomposition operators can be defined by

$$(2.12) \quad \square_k := \mathcal{F}^{-1} \sigma_k \mathcal{F}$$

for $k \in \mathbb{Z}^n$. Now, we give the (discrete) definition of modulation space $M_{p,q}^s$.

Definition 2.3. Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$. The modulation space $M_{p,q}^s$ consists of all $f \in \mathcal{S}'$ such that the (quasi)-norm

$$(2.13) \quad \|f\|_{M_{p,q}^s} := \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \|\square_k f\|_p^q \right)^{1/q}$$

is finite. We write $M_{p,q} := M_{p,q}^0$ for short.

Remark 2.4. We remark that the above definition is independent of the choice of σ (see [26]). So, we can use appropriate σ according to our need. In the definition above, the function sequence $\{\sigma_k(\xi)\}_{k \in \mathbb{Z}^n}$ satisfies $\sigma_k(\xi) = 1$ and $\sigma_k(\xi)\sigma_l(\xi) = 0$ if $k \neq l$, where $|\xi - k|_\infty \leq 1/4$. We also recall that the definition of $M_{p,q}^s$ and $M_{p,q}^s$ are equivalent. In this paper, we mainly use the discrete definition of modulation space, i.e., $M_{p,q}^s$.

Next, we recall function spaces associated with the dyadic decomposition of \mathbb{R}^n . Let $\varphi(\xi)$ be a smooth bump function supported in the ball $\{\xi : |\xi| < \frac{3}{2}\}$ and be equal to 1 on the ball $\{\xi : |\xi| \leq \frac{4}{3}\}$. Denote

$$(2.14) \quad \psi(\xi) = \varphi(\xi) - \varphi(2\xi),$$

and a function sequence

$$(2.15) \quad \begin{cases} \psi_j(\xi) = \psi(2^{-j}\xi), & j \in \mathbb{Z}^+, \\ \psi_0(\xi) = 1 - \sum_{j \in \mathbb{Z}^+} \psi_j(\xi) = \varphi(\xi). \end{cases}$$

For integers $j \in \mathbb{N}$, we define the Littlewood-Paley operators

$$(2.16) \quad \Delta_j = \mathcal{F}^{-1} \psi_j \mathcal{F}.$$

Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'$, set

$$(2.17) \quad \|f\|_{B_{p,q}^s} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\Delta_j f\|_{L_p}^q \right)^{1/q}.$$

The (inhomogeneous) Besov space is the space of all tempered distributions f for which the quantity $\|f\|_{B_{p,q}^s}$ is finite.

Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'$, set

$$(2.18) \quad \|f\|_{F_{p,q}^s} = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\Delta_j f|^q \right)^{1/q} \right\|_{L_p}.$$

The (inhomogeneous) Triebel-Lizorkin space is the space of all tempered distributions f for which the quantity $\|f\|_{F_{p,q}^s}$ is finite.

Let \mathbb{Q}^n be the collection of all cubes $Q_{\nu,k}$ in \mathbb{R}^n with sides parallel to the axes, centered at $2^{-\nu}k$, and with side length $l(Q_{\nu,k}) = 2^{-\nu}$, where $k \in \mathbb{Z}^n$, $\nu \in \mathbb{N}$.

Let $r > 0$. We use rQ to denote the cube in \mathbb{R}^n concentric with Q satisfying $l(rQ) = rl(Q)$. We write $(\nu, k) < (\nu', k')$ if $\nu \geq \nu'$ and

$$(2.19) \quad Q_{\nu,k} \subset 2Q_{\nu',k'} \quad \text{with } Q_{\nu,k}, Q_{\nu',k'} \in \mathbb{Q}^n.$$

For $c \in \mathbb{R}$, we denote $c_+ = \max\{c, 0\}$, and use $[c]$ to represent the largest integer less than or equal to c . We recall some definitions for s -atoms and (Q, s, p, q) -atoms, which are very useful in our proofs.

Definition 2.5. Let $0 < p \leq 1 < q \leq \infty$, $s \in \mathbb{R}$. Let K and L be the integers with

$$(2.20) \quad K \geq ([s] + 1)_+ \quad \text{and} \quad L \geq \max\{[n(1/p - 1)_+ - s], 1\}.$$

(1) A (complex-valued) function $a(x)$ is called an s -atom if

$$(2.21) \quad \text{supp } a \subset 5Q$$

for some $Q = Q_{0k} \in \mathbb{Q}^n$ and

$$(2.22) \quad |D^\alpha a(x)| \leq 1 \quad \text{for } |\alpha| \leq K.$$

(2) Let $Q = Q_{\nu,k} \in \mathbb{Q}^n$. The function $a(x)$ is called a (Q, s, p, q) -atom if (2.21) is satisfied,

$$(2.23) \quad |D^\alpha a(x)| \leq |Q|^{-1/q+s/n-|\alpha|/n} \quad \text{for } |\alpha| \leq K,$$

and

$$(2.24) \quad \int_{\mathbb{R}^n} x^\beta a(x) dx = 0 \quad \text{for } |\beta| \leq L.$$

(3) The distribution $g \in \mathcal{S}'$ is called an (s, p, q) -atom if

$$(2.25) \quad g = \sum_{(\mu,l) < (\nu,k)} d_{\mu l} a_{\mu l}(x) \quad (\text{convergence in } F_{p,q}^s)$$

for some $\nu \in \mathbb{N}$ and $k \in \mathbb{Z}^n$, where $a_{\mu l}$ is a $(Q_{\mu l}, s, p, q)$ -atom and $d_{\mu l}$ are complex numbers with

$$(2.26) \quad \left(\sum_{(\mu,l) < (\nu,k)} |d_{\mu l}|^q \right)^{1/q} \leq |Q_{\nu k}|^{1/q-1/p}$$

with usual modification if $q = \infty$.

We also recall the atomic decomposition for $F_{p,q}^s$. One can find the following lemma and its historic remarks in Triebel's book [23, Section 3.2].

Lemma 2.6. *Let $0 < p \leq 1 < q \leq \infty$, $s \in \mathbb{R}$. Let K and L be fixed integers satisfying (2.20). A distribution $f \in \mathcal{S}'$ is an element of $F_{p,q}^s$ if and only if it can be represented as*

$$(2.27) \quad \sum_{j=1}^{\infty} (\mu_j a_j + \lambda_j g_j) \quad (\text{convergence in } \mathcal{S}')$$

where a_j are s -atoms, g_j are (s,p,q) -atoms, and μ_j and λ_j are complex numbers with

$$(2.28) \quad \left(\sum_{j=1}^{\infty} |\mu_j|^p + |\lambda_j|^p \right)^{1/p} \lesssim \|f\|_{F_{p,q}^s}.$$

We recall inclusion relations between modulation and Besov spaces.

Lemma 2.7 (See [25]). *Let $0 < p, q \leq \infty$. Then the following two statements are true:*

- (1) $M_{p,q}^s \subset B_{p,q} \iff s \geq 0 \vee [n(1/p - 1/q)] \vee [n(1 - 1/p - 1/q)];$
- (2) $B_{p,q} \subset M_{p,q}^s \iff s \leq 0 \wedge [n(1/p - 1/q)] \wedge [n(1 - 1/p - 1/q)].$

The following Bernstein multiplier theorem will be used in our proof.

Lemma 2.8 (Bernstein multiplier theorem). *Let $0 < p \leq 1$, $\partial^\gamma f \in L^2$ for $|\gamma| \leq [n(1/p - 1/2)] + 1$. Then,*

$$(2.29) \quad \|\mathcal{F}^{-1}f\|_{L_p} \lesssim \sum_{|\gamma| \leq [n(1/p - 1/2)] + 1} \|\partial^\gamma f\|_{L_2}.$$

Proof. Denote $N = [n(1/p - 1/2)] + 1$. We use Hölder's inequality to deduce that

$$(2.30) \quad \begin{aligned} \|\mathcal{F}^{-1}f\|_{L_p} &= \|\langle \xi \rangle^{-N} \cdot \langle \xi \rangle^N \mathcal{F}^{-1}f(\xi)\|_{L_p} \\ &\leq \|\langle \xi \rangle^{-N}\|_{L_r} \cdot \|\langle \xi \rangle^N \mathcal{F}^{-1}f(\xi)\|_{L_2} \\ &\lesssim \|\langle \xi \rangle^N \mathcal{F}^{-1}f(\xi)\|_{L_2} \sim \sum_{|\gamma| \leq [n(1/p - 1/2)] + 1} \|\partial^\gamma f\|_{L_2}, \end{aligned}$$

where $1/p = 1/2 + 1/r$ and $rN > n$. □

We recall a convolution inequality for $p \leq 1$.

Lemma 2.9 (Weighted convolution in L_p with $p < 1$; see [22]). *Let $0 < p < 1$, $B(x_0, R) = \{x : |x - x_0| \leq R\}$. Suppose $f, g \in L^p$ with Fourier support in $B(x_0, R)$ and $B(x_1, R)$ respectively. Then there exists a constant $C > 0$ which is independent of x_0, x_1, R, f such that*

$$\||f| * |g|\|_{L_p} \leq CR^{n(1/p-1)} \|f\|_{L_p} \|g\|_{L_p}.$$

Finally, we recall some embedding lemmas.

Lemma 2.10 (Sharpness of embedding, for uniform decomposition). *Suppose $0 < q_1, q_2 \leq \infty$, $s_1, s_2 \in \mathbb{R}$. Then*

$$(2.31) \quad l_{q_1}^{s_1,0} \subset l_{q_2}^{s_2,0}$$

holds if and only if

$$(2.32) \quad \begin{cases} s_2 \leq s_1, \\ \frac{1}{q_2} + \frac{s_2}{n} < \frac{1}{q_1} + \frac{s_1}{n}, \end{cases} \quad \text{or} \quad \begin{cases} s_2 = s_1, \\ q_2 = q_1. \end{cases}$$

Lemma 2.11 (Sharpness of embedding, for dyadic decomposition). *Suppose $0 < q_1, q_2 \leq \infty$, $s_1, s_2 \in \mathbb{R}$. Then*

$$(2.33) \quad l_{q_1}^{s_1,1} \subset l_{q_2}^{s_2,1}$$

holds if and only if

$$(2.34) \quad s_2 < s_1 \quad \text{or} \quad \begin{cases} s_2 = s_1, \\ 1/q_2 \leq 1/q_1. \end{cases}$$

3. PROOFS OF THEOREMS 1.1 AND 1.2

Proposition 3.1. *Let $0 < p < \infty$, $0 < q, r \leq \infty$, $s \in \mathbb{R}$. Then we have*

- (1) $M_{p,q}^s \subset F_{p,r} \implies l_q^{s,0} \subset l_p^{0,0}$,
- (2) $F_{p,r} \subset M_{p,q}^s \implies l_p^{0,0} \subset l_q^{s,0}$.

Proof. Take g to be a smooth function with compact Fourier support on $B(0, 10^{-10})$, $\widehat{g}_k = \widehat{g}(\cdot - k)$. Denote

$$(3.1) \quad G_N = \sum_{k \in \mathbb{Z}^n} a_k T_{Nk} g_k,$$

where $\{a_k\}_{k \in \mathbb{Z}^n}$ is a truncated (only finite nonzero item) sequence of a nonnegative real number, T is the translation operator, and N is some large integer to be chosen later. We have $\square_k G_N = a_k T_{Nk} g_k$. In addition, we have $\|g_k\|_{F_{p,r}} \sim \|g_k\|_{L_p}$.

By the definition of modulation space $M_{p,q}^s$, we have

$$(3.2) \quad \begin{aligned} \|G_N\|_{M_{p,q}^s} &\sim \|\{\|\square_k G_N\|_{L_p}\}\|_{l_q^{s,0}} \sim \|\{a_k \|T_{Nk} g_k\|_{L_p}\}\|_{l_q^{s,0}} \\ &\sim \|\{a_k g\|_{L_p}\}\|_{l_q^{s,0}} \sim \|\{a_k\}\|_{l_q^{s,0}}. \end{aligned}$$

On the other hand, letting $N \rightarrow \infty$, we use the almost orthogonality of $\{a_k T_{Nk} g_k\}_{k \in \mathbb{Z}^n}$ to deduce that

$$(3.3) \quad \begin{aligned} \lim_{N \rightarrow \infty} \|G_N\|_{F_{p,r}} &= \lim_{N \rightarrow \infty} \left(\sum_{k \in \mathbb{Z}^n} a_k^p \|T_{Nk} g_k\|_{F_{p,r}}^p \right)^{1/p} \\ &\sim \left(\sum_{k \in \mathbb{Z}^n} a_k^p \|g_k\|_{L_p}^p \right)^{1/p} \sim \left(\sum_{k \in \mathbb{Z}^n} a_k^p \right)^{1/p} \sim \|\{a_k\}\|_{l_p^{0,0}}. \end{aligned}$$

Thus, if $M_{p,q}^s \subset F_{p,r}$, we deduce that

$$(3.4) \quad \|\{a_k\}\|_{l_p^{0,0}} \sim \lim_{N \rightarrow \infty} \|G_N\|_{F_{p,r}} \lesssim \lim_{N \rightarrow \infty} \|G_N\|_{M_{p,q}^s} \sim \|\{a_k\}\|_{l_q^{s,0}}$$

for any truncated sequence $\{a_k\}_{k \in \mathbb{Z}}$, which implies the desired inclusion relation $l_q^{s,0} \subset l_p^{0,0}$.

On the other hand, if $F_{p,r} \subset M_{p,q}^s$ holds, we conclude

$$(3.5) \quad \|\{a_k\}\|_{l_q^{s,0}} \sim \lim_{N \rightarrow \infty} \|G_N\|_{M_{p,q}^s} \lesssim \lim_{N \rightarrow \infty} \|G_N\|_{F_{p,r}} \sim \|\{a_k\}\|_{l_p^{0,0}},$$

which implies $l_p^{0,0} \subset l_q^{s,0}$. □

Proposition 3.2. *Let $0 < p < \infty$, $0 < q, r \leq \infty$, $s \in \mathbb{R}$. Then we have*

- (1) $M_{p,q}^s \subset F_{p,r} \implies l_q^{s+n/q,1} \subset l_p^{n(1-1/p),1}$,
- (2) $F_{p,r} \subset M_{p,q}^s \implies l_p^{n(1-1/p),1} \subset l_q^{s+n/q,1}$.

Proof. Choose a smooth function h with compact Fourier support on $3/4 \leq |\xi| \leq 4/3$, satisfying $\hat{h}(\xi) = 1$ on $7/8 \leq |\xi| \leq 8/7$. Denote $\widehat{h}_j(\xi) = \hat{h}(\xi/2^j)$,

$$(3.6) \quad \Gamma_j = \{k \in \mathbb{Z}^n : \square_k h_j \neq 0\}, \quad \widetilde{\Gamma_j} = \{k \in \mathbb{Z}^n : \square_k h_j = \mathcal{F}^{-1}\sigma_k\}.$$

Obviously, we have $|\Gamma_j| \sim |\widetilde{\Gamma_j}| \sim 2^{jn}$ for $|j| \geq J$, where J is a sufficient large number. In addition, $k \sim 2^j$ for $k \in \Gamma_j$ or $k \in \widetilde{\Gamma_j}$.

Denote

$$F_N = \sum_{j \geq J}^{\infty} b_j T_{Nj e_0} h_j,$$

where $\{b_j\}_{j=0}^{\infty}$ is a truncated (only finite nonzero item) sequence of a nonnegative real number, $e_0 = (1, 0, \dots, 0)$ is the unit vector of \mathbb{R}^n , and N is a sufficient large number to be chosen later.

We first estimate the norm of $M_{p,q}^s$. Using Lemma 2.9, we deduce that

$$(3.7) \quad \|\square_k h_j\| \lesssim 2^{jn(1/p-1)} \|h_j\|_{L_p} \cdot \|\mathcal{F}^{-1}\sigma_k\|_{L_p} \lesssim 1.$$

Thus,

$$\begin{aligned} \|F_N\|_{M_{p,q}^s} &= \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \|\square_k F_N\|_{L_p}^q \right)^{1/q} \\ &\leq \left(\sum_{j \geq J} \sum_{k \in \Gamma_j} \langle k \rangle^{sq} b_j^q \|\square_k h_j\|_{L_p}^q \right)^{1/q} \\ &\lesssim \left(\sum_{j \geq J} 2^{jsq} b_j^q |\Gamma_j| \right)^{1/q} \lesssim \left(\sum_{j \geq J} 2^{jsq} b_j^q 2^{jn} \right)^{1/q} \sim \|\{b_j\}_{j \geq J}\|_{l_q^{s+n/q,1}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} (3.9) \quad \|F_N\|_{M_{p,q}^s} &\geq \left(\sum_{j \geq J} \sum_{k \in \widetilde{\Gamma_j}} \langle k \rangle^{sq} b_j^q \|\square_k h_j\|_{L_p}^q \right)^{1/q} \\ &= \left(\sum_{j \geq J} \sum_{k \in \widetilde{\Gamma_j}} \langle k \rangle^{sq} b_j^q \|\mathcal{F}^{-1}\sigma_k\|_{L_p}^q \right)^{1/q} \\ &\sim \left(\sum_{j \geq J} 2^{jsq} b_j^q |\widetilde{\Gamma_j}| \right)^{1/q} \sim \left(\sum_{j \geq J} 2^{jsq} b_j^q 2^{jn} \right)^{1/q} \sim \|\{b_j\}_{j \geq J}\|_{l_q^{s+n/q,1}}. \end{aligned}$$

Hence,

$$(3.10) \quad \|F_N\|_{M_{p,q}^s} \sim \|\{b_j\}_{j \geq J}\|_{l_p^{n(1-1/p),1}}.$$

Now, we turn to the estimate of $\|F_N\|_{F_{p,r}}$. By the assumption of h , we have $\Delta_j h_j = h_j$, where $\widehat{h}_j(\xi) = \hat{h}(\xi/2^j)$. By the definition of $F_{p,r}$, we have

$$(3.11) \quad \|F_N\|_{F_{p,r}} = \|\{\Delta_j F_N\}_{j \in \mathbb{N}}\|_{l_r^{0,1}} = \|\{\{b_j h_j\}_{j \geq J}\|_{l_r^{0,1}}.$$

Using the almost orthogonality of $\{a_j T_{Nj e_0} h_j\}_{j=0}^\infty$ as $N \rightarrow \infty$, we deduce

$$(3.12) \quad \begin{aligned} \lim_{N \rightarrow \infty} \|F_N\|_{F_{p,r}} &= \left(\sum_{j \geq J} b_j^p \|h_j\|_{L_p}^p \right)^{1/p} \\ &\sim \left(\sum_{j \geq J} b_j^p 2^{jn(1-1/p)p} \right)^{1/p} \sim \|\{b_j\}_{j \geq J}\|_{l_p^{n(1-1/p),1}}. \end{aligned}$$

If $M_{p,q}^s \subset F_{p,r}$ holds, we have

$$(3.13) \quad \|F_N\|_{F_{p,r}} \lesssim \|F_N\|_{M_{p,q}^s}.$$

Letting $N \rightarrow \infty$, we use the estimates of $\|F_N\|_{F_{p,r}}$ and $\|F_N\|_{M_{p,q}^s}$ obtained above to deduce that

$$(3.14) \quad \|\{b_j\}_{j \geq J}\|_{l_p^{n(1-1/p),1}} \lesssim \|\{b_j\}_{j \geq J}\|_{l_q^{s+n/q,1}}$$

for all truncated sequences $\{b_j\}_{j \in \mathbb{N}}$, which implies the desired inclusion relation $l_q^{s+n/q,1} \subset l_p^{n(1-1/p),1}$.

Similarly, if $F_{p,r} \subset M_{p,q}^s$ holds, we deduce

$$(3.15) \quad \|\{b_j\}_{j \geq J}\|_{l_q^{s+n/q,1}} \lesssim \|\{b_j\}_{j \geq J}\|_{l_p^{n(1-1/p),1}}$$

for all truncated sequences $\{b_j\}_{j \in \mathbb{N}}$, which implies $l_p^{n(1-1/p),1} \subset l_q^{s+n/q,1}$. \square

Proposition 3.3. *Let $0 < p < \infty$, $0 < q, r \leq \infty$. Then we have*

$$M_{p,q} \subset F_{p,r} \implies l_q^{0,1} \subset l_r^{0,1}.$$

Proof. Take h to be a nonzero smooth function with Fourier support $B(0, 10^{-10})$. We choose a sequence $\{\xi_j\}_{j=0}^\infty$ of \mathbb{Z}^n such that $\Delta_j h_j = h_j$ and $\Delta_l h_j = 0$ for $j \in \mathbb{N}$ and $i \neq j$, where $\widehat{h}_j(\cdot) = \widehat{h}(\cdot - \xi_j)$. Obviously, we have $\xi_j \sim 2^j$. Denote

$$(3.16) \quad H = \sum_{j=0}^{\infty} a_j h_j,$$

where $\{a_j\}_{j=0}^\infty$ is a truncated sequence of a nonnegative real number. We have

$$(3.17) \quad \|H\|_{F_{p,r}} = \left\| \left(\sum_{j=0}^{\infty} |a_j|^r |h_j|^r \right)^{1/r} \right\|_{L_p} = \left\| \left(\sum_{j=0}^{\infty} |a_j|^r |h|^r \right)^{1/r} \right\|_{L_p} \sim \|\{a_j\}\|_{l_r^{0,1}}.$$

On the other hand,

$$(3.18) \quad \begin{aligned} \|H\|_{M_{p,q}} &= \left(\sum_{k \in \mathbb{Z}^n} \|\square_k H\|_{L_p}^q \right)^{1/q} \sim \left(\sum_{j=0}^{\infty} |a_j|^q \|h_j\|_{L_p}^q \right)^{1/q} \\ &\sim \left(\sum_{j=0}^{\infty} |a_j|^q \|h\|_{L_p}^q \right)^{1/q} \sim \|\{a_j\}\|_{l_q^{0,1}}. \end{aligned}$$

Thus, $M_{p,q} \subset F_{p,r}$ implies

$$(3.19) \quad \|\{a_j\}\|_{l_r^{0,1}} \sim \|H\|_{F_{p,r}} \lesssim \|H\|_{M_{p,q}} \sim \|\{a_j\}\|_{l_q^{0,1}},$$

which implies the desired conclusion. \square

We recall a lemma for the local property of $M_{p,q}$, where one can see a proof for $1 \leq p, q \leq \infty$ in [17]. The proof for $0 < p, q \leq \infty$ is similar and we omit the details here.

Lemma 3.4. *Let $0 < p, q \leq \infty$, $r > 0$, and f be a tempered distribution supported on $B(x_0, r)$ for some $x_0 \in \mathbb{R}^n$. Then $f \in M_{p,q}$ if and only if $\widehat{f} \in L_q$. Moreover, we have*

$$(3.20) \quad \|f\|_{M_{p,q}} \sim \|\widehat{f}\|_{L_q}$$

in this case.

Proposition 3.5. *Let $0 < p \leq 1$. We have the following inclusion relation:*

$$(3.21) \quad F_{p,\infty}^{n(2/p-1)} \subset M_{p,p}.$$

Proof. We first verify

$$(3.22) \quad \|a\|_{M_{p,p}} \lesssim 1$$

for any $n(2/p-1)$ -atom a . Take a to be an $n(2/p-1)$ -atom as in Definition 2.5 (with $s = n(2/p-1)$). Observing that $K \geq [n(2/p-1)] + 1 \geq [n(1/p-1/2)] + 1$, we have

$$(3.23) \quad |\partial^\gamma a| \leq 1$$

for $|\gamma| \leq [n(1/p-1/2)] + 1$. By the Bernstein multiplier theorem and Lemma 3.4, we have the following estimate of a :

$$(3.24) \quad \|a\|_{M_{p,p}} \sim \|\mathcal{F}^{-1}a\|_{L_p} \lesssim \sum_{|\gamma| \leq [n(1/p-1/2)] + 1} \|\partial^\gamma a\|_{L_2} \lesssim 1.$$

Next, we turn to the estimate of an $(n(2/p-1), p, \infty)$ -atom for $F_{p,\infty}^{n(2/p-1)}$. By Definition 2.5, an $(n(2/p-1), p, \infty)$ -atom g can be represented by

$$(3.25) \quad g = \sum_{(\mu,l) < (\nu,k)} d_{\mu l} a_{\mu l}(x) \quad (\text{convergence in } F_{p,\infty}^{n(2/p-1)})$$

for some $\nu \in \mathbb{N}$ and $k \in \mathbb{Z}^n$, where $a_{\mu l}$ are $(Q_{\mu l}, n(2/p-1), p, \infty)$ -atoms and $d_{\mu l}$ are complex numbers with

$$(3.26) \quad \sup_{(\mu,l) < (\nu,k)} |d_{\mu l}| \leq |Q_{\nu k}|^{-1/p}.$$

For a fixed $\tau \leq \nu$, we denote

$$(3.27) \quad g_\tau = \sum_{(\tau,l) < (\nu,k)} d_{\tau l} a_{\tau l}(x).$$

Then, g can be represented by

$$(3.28) \quad g = \sum_{\tau \geq \nu} g_\tau \quad (\text{convergence in } F_{p,\infty}^{n(2/p-1)}).$$

We now concentrate on the estimate of g_τ . By Definition 2.5, we have

$$(3.29) \quad |\partial^\gamma a_{\tau l}| \leq |Q_{\tau l}|^{(2/p-1)-|\gamma|/n} = |2^{-\tau n}|^{(2/p-1)-|\gamma|/n}$$

for $|\gamma| \leq [n(1/p - 1/2)] + 1 \leq K$. Recalling $\text{supp } a_{\tau l} \subset 5Q_{\tau l}$, we use (3.29) and the almost orthogonality of $a_{\tau l}$ to deduce that

$$\begin{aligned} |\partial^\gamma g_\tau| &= \left| \sum_{(\tau,l) < (\nu,k)} d_{\tau l} \partial^\gamma a_{\tau l}(x) \right| \\ (3.30) \quad &\lesssim \sup_{(\tau,l) < (\nu,k)} |d_{\mu l}| |2^{-\tau n}|^{(2/p-1)-\gamma/n} \\ &\lesssim |Q_{\nu k}|^{-1/p} |2^{-\tau n}|^{(2/p-1)-\gamma/n} \end{aligned}$$

for all $|\gamma| \leq [n(1/p - 1/2)] + 1$. By the Bernstein multiplier theorem and Lemma 3.4, we deduce that

$$\begin{aligned} \|g_\tau\|_{M_{p,p}} &\sim \|\mathcal{F}^{-1} g_\tau\|_{L_p} \\ (3.31) \quad &\lesssim \sum_{|\gamma| \leq [n(1/p - 1/2)] + 1} \|\partial^\gamma g_\tau\|_{L_2} \\ &\lesssim \sum_{|\gamma| \leq [n(1/p - 1/2)] + 1} |Q_{\nu k}|^{1/2-1/p} |2^{-\tau n}|^{(2/p-1)-\gamma/n}. \end{aligned}$$

By a dilation argument, we have

$$(3.32) \quad \|g_\tau\|_{M_{p,p}} \sim \|\mathcal{F}^{-1} g_\tau\|_{L_p} \lesssim |Q_{\nu k}|^{1/2-1/p} |2^{-\tau n}|^{1/p-1/2}.$$

Thus,

$$\begin{aligned} \|g\|_{M_{p,p}}^p &= \left\| \sum_{\tau \geq \nu} g_\tau \right\|_{M_{p,p}}^p \leq \sum_{\tau \geq \nu} \|g_\tau\|_{M_{p,p}}^p \\ (3.33) \quad &\leq |Q_{\nu k}|^{p/2-1} \sum_{\tau \geq \nu} |2^{-\tau n}|^{1-p/2} \\ &\lesssim |Q_{\nu k}|^{p/2-1} \cdot |2^{-\nu n}|^{1-p/2} \sim 1. \end{aligned}$$

By Lemma 2.6, we have

$$\begin{aligned} \|f\|_{M_{p,p}} &= \left\| \sum_{j=1}^{\infty} (\mu_j a_j + \lambda_j g_j) \right\|_{M_{p,p}} \\ (3.34) \quad &\leq \left(\left\| \sum_{j=1}^{\infty} (\mu_j^p \|a_j\|_{M_{p,p}}^p + \lambda_j^p \|g_j\|_{M_{p,p}}^p) \right\|^p \right)^{1/p} \\ &\lesssim \left(\left\| \sum_{j=1}^{\infty} \mu_j^p + \lambda_j^p \right\|^p \right)^{1/p} \lesssim \|f\|_{F_{p,\infty}^{n(2/p-1)}}, \end{aligned}$$

which is the desired conclusion. \square

Now we are in the position to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We divide this proof into two parts.

(Sufficiency): For $1/p \leq 1/q$, we have $1/r \leq 1/q$, and then $F_{p,q} \subset F_{p,r}$. Using Lemma 2.7, we obtain $M_{p,q} \subset B_{p,q}$. Thus, we deduce that

$$(3.35) \quad M_{p,q} \subset B_{p,q} \subset F_{p,q} \subset F_{p,r},$$

which is the desired conclusion.

For $1/p > 1/q$, use $M_{p,p} \subset F_{p,p}$ obtained above to deduce that

$$(3.36) \quad M_{p,q}^{n(1/p-1/q)+2\epsilon} \subset M_{p,p}^\epsilon \subset F_{p,p}^\epsilon \subset F_{p,r}$$

for any $\epsilon > 0$, $r \in (0, \infty]$.

(Necessity): For $1/p \leq 1/q$, using Proposition 3.1 and Lemma 2.10, we deduce that $l_q^{s,0} \subset l_p^{0,0}$, which implies $s \geq 0$.

On the other hand, if $M_{p,q} \subset F_{p,r}$ holds, we use Proposition 3.3 and Lemma 2.10 to deduce that $1/r \leq 1/q$.

For $1/p > 1/q$, we use Proposition 3.1 and Lemma 2.11 to deduce $l_q^{s,0} \subset l_p^{0,0}$, which implies $s > n(1/p - 1/q)$. \square

Proof of Theorem 1.2. We divide this proof into two parts.

(Sufficiency): For $1/q \leq 1/p$, by Lemma 2.7, we obtain $B_{p,\infty} \subset M_{p,\infty}^{n(1-1/p)}$. Using the fact $F_{p,\infty} \subset B_{p,\infty}$, we deduce that

$$(3.37) \quad F_{p,\infty} \subset M_{p,\infty}^{n(1-1/p)}.$$

In addition, we have $F_{p,\infty}^{n(2/p-1)} \subset M_{p,p}$ by Proposition 3.5. By potential lifting, we obtain

$$(3.38) \quad F_{p,\infty} \subset M_{p,p}^{n(1-2/p)}.$$

Thus, the desired conclusion can be deduced by a standard interpolation argument between (3.37) and (3.38).

For $1/p > 1/q$, recalling $F_{p,\infty} \subset M_{p,p}^{n(1-2/p)}$ obtained in Proposition 3.5, we deduce that

$$(3.39) \quad F_{p,r} \subset F_{p,\infty} \subset M_{p,p}^{n(1-2/p)} \subset M_{p,q}^{n(1-1/p-1/q)-\epsilon}$$

for any $\epsilon > 0$, $r \in (0, \infty]$.

(Necessity): We use Proposition 3.1 to deduce inclusion relation $l_p^{n(1-1/p),1} \subset l_q^{s+n/q,1}$. Then, Lemma 2.11 yields that $s \leq n(1 - 1/p - 1/q)$ for $1/q \leq 1/p$, while the inequality is strict for $1/q > 1/p$. \square

An open problem and its difficulties. Inspired by the article [13], one may ask a natural question: can we improve and extend the embedding results in [13] to the more general frame of Triebel-Lizorkin spaces? More exactly, can we establish the sharp conditions for the following two embedding relations:

$$(3.40) \quad (i) \quad M_{p,q}^s \subset F_{p,r}, \quad (ii) \quad F_{p,r} \subset M_{p,q}^s,$$

for $1 < p < \infty$, $0 < r, q \leq \infty$, $s \in \mathbb{R}$? A key problem is how to find the optimal r for the embedding relations $F_{p,r} \subset M_{p,q}^{n(1-1/p-1/q)}$ in the set $P =: \{(p, q) : 1 < p < 2, 1 - 1/p < 1/q \leq 1/p\}$. Unfortunately, due to the following two reasons, it is quite hard to establish such conditions:

- (1) **The “useless” of complex interpolation:** An interpolation argument is quite useful in the L_p ($p > 1$) case as in [13], while the situation becomes more complicated in the framework of Triebel-Lizorkin space. In this paper, we obtain

$$(3.41) \quad F_{p,\infty} \subset M_{p,q}^{n(1-1/p-1/q)}, \quad p \leq 1, 0 \leq 1/q \leq 1/p.$$

Obviously, $r = \infty$ is optimal in this case. However, when $p = q = 2$, the optimal r is 2 rather than ∞ . In fact, in the embedding relation

$$(3.42) \quad F_{2,2} \subset M_{2,2},$$

$r = 2$ is optimal. If we apply complex interpolation between (3.41) and (3.42), we only have

$$(3.43) \quad F_{p,p/(p-1)} \subset M_{p,q}^{n(1-1/p-1/q)}, \quad (p, q) \in P.$$

Now, a new question is: how do we determine whether $r = p/(p-1)$ is the optimal one in the above embedding relation (3.43)? If $r = p/(p-1)$ is not the optimal one, we have to establish the inclusion relation $F_{p,r} \subset M_{p,q}^{n(1-1/p-1/q)}$ for some $r > p/(p-1)$, $p, q \in P$ directly (without the help of interpolation). However, we face new difficulties as follows.

- (2) **The lack of appropriate atoms:** In this paper, we use an atom decomposition for inhomogeneous Triebel-Lizorkin spaces (see Lemma 2.6). However, this atom decomposition only works for $0 < p \leq 1 < q \leq \infty$. When we handle the case $p > 1$, the lack of appropriate atoms leads to new difficulties.

REFERENCES

- [1] Árpád Bényi, Karlheinz Gröchenig, Kasso A. Okoudjou, and Luke G. Rogers, *Unimodular Fourier multipliers for modulation spaces*, J. Funct. Anal. **246** (2007), no. no. 2, 366–384. MR2321047
- [2] H. G. Feichtinger, *Modulation spaces on locally compact Abelian group*, Technical Report, University of Vienna, 1983. Published in: Proc. Internat. Conf. on Wavelet and Applications, 99–140. New Delhi Allied Publishers, India, 2003.
- [3] Hans G. Feichtinger, *Modulation spaces: looking back and ahead*, Sampl. Theory Signal Image Process. **5** (2006), no. no. 2, 109–140. MR2233968
- [4] H. G. Feichtinger, *Banach convolution algebras of Wiener type*, Functions, series, operators, Vol. I, II (Budapest, 1980), Colloq. Math. Soc. János Bolyai, vol. 35, North-Holland, Amsterdam, 1983, pp. 509–524. MR751019
- [5] P. Gröbner, *Banachräume Glatter Funktionen und Zerlegungsmethoden*, Doctoral Thesis, University of Vienna, 1992.
- [6] Karlheinz Gröchenig, *Foundations of time-frequency analysis*, Applied and Numerical Harmonic Analysis, Birkhäuser Boston, Inc., Boston, MA, 2001. MR1843717
- [7] David Marc Goldberg, *A local version of real hardy spaces*, ProQuest LLC, Ann Arbor, MI, 1978. Thesis (Ph.D.)–Princeton University. MR2627942
- [8] Weichao Guo and Jiecheng Chen, *Stability of nonlinear Schrödinger equations on modulation spaces*, Front. Math. China **9** (2014), no. no. 2, 275–301. MR3171501
- [9] W. Guo, J. Chen, D. Fan, G. Zhao, Characterization of some properties on weighted modulation spaces, arXiv:1602.02871 (2016).
- [10] WeiChao Guo, DaShan Fan, HuoXiong Wu, and GuoPing Zhao, *Sharpness of some properties of weighted modulation spaces*, Sci. China Math. **59** (2016), no. no. 1, 169–190. MR3437002
- [11] Jinsheng Han and Baoxiang Wang, *α -modulation spaces (I) scaling, embedding and algebraic properties*, J. Math. Soc. Japan **66** (2014), no. no. 4, 1315–1373. MR3272601
- [12] Masaharu Kobayashi, Akihiko Miyachi, and Naohito Tomita, *Embedding relations between local Hardy and modulation spaces*, Studia Math. **192** (2009), no. no. 1, 79–96. MR2491791
- [13] Masaharu Kobayashi and Mitsuru Sugimoto, *The inclusion relation between Sobolev and modulation spaces*, J. Funct. Anal. **260** (2011), no. no. 11, 3189–3208. MR2776566
- [14] Akihiko Miyachi, Fabio Nicola, Silvia Rivetti, Anita Tabacco, and Naohito Tomita, *Estimates for unimodular Fourier multipliers on modulation spaces*, Proc. Amer. Math. Soc. **137** (2009), no. no. 11, 3869–3883. MR2529896
- [15] Kasso A. Okoudjou, *Embedding of some classical Banach spaces into modulation spaces*, Proc. Amer. Math. Soc. **132** (2004), no. no. 6, 1639–1647. MR2051124

- [16] Michael Ruzhansky, Mitsuru Sugimoto, and Baoxiang Wang, *Modulation spaces and nonlinear evolution equations*, Evolution equations of hyperbolic and Schrödinger type, Progr. Math., vol. 301, Birkhäuser/Springer Basel AG, Basel, 2012, pp. 267–283. MR3014810
- [17] Michael Ruzhansky, Mitsuru Sugimoto, Joachim Toft, and Naohito Tomita, *Changes of variables in modulation and Wiener amalgam spaces*, Math. Nachr. **284** (2011), no. no. 16, 2078–2092. MR2844679
- [18] Mitsuru Sugimoto and Naohito Tomita, *The dilation property of modulation spaces and their inclusion relation with Besov spaces*, J. Funct. Anal. **248** (2007), no. no. 1, 79–106. MR2329683
- [19] Joachim Toft and Patrik Wahlberg, *Embeddings of α -modulation spaces*, Pliska Stud. Math. Bulgar. **21** (2012), 25–46. MR3114337
- [20] Joachim Toft, *Continuity properties for modulation spaces, with applications to pseudo-differential calculus. I*, J. Funct. Anal. **207** (2004), no. no. 2, 399–429. MR2032995
- [21] H. Triebel, *Modulation spaces on the Euclidean n -space* (English, with German and Russian summaries), Z. Anal. Anwendungen **2** (1983), no. no. 5, 443–457. MR725159
- [22] Hans Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983. MR781540
- [23] Hans Triebel, *Theory of function spaces. II*, Monographs in Mathematics, vol. 84, Birkhäuser Verlag, Basel, 1992. MR1163193
- [24] Wang Baoxiang, Zhao Lifeng, and Guo Boling, *Isometric decomposition operators, function spaces $E_{p,q}^\lambda$, and applications to nonlinear evolution equations*, J. Funct. Anal. **233** (2006), no. no. 1, 1–39. MR2204673
- [25] Baoxiang Wang and Chunyan Huang, *Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations*, J. Differential Equations **239** (2007), no. no. 1, 213–250. MR2341554
- [26] Baoxiang Wang, Zhaojun Huo, Chengchun Hao, and Zihua Guo, *Harmonic analysis method for nonlinear evolution equations. I*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011. MR2848761
- [27] Baoxiang Wang and Henryk Hudzik, *The global Cauchy problem for the NLS and NLKG with small rough data*, J. Differential Equations **232** (2007), no. no. 1, 36–73. MR2281189
- [28] Guoping Zhao, Jiecheng Chen, Dashan Fan, and Weichao Guo, *Sharp estimates of unimodular multipliers on frequency decomposition spaces*, Nonlinear Anal. **142** (2016), 26–47. MR3508056

SCHOOL OF MATHEMATICS AND INFORMATION SCIENCES, GUANGZHOU UNIVERSITY, GUANGZHOU, 510006, PEOPLE'S REPUBLIC OF CHINA

E-mail address: weichaoguomath@gmail.com

SCHOOL OF MATHEMATICAL SCIENCES, XIAMEN UNIVERSITY, XIAMEN, 361005, PEOPLE'S REPUBLIC OF CHINA

E-mail address: huoxwu@xmu.edu.cn

SCHOOL OF APPLIED MATHEMATICS, XIAMEN UNIVERSITY OF TECHNOLOGY, XIAMEN, 361024, PEOPLE'S REPUBLIC OF CHINA

E-mail address: guopingzhaomath@gmail.com