

SOME ELEMENTARY EXAMPLES OF NON-LIFTABLE VARIETIES

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ABSTRACT. We present some simple examples of smooth projective varieties in positive characteristic, arising from linear algebra, which do not admit a lifting neither to characteristic zero, nor to the ring of Witt vectors of length 2. Our first construction is the blow-up of the graph of the Frobenius morphism of a homogeneous space. The second example is a blow-up of \mathbb{P}^3 in a ‘purely characteristic- p ’ configuration of points and lines.

1. INTRODUCTION

Various theorems in modern algebraic geometry are proved using characteristic p methods along the following lines. Given a complex algebraic variety X , one reduces the variety mod p , exploits the Frobenius morphism on the reduction X_p , and deduces statements about the original X . A remarkable application of this approach is the Mori’s bend and break technique [Mor79] used for the proof of the conjecture of Hartshorne. Similarly, characteristic zero (particularly complex analytic) methods are employed to study varieties in positive characteristic. The main technical obstacle is that, while every characteristic zero variety can be reduced mod p , not every variety in positive characteristic arises as the reduction mod p of a variety in characteristic zero. The first example of such a variety was given by Serre [Ser61].

It turns out that for many purposes, one does not need to lift a given variety all the way to characteristic zero, and it suffices to have a lifting modulo p^2 . For example, Deligne and Illusie [DI87] showed that for a smooth variety X over a perfect field k of characteristic $p > \dim X$ admitting a lifting to $W_2(k)$ (the ring of Witt vectors of length 2), the Hodge–de Rham spectral sequence degenerates, and the Kodaira vanishing theorem holds. More recently, Langer [Lan16] showed that the logarithmic Bogomolov–Miyazaki–Yau inequality holds for surfaces liftable to $W_2(k)$ (as long as $p > 2$). Counterexamples to Kodaira vanishing in positive characteristic given by Raynaud [Ray78] give the first example of varieties which do not lift to $W_2(k)$. Subsequently, a rational example was given by Lauritzen and

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Rao [LR97]. In the positive direction, it is known that every Frobenius split variety lifts to $W_2(k)$ [Lan15, Proposition 8.4].

In this paper, we construct new examples of smooth projective varieties that do not admit lifts neither to characteristic zero, nor to $W_2(k)$ (some of them do not even lift to any ring A with $pA \neq 0$). However it turns out that they avoid standard characteristic p pathologies, in particular they satisfy the following

Good properties:

- (1) they are smooth, projective, rational, and simply connected,
- (2) their classes in the Grothendieck ring of varieties are polynomials in the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1]$ with non-negative integer coefficients,
- (3) their ℓ -adic integral cohomology rings are generated by algebraic cycles,
- (4) their integral crystalline cohomology groups are torsion-free F -crystals,
- (5) their Hodge–de Rham and conjugate spectral sequences degenerate, they are ordinary in the sense of Bloch–Kato, and of Hodge–Witt type (cf. §2.5 for the relevant definitions).

Since our constructions are very simple, we try to aim our exposition at non-experts, and go for elementary arguments whenever possible.

The first construction is given by the blow-up of the two-fold self product of a suitable projective homogeneous space $\neq \mathbb{P}^n$ along the graph of its Frobenius morphism. The easiest examples of such homogeneous spaces being the three-dimensional complete flag variety SL_3/B (isomorphic to the incidence variety $\{x_0y_0 + x_1y_1 + x_2y_2 = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^2$) and the four-dimensional smooth quadric hypersurface $Q = \{x_0^2 + x_1x_2 + x_3x_4 = 0\} \subseteq \mathbb{P}^5$ (isomorphic to the Grassmannian $G(2, 4)$ via the Plücker embedding), the smallest non-liftable examples given by the construction are of dimension six and eight, with Picard numbers five and three, respectively.

Theorem (cf. Theorem 3.2). *The blow-up $\mathrm{Bl}_{\Gamma_F}(Y \times Y)$ of a suitably chosen homogeneous space Y along the graph of its Frobenius morphism satisfies the properties (1)–(5), but does not admit a lifting neither to $W_2(k)$, nor to characteristic zero.*

The second construction is the following. Let X be the variety obtained from \mathbb{P}^3 by (1) blowing up all \mathbb{F}_p -rational points, and (2) blowing up the strict transforms of all lines connecting \mathbb{F}_p -rational points.

Theorem (Theorem 4.1). *The variety described above satisfies the properties (1)–(5), but does not admit a lift to any ring A with $pA \neq 0$.*

Both in Theorem 3.2 and Theorem 4.1, the proofs of non-liftability use the key observation (cf. [LS14], and Proposition 2.3 below) that if the blow-up of a smooth variety X along a smooth subvariety Z lifts, then both X and Z lift. In Theorem 3.2, if X was liftable, the homogeneous space Y would be liftable together with Frobenius, which is known to be impossible by the work of Paranjape–Srinivas [PS89] (for lifts to characteristic zero) and Buch–Thomsen–Lauritzen–Mehta [BTLM97] (for lifts to $W_2(k)$). In Theorem 4.1, we show that the liftability of X would imply the liftability of the arrangement of all \mathbb{F}_p -rational points in \mathbb{P}^2 preserving the incidence relations; thus non-liftability is established by means of elementary linear algebra. The properties (1)–(5) in both theorems are established quite easily using standard formulas expressing the cohomology of a blow-up which we recall in §2.4.

1.1. **Notation.** Throughout k denotes a perfect field of characteristic $p > 0$. For any k -scheme X by $X^{(1)}$ we denote the Frobenius pullback $X^{(1)} \stackrel{\text{def}}{=} X \times_{\text{Spec}(k), F} \text{Spec}(k)$ and by $F_{X/k} : X \rightarrow X^{(1)}$ the relative Frobenius of X over k . We say that a scheme X/k admits a $W_2(k)$ -lifting if there exists a flat $W_2(k)$ -scheme \tilde{X} such that $\tilde{X} \times_{\text{Spec}(W_2(k))} \text{Spec}(k) \simeq X$. Finally, we say that a scheme X lifts to $W_2(k)$ compatibly with Frobenius if there exists a $W_2(k)$ -lifting \tilde{X} of X together with a morphism $\widetilde{F_{X/k}} : \tilde{X} \rightarrow \tilde{X}^{(1)} \stackrel{\text{def}}{=} \tilde{X} \times_{\text{Spec}(W_2(k)), \sigma} \text{Spec}(W_2(k))$ restricting to the relative Frobenius morphism $F_{X/k} : X \rightarrow X^{(1)}$. For schemes defined over the field \mathbb{F}_p the absolute Frobenius morphism is in fact \mathbb{F}_p -linear and therefore the relative Frobenius morphism can be interpreted as an endomorphism $F_X : X \rightarrow X^{(1)} \simeq X$.

By $\mathbb{L}_{X/k}$ we denote the cotangent complex of a scheme X over k . Moreover, by Def_X we mean the deformation functor of X , that is, a covariant functor from the category $\mathbf{Art}_{W(k)}(k)$ of Artinian local $W(k)$ -algebras with residue field k to the category of sets defined by the formula:

$$\mathbf{Art}_{W(k)}(k) \ni (A, \mathfrak{m}_A) \mapsto \text{Def}_X(A) \stackrel{\text{def}}{=} \left\{ \begin{array}{l} \text{isomorphism classes of flat} \\ \text{deformations of } X \text{ over } \text{Spec}(A) \end{array} \right\}.$$

Similarly, if $Z = \{Z_i\}_{i \in I}$ is a family of closed subschemes of X indexed by a preorder I (i.e., a set with a reflexive and transitive binary relation), such that Z_i is a closed subscheme of Z_j whenever $i \leq j$ (in other words, I is a small category whose morphism sets have at most one element, and Z is a functor from I to the category of closed subschemes of X), we denote by $\text{Def}_{X,Z}$ the functor of flat deformations of X together with compatible embedded deformations of the Z_i , preserving the inclusion relations given by the relation \leq . If $f : X \rightarrow Y$ is a map of k -schemes, we denote by Def_f the functor of flat deformations of X , and Y along with a deformation of f .

2. TECHNICAL BACKGROUND

Here we review the necessary technical results regarding deformation theory of products (§2.1), descending deformations along morphisms (§2.2), cohomology of blowing up (§2.4), and Hodge–de Rham degeneration, ordinarity, and the Hodge–Witt property of blow-ups (§2.5).

2.1. **Deformations of products.** Our goal is to show that given two k -schemes X and Y such that $H^1(X, \mathcal{O}_X) = H^1(Y, \mathcal{O}_Y) = 0$, then every deformation of $X \times Y$ comes from a pair of deformations of X and Y (Proposition 2.2). We begin with a few remarks concerning deformation obstruction classes. First, observe that by [Ill71] we know that for any k -scheme Z the obstruction class to lifting an element

$$(\tilde{f} : \tilde{Z} \rightarrow \text{Spec}(A)) \in \text{Def}_Z(A)$$

to a thickening $\eta : 0 \rightarrow I \rightarrow (B, \mathfrak{m}_B) \rightarrow (A, \mathfrak{m}_A) \rightarrow 0$, satisfying $\mathfrak{m}_B I = 0$, is given by a class in $\text{Ext}^2(\mathbb{L}_{Z/k}, \mathcal{O}_Z) \otimes_k I$ defined as the Yoneda composition of Kodaira–Spencer class $K_{\tilde{Z}/A/Z} \in \text{Ext}^1(\mathbb{L}_{\tilde{Z}/A}, L\tilde{f}^* \mathbb{L}_{A/Z}[1])$ and the pullback $\tilde{f}^* \eta$ of

the extension class $\eta \in \text{Ext}^1(\mathbb{L}_{A/\mathbb{Z}}, I)$. Moreover, by a simple diagram chase based on the properties of the cotangent complex we obtain:

Lemma 2.1 (Additivity of Kodaira–Spencer). *Let $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ be morphisms of S -schemes. Let $p_X : X \times_Z Y \rightarrow X$ and $p_Y : X \times_Z Y \rightarrow Y$ denote the projections, $h : X \times_Z Y \rightarrow Z$ the composition $h = f \circ p_X = g \circ p_Y$. Then, Kodaira–Spencer class:*

$$K_{X \times_Z Y/Z/S} \in \text{Ext}^1(\mathbb{L}_{X \times_Z Y/Z}, Lh^* \mathbb{L}_{Z/S})$$

equals the direct sum of pullbacks of Kodaira–Spencer classes:

$$K_{X/Z/S} \in \text{Ext}^1(\mathbb{L}_{X/Z}, Lf^* \mathbb{L}_{Z/S}) \text{ and } K_{Y/Z/S} \in \text{Ext}^1(\mathbb{L}_{Y/Z}, Lg^* \mathbb{L}_{Z/S}).$$

Equipped with the above description, we are ready to prove:

Proposition 2.2. *The morphism of deformation functors*

$$\text{prod}_{X,Y} : \text{Def}_X \times \text{Def}_Y \rightarrow \text{Def}_{X \times Y}, \quad (\tilde{X}, \tilde{Y}) \mapsto \tilde{X} \times_{\text{Spec}(A)} \tilde{Y}$$

is smooth (in particular levelwise surjective) if $H^1(X, \mathcal{O}_X) = H^1(Y, \mathcal{O}_Y) = 0$.

Proof. By the above general considerations and the additivity of Kodaira–Spencer class we see that the morphisms on tangent and obstruction space

$$\begin{aligned} T_{\text{prod}_{X,Y}} &: \text{Ext}^1(\mathbb{L}_{X/k}, \mathcal{O}_X) \oplus \text{Ext}^1(\mathbb{L}_{Y/k}, \mathcal{O}_Y) \\ &\rightarrow \text{Ext}^1(Lp_X^* \mathbb{L}_{X/k} \oplus Lp_Y^* \mathbb{L}_{Y/k}, \mathcal{O}_{X \times Y}), \end{aligned}$$

$$\begin{aligned} \text{Ob}_{\text{prod}_{X,Y}} &: \text{Ext}^2(\mathbb{L}_{X/k}, \mathcal{O}_X) \oplus \text{Ext}^2(\mathbb{L}_{Y/k}, \mathcal{O}_Y) \\ &\rightarrow \text{Ext}^2(Lp_X^* \mathbb{L}_{X/k} \oplus Lp_Y^* \mathbb{L}_{Y/k}, \mathcal{O}_{X \times Y}), \end{aligned}$$

are given as direct sums of morphisms:

$$\begin{aligned} \text{Ext}^\bullet(\mathbb{L}_{X/k}, \mathcal{O}_X) &\rightarrow \text{Ext}^\bullet(\mathbb{L}_{X/k}, Rp_{X*} \mathcal{O}_{X \times Y}) \simeq \text{Ext}^\bullet(Lp_X^* \mathbb{L}_{X/k}, \mathcal{O}_{X \times Y}); \\ \text{Ext}^\bullet(\mathbb{L}_{Y/k}, \mathcal{O}_Y) &\rightarrow \text{Ext}^\bullet(\mathbb{L}_{Y/k}, Rp_{Y*} \mathcal{O}_{X \times Y}) \simeq \text{Ext}^\bullet(Lp_Y^* \mathbb{L}_{Y/k}, \mathcal{O}_{X \times Y}), \end{aligned}$$

which arise from the natural distinguished triangles

$$\mathcal{O}_X \longrightarrow Rp_{X*} \mathcal{O}_{X \times Y} \longrightarrow C_{p_X} \quad \text{and} \quad \mathcal{O}_Y \longrightarrow Rp_{Y*} \mathcal{O}_{X \times Y} \longrightarrow C_{p_Y},$$

induced by the structure morphisms $p_X^\#$ and $p_Y^\#$ of the projections. Here C_{p_X} and C_{p_Y} denote mapping cones in the derived category.

By the assumptions and the spectral sequence:

$$E_{ij}^2 = \text{Ext}^i(\mathbb{L}_{X/k}, \mathcal{H}^j(C_{p_X})) \Rightarrow \text{Ext}^{i+j}(\mathbb{L}_{X/k}, C_{p_X})$$

we see that $\text{Ext}^1(\mathbb{L}_{X/k}, C_{p_X}) = H^1(Y, \mathcal{O}_Y) \otimes_k \text{Ext}^0(\mathbb{L}_{X/k}, \mathcal{O}_X) = 0$. Analogously we obtain $\text{Ext}^1(\mathbb{L}_{Y/k}, C_{p_Y}) = 0$. Therefore $T_{\text{prod}_{X,Y}}$ is surjective and $\text{Ob}_{\text{prod}_{X,Y}}$ is injective, which by [FM98, Lemma 6.1] implies that $\text{prod}_{X,Y}$ is a smooth morphism of deformation functors. \square

2.2. Descending deformations along morphisms. One of our main tools is the following proposition, which one can prove along the same lines as [LS14, Proposition 2.2]. See also [CvS09] and [Wah79], where this idea appeared previously.

Proposition 2.3. (1) *Let $f : Y \rightarrow X$ be a map satisfying $Rf_*\mathcal{O}_Y = \mathcal{O}_X$. Then there exists a natural transformation $\text{Def}_Y \rightarrow \text{Def}_X$. More generally, if $W = \{W_i\}_{i \in I}$ (resp. $Z = \{Z_i\}_{i \in I}$) is a family of closed subschemes of Y (resp. X) parametrized by a preorder I (cf. §1.1), and if $Rf_*\mathcal{O}_{W_i} = \mathcal{O}_{Z_i}$ (in particular, $Z_i = f(W_i)$), then there exists a natural transformation $\text{Def}_{Y,W} \rightarrow \text{Def}_{X,Z}$.*

(2) *Let X be a smooth scheme, $Z \subseteq X$ a smooth closed subscheme of codimension ≥ 2 , $f : Y = \text{Bl}_Z X \rightarrow X$ the blow-up of X along Z , $E = \{E_j\}_{j \in J}$ the set of connected components of $f^{-1}(Z)$. Then the forgetful transformation $\text{Def}_{Y,E} \rightarrow \text{Def}_Y$ is an isomorphism (here the index set J is given the trivial order). More generally, if $W = \{W_i\}_{i \in I}$ is a family of closed subschemes of Y , then the forgetful transformation $\text{Def}_{Y,W \sqcup E} \rightarrow \text{Def}_{Y,W}$ is an isomorphism. Here by $W \sqcup E$ we mean the family $\{W_i\}_{i \in I} \sqcup \{E_j\}_{j \in J}$ parametrized by $I \sqcup J$ with no non-trivial relations between I and J .*

As a simple corollary we obtain:

Proposition 2.4. *Let $f : X \rightarrow Y$ be a morphism of schemes over a field k satisfying $H^1(X, \mathcal{O}_X) = H^1(Y, \mathcal{O}_Y) = 0$. If $\text{Bl}_{\Gamma_f}(X \times Y)$ lifts to $A \in \mathbf{Art}_{W(k)}(k)$, then there exist A -liftings of X and Y together with a lifting of f .*

Proof. Assume $\text{Bl}_{\Gamma_f}(X \times Y)$ lifts to A . By Proposition 2.3 there exists a deformation $\widetilde{X \times Y}$ of the product $X \times Y$ together with an embedded deformation $\widetilde{\Gamma_f}$ of Γ_f . By Proposition 2.2 the A -scheme $\widetilde{X \times Y}$ is isomorphic to $\widetilde{X} \times_{\text{Spec}(A)} \widetilde{Y}$ for some deformations of X and Y . The restriction of the projection $\widetilde{p}_X : \widetilde{X} \times_{\text{Spec}(A)} \widetilde{Y} \rightarrow \widetilde{X}$ to $\widetilde{\Gamma_f}$ is an isomorphism (as its restriction to $\text{Spec}(k)$ is an isomorphism) and therefore the tuple $(\widetilde{X}, \widetilde{Y}, \widetilde{p}_Y \circ (\widetilde{p}_X|_{\widetilde{\Gamma_f}})^{-1})$ gives the desired pair of liftings of X and Y together with a lifting of f . □

2.3. Regular sequences and flatness. In the proof of Theorem 4.1, we need the following simple claim: if R is a three-dimensional regular local k -algebra with residue field k , L and H a smooth curve and a smooth hypersurface in $X = \text{Spec } R$ intersecting transversally at the closed point P , then any embedded deformation of (X, L, H) induces a deformation of P inside L and H . This claim is implied by the following general results regarding deformations and regular sequences.

Lemma 2.5. *Suppose (A, \mathfrak{m}_A) is an element of $\mathbf{Art}_{W(k)}(k)$ and R is a local k -algebra. Moreover let S be an A -flat local ring such that $S \otimes_A k = R$. Then for any element $f \in S$ such that $\bar{f} \in R$ (we denote by \bar{f} the image of f under the natural map $S \rightarrow R$) is a non-zero divisor the following assertions hold true:*

- (1) *the element f is a non-zero divisor in S ,*
- (2) *the quotient ring $S/(f)$ is A -flat.*

Proof. The proof of the first claim follows by induction with respect to the length of A . For the case $\text{len}(A) = 1$ we know that $A = k$ and therefore the claim is clear. For $\text{len}(A) > 1$, we observe that (A, \mathfrak{m}_A) is an extension of $(A', \mathfrak{m}_{A'}) \in \mathbf{Art}_{W(k)}(k)$ of $\text{len}(A') = \text{len}(A) - 1$ by a principal ideal $I = (s)$ satisfying $\mathfrak{m}_A I = 0$.

Now, take an element $g \in S$ such that $gf = 0$. By the induction hypothesis applied for A' the element $[f] \in S' \stackrel{\text{def}}{=} S \otimes_A A'$ (we denote by $[g]$ the image of g under the natural map $S \rightarrow S'$) is a non-zero divisor in S' which implies that $[g] = 0$ and therefore there exists an element $g' \in S$ such that $g = sg'$. Consequently from the relation $sg'f = 0$ we infer that $g'f \in \text{Ann}(s)$ which by the A -flatness of S means that $g'f \in \mathfrak{m}_A S$, that is, $\overline{g'} \cdot \overline{f} = 0$. By the induction hypothesis applied for $A = k$ we see that $g' \in \mathfrak{m}_A S$ which yields that $g \in \mathfrak{m}_A I \cdot S = (0)$. This implies that f is a non-zero divisor and thus proves the first part of the lemma.

The proof of the second claim is a standard application of local criterion of flatness applied to an A -flat resolution:

$$0 \longrightarrow S \xrightarrow{f \cdot} S \longrightarrow S/(f) \longrightarrow 0$$

implied by the first claim. □

Corollary 2.6. *Let the rings (A, \mathfrak{m}_A) , R and S be as above. Moreover, let $f_1, \dots, f_k \in S$ for $k \geq 1$ be a sequence of elements such that their reductions $\overline{f_1}, \dots, \overline{f_k} \in R$ form a regular sequence in R . Then f_1, \dots, f_k is a regular sequence in S and $S/(f_1, \dots, f_k)$ is an A -flat lifting of $R/(\overline{f_1}, \dots, \overline{f_k})$.*

Proof. The proof follows from Lemma 2.5 by induction with respect to the parameter k . □

2.4. Blow-up formulas. In this section, we review formulas for the cohomology of the blow-up of a smooth proper scheme X along a smooth subscheme Z , and deduce statements regarding Hodge–de Rham degeneration, ordinarity, and the Hodge–Witt property. It is best to deduce the blow-up formulas for different cohomology theories from a single motivic statement.

Proposition 2.7 (cf. [Voe00, 3.5.3]). *Suppose that X is a smooth proper scheme over a field k , $Z \subseteq X$ a smooth closed subscheme of codimension c . Then there is a decomposition of Chow motives*

$$M(\text{Bl}_Z X) = M(X) \oplus \bigoplus_{i=1}^{c-1} M(Z)(i)[2i].$$

In particular, $[\text{Bl}_Z X] = [X] + (\mathbb{L} + \mathbb{L}^2 + \dots + \mathbb{L}^{c-1})[Z]$ in the Grothendieck ring of varieties, where $\mathbb{L} = [\mathbb{A}_k^1]$.

Corollary 2.8. *Suppose that X is a smooth proper scheme over a field k , $Z \subseteq X$ a smooth closed subscheme of codimension c . Let H^n denote one of the following families of functors of smooth projective varieties X :*

- (1) $H^n(X \otimes \bar{k}, \mathbb{Z}_\ell)$ for some ℓ invertible in k , treated as a $\text{Gal}(\bar{k}/k)$ -module,
- (2) (if k is perfect of characteristic $p > 0$) $H^n(X/W(k))$, the integral crystalline cohomology, a $W(k)$ -module with a σ -linear endomorphism induced by the Frobenius,
- (3) $H_{dR}^n(X) = H^n(X, \Omega_{X/k}^\bullet)$, the algebraic de Rham cohomology, endowed with the Hodge filtration,
- (4) $H_{Hdg}^n(X) = \bigoplus_{p+q=n} H^q(X, \Omega_{X/k}^p)$, Hodge cohomology, a graded k -vector space,
- (5) $H_{HW}^n(X) = \bigoplus_{p+q=n} H^q(X, W\Omega_X^p)$, Hodge–Witt cohomology, a graded $W(k)$ -module endowed with $\sigma^{\pm 1}$ -linear endmorphisms F and V satisfying $FV = p = VF$,

(6) (if n is even) $A^{n/2}(X)$, the $\frac{n}{2}$ -th Chow group of X (treated as an abelian group). Moreover, let $-(n)$ denote the Tate twist, i.e., the tensor product with $H^2(\mathbb{P}^1)^{\otimes n}$ in the appropriate tensor category. Then there is a natural isomorphism of objects in the appropriate category as listed above

$$H^n(\mathrm{Bl}_Z(X)) = H^n(X) \oplus \bigoplus_{i=1}^{c-1} H^{n-2i}(Z)(i).$$

Proof. This follows from Proposition 2.7 and the fact that the cohomology theories H above all admit cycle class maps and actions by correspondences. For ℓ -adic and crystalline cohomology this is well known, and for Hodge and Hodge–Witt cohomology it follows from the work of Chatzistamatiou and Rülling [CR11]. \square

2.5. Hodge–de Rham degeneration, ordinarity, and the Hodge–Witt property. Let X be a smooth proper scheme over k . The first hypercohomology spectral sequence of the de Rham complex $\Omega_{X/k}^\bullet$,

$$E_1^{i,j} = H^j(X, \Omega_{X/k}^i) \Rightarrow H_{dR}^{i+j}(X/k) := H^{i+j}(X, \Omega_{X/k}^\bullet),$$

is called the *Hodge–de Rham spectral sequence* of X . We say that it *degenerates* if it degenerates on the first page, i.e., there are no non-zero differentials. As X is proper, the cohomology groups are finite dimensional, and hence the degeneration is equivalent to the condition that

$$(1) \quad \dim H_{dR}^n(X/k) = \sum_{p+q=n} \dim H^q(X, \Omega_{X/k}^p) \quad \text{for all } n \geq 0.$$

The Hodge–de Rham spectral sequence of X degenerates if k is of characteristic zero, or if $\dim X \leq p = \mathrm{char} k$ and X lifts to $W_2(k)$ [DI87, Corollaire 2.4].

The scheme X is called *ordinary* (in the sense of Bloch and Kato) if the Frobenius $F : H^q(X, W\Omega_X^p) \rightarrow H^q(X, W\Omega_X^p)$ on Hodge–Witt cohomology is bijective for all p and q (cf. [BK86, Proposition 7.3]) for several equivalent criteria). It is called *Hodge–Witt* if the Hodge–Witt groups $H^q(X, W\Omega_X^p)$ are finitely generated $W(k)$ -modules. It follows from [IR83, IV 4] that X is Hodge–Witt if it is ordinary, and that $X \times Y$ is ordinary if X and Y are.

Corollary 2.9. *Suppose that X is a smooth proper scheme over a field k , $Z \subseteq X$ a smooth closed subscheme of codimension > 1 . Then*

- (1) *The Hodge–de Rham spectral sequences of Z and X degenerate if and only if the Hodge–de Rham sequence of $\mathrm{Bl}_Z X$ degenerates.*
- (2) *The scheme $\mathrm{Bl}_Z(X)$ is ordinary (resp. Hodge–Witt) if and only if both X and Y are ordinary (resp. Hodge–Witt).*

Proof. The first assertion follows from Corollary 2.8 for H_{dR}^n and H_{Hdg}^n and (1). For the latter, use Corollary 2.8 for H_{HW}^n and the characterizations given above. \square

3. THE FIRST CONSTRUCTION

We fix a semisimple algebraic group G over $k = \mathbb{F}_p$, a reduced parabolic subgroup $P \subseteq G$, and set $Y = G/P$. We assume that either G is of type A and Y is not a projective space, or that P is contained in a maximal parabolic subgroup as listed in [BTLM97, 4.3.1–4.3.7] (these are the cases in which we know that Y does not lift to $W_2(k) = \mathbb{Z}/p^2\mathbb{Z}$ together with Frobenius). For example, Y could be the

Grassmannian $\text{Gr}(n, k)$ ($1 < k < n - 1$) or the full flag variety SL_n/B ($n \geq 3$, $B =$ upper-triangular matrices), or a smooth quadric hypersurface in \mathbb{P}^n , $n \geq 4$. Presumably all homogeneous spaces which are not toric (i.e., not a product of projective spaces) do not admit a lift to $W_2(k)$ together with Frobenius.

Proposition 3.1. *Let Y be a homogeneous space as described above. Then, Y does not admit a $W_2(k)$ -lifting compatible with Frobenius. Moreover, it satisfies $H^1(Y, \mathcal{T}_Y) = 0$ and $H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$.*

Proof. For the part of the proof concerning Frobenius liftability see [BTLM97, Theorem 6]. Vanishing of $H^1(Y, \mathcal{T}_Y)$ follows from [Dem77, Théorème 2]. Finally, $H^i(Y, \mathcal{O}_Y) = 0$ is the consequence of Kempf vanishing (i.e., a characteristic p analogue of the Borel–Weil–Bott theorem) as 0 is a dominant weight for the parabolic subgroup of G corresponding to Y . \square

Theorem 3.2. *Let $\Gamma_F \subseteq Y \times Y$ be the graph of the Frobenius morphism $F_Y : Y \rightarrow Y$. Let $X = \text{Bl}_{\Gamma_F}(Y \times Y)$ be the blow-up of $Y \times Y$ along Γ_F , and $X' = \text{Bl}_{\Delta_Y}(Y \times Y)$ the blow-up of $Y \times Y$ along the diagonal. Then X and X' share the properties (1)–(5) of §1 and moreover:*

- (a) *they are ‘étale homeomorphic’, i.e., their étale sites are equivalent;*
- (b) *their ℓ -adic integral cohomology rings are isomorphic as Galois representations;*
- (c) *their integral crystalline cohomology groups are isomorphic torsion-free F -crystals.*

However, X' admits a projective lift to $W(k)$, while X lifts neither to characteristic zero (even formally), nor to $W_2(k)$.

Proof. The property (1) follows from Bruhat decomposition and the birational invariance of the étale fundamental group of smooth varieties. Properties (2)–(5) follow from the results of §§2.4–2.5. Property (a) follows from the existence of the following cartesian diagram:

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ f \downarrow & & \downarrow f' \\ Y \times Y & \xrightarrow{\text{id} \times F_Y} & Y \times Y, \end{array}$$

where f and f' are the respective blow-up maps. Indeed, the Frobenius map $F_{X'} : X' \rightarrow X'$ and the composition $(F_Y \times \text{id}) \circ f' : X' \rightarrow Y \times Y$ yield a map $v : X' \rightarrow X$ making the diagram

$$\begin{array}{ccccccc} & & F_{X'} & & F_X & & \\ & \xrightarrow{\quad} & & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\ X' & \xrightarrow{v} & X & \xrightarrow{u} & X' & \xrightarrow{v} & X \\ & \downarrow f' & \downarrow f & & \downarrow f' & & \downarrow f \\ Y \times Y & \xrightarrow{F_Y \times \text{id}} & Y \times Y & \xrightarrow{\text{id} \times F_Y} & Y \times Y & \xrightarrow{F_Y \times \text{id}} & Y \times Y \\ & & F_{Y \times Y} & & F_{Y \times Y} & & \end{array}$$

commute. In particular, $v \circ u = F_X$ and $u \circ v = F_{X'}$. Since F_X and $F_{X'}$ are étale homeomorphisms by [Gro77, XIV=XV §1 $n^\circ 2$, Pr. 2(c)], u and v are étale homeomorphisms as well. Property (b) follows from (a). Finally, property (c) follows from

the blow-up formula in §2.4. We remark that the crystalline cohomology algebras $H_{\text{cris}}^*(X/W)$ and $H_{\text{cris}}^*(X'/W)$ are not isomorphic, but become so after inverting p .

We now prove that X' lifts to $W(k)$ projectively and that X does not lift either to $W_2(k)$ or any ramified extension of $W(k)$. For the first claim, we observe that Y lifts to a projective scheme \mathcal{Y} over $W(k)$ and consequently $\mathcal{X}' = \text{Bl}_{\Delta_{\mathcal{Y}}}(\mathcal{Y} \times_{W(k)} \mathcal{Y})$ is a projective lifting of X' .

For the second claim we suppose that there exists a $W_2(k)$ -lifting of $X = \text{Bl}_{\Gamma_F}(Y \times Y)$. By Proposition 2.4 there exist two liftings \tilde{Y} and \tilde{Y}' of Y together with a lifting $\tilde{F}_Y : \tilde{Y} \rightarrow \tilde{Y}'$ of $F_Y : Y \rightarrow Y$. However, by the property $H^1(Y, \mathcal{T}_Y) = 0$ the homogeneous space Y is rigid, which implies that the lifting \tilde{Y}' is isomorphic to \tilde{Y} . This implies that Y is $W_2(k)$ -liftable compatibly with Frobenius, which contradicts Proposition 3.1.

Finally, we address characteristic zero non-liftability of X . Again, we reason by contradiction. Any characteristic zero lifting of X induces a formal lifting of X which by Proposition 2.4 and rigidity of Y gives a formal lifting of a non-trivial endomorphism $F_Y : Y \rightarrow Y$. By the Grothendieck algebraization theorem the formal lifting of the finite morphism F_Y extends to an algebraic lifting which contradicts the final result of [PS89] stating that homogeneous spaces in characteristic zero not isomorphic to products of projective spaces admit no non-trivial endomorphisms. □

4. SECOND CONSTRUCTION

We work over an algebraically closed field k of characteristic p . Let $P = \mathbb{P}^3(\mathbb{F}_p) \subseteq \mathbb{P}_k^3$ be the set of all \mathbb{F}_p -rational points, and let $Y = \text{Bl}_P \mathbb{P}_k^3$, and let L be the set of all lines in \mathbb{P}_k^3 meeting P at least twice. Finally, let $\tilde{L} \subseteq Y$ be the set of the strict transforms of all elements of L , and let $X = \text{Bl}_{\tilde{L}} Y$.

Theorem 4.1. *The three-fold X has the properties (1)–(5) from the introduction, but does not admit a lift to any ring A with $pA \neq 0$.*

For the properties (1)–(5), we argue exactly as in the previous section. The proof that X does not deform to any algebra A with $pA \neq 0$ consists of the following three propositions.

Proposition 4.2. *Let A be an object of $\mathbf{Art}_{W(k)}(k)$, and suppose that X lifts to A . Then \mathbb{P}_k^3 lifts to A together with all \mathbb{F}_p -rational points and lines, preserving the incidence relations.*

Proof. Let E be the set consisting of the preimages in Y of the elements of P , F the set of preimages in X of the elements of \tilde{L} . Finally, let $Q = (\bigcup \tilde{L}) \cap (\bigcup E)$ (treated as a set of closed points). We have the following chain of natural transformations between various deformation functors:

$$\begin{aligned} \text{Def}_X &\xleftarrow{\cong} \text{Def}_{X,F} \longrightarrow \text{Def}_{Y,\tilde{L}} \xleftarrow{\cong} \text{Def}_{Y,\tilde{L} \cup E} \xleftarrow{\cong} \text{Def}_{Y,\tilde{L} \cup E \cup Q} \\ &\hspace{20em} \longrightarrow \text{Def}_{\mathbb{P}_k^3, L \cup P}. \end{aligned}$$

We remind the reader of our convention (cf. §1.1) that for a family of closed subschemes $Z = \{Z_i\}_{i \in I}$ of a scheme X indexed by a preorder I , $\text{Def}_{X,Z}$ is the functor of deformations of X , together with embedded deformations of Z_i , preserving the inclusion relations $Z_i \subseteq Z_{i'}$ for $i \leq i'$. Above, we give the families $F, \tilde{L}, \tilde{L} \cup E$ the

trivial order, and order $\tilde{L} \cup E \cup Q$ and $L \cup P$ by inclusion. In particular, the functor $\text{Def}_{Y, \tilde{L} \cup E \cup Q}$ parametrizes deformations of Y together with the strict transforms of the \mathbb{F}_p -rational lines (i.e., \tilde{L}) and the preimages of the \mathbb{F}_p -rational points (i.e., E) in \mathbb{P}_k^3 such that their mutual intersections are flat over the base (i.e., induce a compatible embedded deformation of Q). Similarly, $\text{Def}_{\mathbb{P}_k^3, P \cup L}$ is the functor of deformations of \mathbb{P}_k^3 together with all the \mathbb{F}_p -rational points and lines, preserving the incidence relations. We discuss the maps in this chain below.

The maps $\text{Def}_{X, F} \rightarrow \text{Def}_X$, $\text{Def}_{Y, \tilde{L} \cup E} \rightarrow \text{Def}_{Y, \tilde{L}}$, and $\text{Def}_{Y, \tilde{L} \cup E \cup Q} \rightarrow \text{Def}_{Y, \tilde{L} \cup E}$ are the forgetful transformations. The first two are isomorphisms by Proposition 2.3(2), and the last map is an isomorphism by Corollary 2.6 of Lemma 2.5 applied to the local equations of E and \tilde{L} .

The maps $\text{Def}_{X, F} \rightarrow \text{Def}_{Y, \tilde{L}}$ and $\text{Def}_{Y, \tilde{L} \cup E \cup Q} \rightarrow \text{Def}_{\mathbb{P}_k^3, L \cup P}$ are the maps of Proposition 2.3(1). For the latter, strictly speaking, Proposition 2.3(1) yields a map $\text{Def}_{Y, \tilde{L} \cup E \cup Q} \rightarrow \text{Def}_{\mathbb{P}_k^3, Z}$, where $Z = \{Z_s\}_{s \in S}$ is the ‘image’ of $\tilde{L} \cup E \cup Q$, defined as follows. Let $S = L \sqcup P \sqcup K$ where $K = \{(x, \ell) \in P \times L : x \in \ell\}$, given the ordering whose non-trivial relations are $(\ell, x) \leq \ell$ and $(\ell, x) \leq x$ for $x \in P, \ell \in L, (x, \ell) \in K$. Then set $Z_\ell = \ell$ for $\ell \in L$, $Z_x = x$ for $x \in P$, and $Z_{(x, \ell)} = x$ for $(x, \ell) \in K$. For an algebra A , an element of $\text{Def}_{\mathbb{P}_k^3, Z}$ is thus given by a deformation of \mathbb{P}_k^3 together with deformations of the $\ell \in L, x \in X$, and $Z_{(x, \ell)} = x$ for $(x, \ell) \in K$, preserving the relations $\tilde{Z}_{(x, \ell)} \subseteq \tilde{x}$ and $\tilde{Z}_{(x, \ell)} \subseteq \tilde{\ell}$ for $(x, \ell) \in K$ (here the tildes mean the corresponding deformations over A). But each x is a point, so $\tilde{Z}_{(x, \ell)} \subseteq \tilde{x}$ implies $\tilde{Z}_{(x, \ell)} = \tilde{x}$, and the deformation of (\mathbb{P}_k^3, Z) simplifies to a deformation of $(\mathbb{P}_k^3, L \cup P)$ preserving the incidence relations. Thus $\text{Def}_{\mathbb{P}_k^3, Z}$ can be identified with $\text{Def}_{\mathbb{P}_k^3, L \cup P}$. \square

Remark 4.3. Since we will have to deal with a little bit of elementary projective geometry and matroid representability over arbitrary rings, let us fix some conventions. Let A be a local ring with residue field k . A projective n -space \mathbb{P} over A is an A -scheme of the form $\mathbb{P}_A(V)$ for a finitely generated free A -module V , and a d -dimensional linear subspace L of \mathbb{P} is the image of a map $\mathbb{P}_A(W) \rightarrow \mathbb{P}_A(V)$ induced by a surjective map $V \rightarrow W$. Zero-dimensional linear subspaces of \mathbb{P} can be identified with the set $\mathbb{P}(A)$. If $x, y \in \mathbb{P}(A)$ are points, given by the surjections $V \rightarrow L_x$ and $V \rightarrow L_y$ respectively, whose images in $\mathbb{P}(k)$ are distinct, there exists a unique line (i.e., a one-dimensional linear subspace) $\ell(x, y)$ containing both x and y induced by the surjection $V \rightarrow L_x \oplus L_y$. We say that points x, y, z are collinear (resp. coplanar) if they lie on one line (resp. two-dimensional subspace). If x_0, \dots, x_n, z are points whose images in $\mathbb{P}(k)$ are in general position, there exists a unique isomorphism $\phi : \mathbb{P} \rightarrow \mathbb{P}_A^n$ such that $\phi(x_i) = e_i := (0 : \dots : 0 : 1 : 0 : \dots : 0)$ (with 1 on the i -th coordinate) and $\phi(z) = f := (1 : \dots : 1)$. In particular, if $A \in \mathbf{Art}_{W(k)}(k)$, and S is a configuration of linear subspaces of \mathbb{P}_k^n ordered by inclusion, containing the points $e'_i = (0 : \dots : 0 : 1 : 0 : \dots : 0)$ and $f' = (1 : \dots : 1)$, we can identify the deformation functor $\text{Def}_{\mathbb{P}_k^n, S}(A)$ with the set of all families of linear subspaces \tilde{S} in \mathbb{P}_A^n which yield the given S upon restriction to k , and such that $\tilde{e}'_i = e_i$ and $\tilde{f}' = f$.

Proposition 4.4. *Suppose that \mathbb{P}_k^3 lifts to an Artinian $W(k)$ -algebra A together with all \mathbb{F}_p -rational points, preserving collinearity. Then \mathbb{P}_k^2 lifts to A together with all \mathbb{F}_p -rational points, preserving collinearity as well.*

Proof. The key observation is that coplanarity is also preserved, i.e., that $\text{Def}_{\mathbb{P}_k^3, LUP} = \text{Def}_{\mathbb{P}_k^3, H \cup L \cup P}$, where H denotes the set of all \mathbb{F}_p -rational hyperplanes in \mathbb{P}_k^3 (with $H \cup L \cup P$ ordered by inclusion). Indeed, let A be an object of $\mathbf{Art}_{W(k)}(k)$, and suppose we are given an element of $\text{Def}_{\mathbb{P}_k^3, LUP}(A)$, which by simple rigidification (e.g., the requirement that the points $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$, and $(1 : 1 : 1)$ do not deform) can be identified with a configuration of points \tilde{x} and lines $\tilde{\ell}$ in \mathbb{P}_A^3 , indexed by P and L respectively, such that $\tilde{x} \subseteq \tilde{\ell}$ whenever $x \in \ell$. To get an element of $\text{Def}_{\mathbb{P}_k^3, H \cup L \cup P}$, it suffices to show that whenever $x_1, x_2, x_3, x_4 \in P$ is a quadruple of coplanar points, the points $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \in \mathbb{P}^3(A)$ are coplanar. If two of the points x_i coincide, there is nothing to show, and similarly if all four lie on a line. Otherwise, let $\ell_{12} = \ell(x_1, x_2)$ and $\ell_{34} = \ell(x_3, x_4)$; then $\tilde{\ell}_{12} = \ell(\tilde{x}_1, \tilde{x}_2)$ and $\tilde{\ell}_{34} = \ell(\tilde{x}_3, \tilde{x}_4)$. Since the x_i are coplanar, the lines ℓ_{12} and ℓ_{34} intersect in a unique point $y \in P$. Then $\tilde{y} \in \tilde{\ell}_{12} \cap \tilde{\ell}_{34} = \ell(\tilde{x}_1, \tilde{x}_2) \cap \ell(\tilde{x}_3, \tilde{x}_4)$. Thus the hyperplane through $\tilde{y}, \tilde{x}_1, \tilde{x}_2$ yields a lift of the hyperplane through x_1, x_2, x_3, x_4 .

Since coplanarity is preserved, we can forget everything except for the plane $x_0 = 0$ (say) and get a desired lifting of \mathbb{P}_k^2 . Equivalently, we could have used a projection from one of the \mathbb{F}_p -rational points. \square

To finish, we prove that the matroid $\mathbb{P}^2(\mathbb{F}_p)$ does not admit a projective representation over any ring A with $pA \neq 0$. For A a field, this is well known (cf. e.g. [Gor88, §2]), but we need to make sure that the proof works for arbitrary rings.

Proposition 4.5. *Let A be a ring, $\rho : \mathbb{P}^2(\mathbb{F}_p) \rightarrow \mathbb{P}^2(A)$ a map taking triples of collinear points to triples of collinear points. Then $pA = 0$.*

Proof. Changing coordinates in $\mathbb{P}^2(A)$, we can assume that

$$\rho(1 : 0 : 0) = (1 : 0 : 0), \quad \rho(0 : 1 : 0) = (0 : 1 : 0), \quad \rho(0 : 0 : 1) = (0 : 0 : 1),$$

and $\rho(1 : 1 : 1) = (1 : 1 : 1)$. Thus

$$\rho(1 : 1 : 0) = \rho(\ell((0 : 0 : 1), (1 : 1 : 1)) \cap \ell((1 : 0 : 0), (0 : 1 : 0))) = (1 : 1 : 0)$$

as well. For $n \in \mathbb{Z}$, let $P_n = (n : 0 : 1)$, $Q_n = (n + 1 : 1 : 1) \in \mathbb{P}^2(\mathbb{F}_p)$, and let $P'_n, Q'_n \in \mathbb{P}^2(A)$ be the points with the same coordinates as P_n, Q_n . We check by induction on $n \geq 0$ that $\rho(P_n) = P'_n$ and $\rho(Q_n) = Q'_n$: the base case is ok, and for the induction step we note that $P_n = \ell(Q_{n-1}, (0 : 1 : 0)) \cap \ell(P_0, (1 : 0 : 0))$, $Q_n = \ell(P_n, (1 : 1 : 0)) \cap \ell(Q_0, (1 : 0 : 0))$, and that the same statements hold with the primes (see Figure 1). Thus $\rho(p : 0 : 1) = \rho(p : 0 : 1) = \rho(0 : 0 : 1) = (0 : 0 : 1)$, and hence $p = 0$ in A . \square

Remark 4.6. Note that the proof exhibits a sub-matroid (denoted M_p in [Gor88]) consisting of $2p + 3$ points sharing the desired property of $\mathbb{P}^2(\mathbb{F}_p)$. This means that in our second non-liftable example we could have blown up a smaller configuration of $2p + 4$ points and (strict transforms of) $4p + 7$ lines between them.

Remark 4.7. With the same proof, one can construct similar examples in higher dimensions: blow-up \mathbb{P}_k^n ($n \geq 3$) in all \mathbb{F}_p -rational points, (strict transforms of) lines, planes, and so on. Such varieties were studied in [RTW13, Definition 1.2] in relation to automorphisms of the Drinfeld half-space.

Remark 4.8. We also remark that in [Lan16, Proposition 8.4] it is proved that a pair (X, D) where X is the blow-up of \mathbb{P}_k^2 in all \mathbb{F}_p -rational points and D is a union

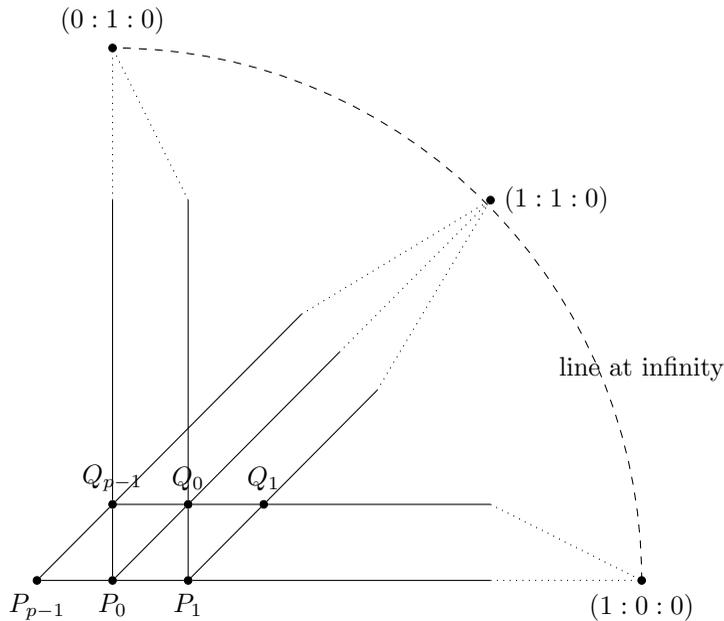


FIGURE 1. Proof of Proposition 4.5

of strict transforms of at least $4p - 3$ \mathbb{F}_p -rational lines does not lift to $W_2(k)$. The argument above proves that the matroid M_p leads to a non-liftable example with a fewer number of lines equal to $2p + 3$. We do not know whether $2p + 3$ is the minimal number of lines necessary to exhibit $W_2(k)$ non-liftability.

Remark 4.9. Related non-liftable examples also appear in [BHH87, Section 3.5J] and [Eas08]. The constructions are based on ramified coverings along purely characteristic p line configurations instead of blow-ups.

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REFERENCES

- [BHH87] Gottfried Barthel, Friedrich Hirzebruch, and Thomas Höfer, *Geradenkonfigurationen und Algebraische Flächen* (German), Aspects of Mathematics, D4, Friedr. Vieweg & Sohn, Braunschweig, 1987. MR912097
- [BK86] Spencer Bloch and Kazuya Kato, *p -adic étale cohomology*, Inst. Hautes Études Sci. Publ. Math. **63** (1986), 107–152. MR849653
- [BTLM97] Anders Buch, Jesper F. Thomsen, Niels Lauritzen, and Vikram Mehta, *The Frobenius morphism on a toric variety*, Tohoku Math. J. (2) **49** (1997), no. 3, 355–366, DOI 10.2748/tmj/1178225109. MR1464183
- [CR11] Andre Chatzistamatiou and Kay Rülling, *Higher direct images of the structure sheaf in positive characteristic*, Algebra Number Theory **5** (2011), no. 6, 693–775, DOI 10.2140/ant.2011.5.693. MR2923726

- [CvS09] Sławomir Cynk and Duco van Straten, *Small resolutions and non-liftable Calabi-Yau threefolds*, Manuscripta Math. **130** (2009), no. 2, 233–249, DOI 10.1007/s00229-009-0293-0. MR2545516
- [Dem77] M. Demazure, *Automorphismes et déformations des variétés de Borel*, Invent. Math. **39** (1977), no. 2, 179–186, DOI 10.1007/BF01390108. MR0435092
- [DI87] Pierre Deligne and Luc Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham* (French), Invent. Math. **89** (1987), no. 2, 247–270, DOI 10.1007/BF01389078. MR894379
- [Eas08] Robert W. Easton, *Surfaces violating Bogomolov-Miyaoka-Yau in positive characteristic*, Proc. Amer. Math. Soc. **136** (2008), no. 7, 2271–2278, DOI 10.1090/S0002-9939-08-09466-5. MR2390492
- [FM98] Barbara Fantechi and Marco Manetti, *Obstruction calculus for functors of Artin rings. I*, J. Algebra **202** (1998), no. 2, 541–576, DOI 10.1006/jabr.1997.7239. MR1617687
- [Gor88] Gary Gordon, *Algebraic characteristic sets of matroids*, J. Combin. Theory Ser. B **44** (1988), no. 1, 64–74, DOI 10.1016/0095-8956(88)90096-2. MR923266
- [Gro77] Alexandre Grothendieck, *Séminaire de géométrie algébrique du bois-marie 1965-66, cohomologie l-adique et fonctions l, sga5*, Springer Lecture Notes, vol. 589, Springer-Verlag, 1977.
- [Ill71] Luc Illusie, *Complexe cotangent et déformations. I* (French), Lecture Notes in Mathematics, Vol. 239, Springer-Verlag, Berlin-New York, 1971. MR0491680
- [IR83] Luc Illusie and Michel Raynaud, *Les suites spectrales associées au complexe de de Rham-Witt* (French), Inst. Hautes Études Sci. Publ. Math. **57** (1983), 73–212. MR699058
- [Lan15] Adrian Langer, *Bogomolov’s inequality for Higgs sheaves in positive characteristic*, Invent. Math. **199** (2015), no. 3, 889–920, DOI 10.1007/s00222-014-0534-z. MR3314517
- [Lan16] Adrian Langer, *The Bogomolov-Miyaoka-Yau inequality for logarithmic surfaces in positive characteristic*, Duke Math. J. **165** (2016), no. 14, 2737–2769, DOI 10.1215/00127094-3627203. MR3551772
- [LR97] N. Lauritzen and A. P. Rao, *Elementary counterexamples to Kodaira vanishing in prime characteristic*, Proc. Indian Acad. Sci. Math. Sci. **107** (1997), no. 1, 21–25, DOI 10.1007/BF02840470. MR1453823
- [LS14] Christian Liedtke and Matthew Satriano, *On the birational nature of lifting*, Adv. Math. **254** (2014), 118–137, DOI 10.1016/j.aim.2013.10.030. MR3161094
- [Mor79] Shigefumi Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) **110** (1979), no. 3, 593–606, DOI 10.2307/1971241. MR554387
- [PS89] K. H. Paranjape and V. Srinivas, *Self-maps of homogeneous spaces*, Invent. Math. **98** (1989), no. 2, 425–444, DOI 10.1007/BF01388861. MR1016272
- [Ray78] M. Raynaud, *Contre-exemple au “vanishing theorem” en caractéristique $p > 0$* (French), C. P. Ramanujam—a tribute, Tata Inst. Fund. Res. Studies in Math., vol. 8, Springer, Berlin-New York, 1978, pp. 273–278. MR541027
- [RTW13] Bertrand Rémy, Amaury Thuillier, and Annette Werner, *Automorphisms of Drinfeld half-spaces over a finite field*, Compos. Math. **149** (2013), no. 7, 1211–1224, DOI 10.1112/S0010437X12000905. MR3078645
- [Ser61] Jean-Pierre Serre, *Exemples de variétés projectives en caractéristique p non relevables en caractéristique zéro* (French), Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 108–109. MR0132067
- [Voe00] Vladimir Voevodsky, *Triangulated categories of motives over a field*, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 188–238. MR1764202
- [Wah79] Jonathan M. Wahl, *Simultaneous resolution and discriminantal loci*, Duke Math. J. **46** (1979), no. 2, 341–375. MR534056

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