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THE STABILITY OF FUBINI-STUDY METRIC ON $\mathbb{CP}^n$

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ABSTRACT. In this note, we study the stability of a critical point of a conformally invariant functional $\mathcal{F}$. For $n \geq 3$, by use of the variational formulas, we prove that the Fubini-Study metric on $\mathbb{CP}^n$ is a strictly stable critical point of $\mathcal{F}$.

1. Introduction

Let M be an n-dimensional closed and smooth manifold. Denote by $\mathscr{M}(M)$ and $\mathscr{G}(M)$ the space of smooth Riemannian metrics and the diffeomorphism group of M, respectively. We recall that a functional $\mathcal{F} : M \to R$ is called Riemannian if $\mathcal{F}$ is invariant under the action of $\mathscr{G}(M)$, i.e., $\mathcal{F}(\varphi^* g) = \mathcal{F}(g)$ for each $\varphi \in \mathscr{G}(M)$ and $g \in \mathscr{M}(M)$.

There are many results about the study of Riemannian functionals in the literature; for example, see [1–4, 7, 10, 11].

In [8], Kobayashi considered the following conformally invariant functional:

$$\mathcal{F}(g) = \frac{2}{n} \int_M \mid W \mid^{\frac{n}{2}} \mathrm{dVol}_g \tag{1.1}$$

where W is the Weyl conformal curvature tensor. His main subject in [8] was to determine $\inf\{\mathcal{F}(g), g \in \mathscr{M}(M)\}$, and he proved the following result:

Theorem 1.1 ([8]).

$$\mathcal{F}(g_{FS}) = \inf\{\mathcal{F}(g), g \in \mathscr{M}(\mathbb{CP}^2)\}.$$

Here g_{FS} is the Fubini-Study metric on $\mathbb{CP}^2$.

To determine $\inf\{\mathcal{F}(g), g \in \mathscr{M}(M)\}$ is not easy, so Kobayashi used variational propositions of $\mathcal{F}$ to study the stability of some critical points.

Definition 1.2. Let $g \in \mathscr{M}(M)$ be a critical point of the functional $\mathcal{F}(g)$. Then g is said to be stable if

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{F}(g) \geq 0 \tag{1.2}$$

for all smooth variations g_t with $g_0 = g$. Moreover, g is said to be strictly stable if g is stable and if equality of (1.2) holds only when $\frac{dg_t}{dt}|_{t=0} \in \mathcal{S}_1(g)$ (see (2.5)).

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For dimension 4, Besse proved that half-conformally flat metrics and metrics which are locally conformal to an Einstein metric are critical points of $\mathcal{F}$ in [1]. In [8], Kobayashi calculated the second variation of $\mathcal{F}$ at critical point, and he obtained the following stable result:

Theorem 1.3 ([8]). *Let g be the standard Einstein metric on $S^2(1) \times S^2(1)$, $g = \bar{g} + \bar{\bar{g}}$, where $\bar{g}$ and $\bar{\bar{g}}$ are Riemannian metrics on $S^2(1)$ with constant Gauss curvature 1. Then, g is a strictly stable critical point of the functional $\mathcal{F}$.*

We generalized the result:

Theorem 1.4 ([5]). *Let g be the standard Einstein metric on $S^3(1)\times S^3(1)$, that is, $g = \bar{g} + \bar{\bar{g}}$, where $\bar{g}$ and $\bar{\bar{g}}$ are Riemannian metrics on $S^3(1)$ with constant sectional curvature 1. Then g is a strictly stable critical point of the functional $\mathcal{F}$.*

For $n \geq 3$, it is still an unsolved problem whether the Fubini-Study metric g_{FS} on $\mathbb{CP}^n$ is the minimum point of $\mathcal{F}$ on $\mathbb{CP}^n$. In this note, we consider its stability and prove the following result:

Theorem 1.5. *Let g_{FS} be the Fubini-Study metric on $\mathbb{CP}^n$. Then g_{FS} is a strictly stable critical point of $\mathcal{F}$ for $n \geq 3$.*

2. Preliminaries and notation

Let (M, g) be an n-dimensional Riemannian manifold. We choose a local orthonormal vector field $\{e_1, \cdots, e_n\}$ adapted to the Riemannian metric g. The Riemannian curvature tensor is defined by

$$R(e_i, e_j, e_k, e_l) = g(\nabla_{e_i}\nabla_{e_j}e_l - \nabla_{e_j}\nabla_{e_i}e_l - \nabla_{[e_i,e_j]}e_l, e_k); \tag{2.1}$$

here ∇ is the Levi-Civita connection of g. Let W_{ijkl} denote the components of the Weyl curvature tensor of (M, g),

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(C_{ik}g_{jl} - C_{il}g_{jk} + C_{jl}g_{ik} - C_{jk}g_{il}). \tag{2.2}$$

Here C is a symmetric (0,2)-tensor defined by

$$C = Ric - \frac{r}{2(n-1)}g, \tag{2.3}$$

with Ric and r denoting the Ricci curvature tensor and scalar curvature of g, respectively. C is called the *Schouten* tensor.

By denoting $S_2(M)$ the vector space of all symmetric (0,2)-tensor fields on M, we know that $S_2(M) = \mathcal{S}_0(g) \oplus \mathcal{S}_1(g)$ from Lemma 3.6 in [8], where

$$\mathcal{S}_0(g) = \{h \in S_2M, divh = 0, trh = 0\}, \tag{2.4}$$

$$\mathcal{S}_1(g) = \{L_Xg + fg, X \in TM, f \in C^\infty(M)\}, \tag{2.5}$$

and this decomposition is orthogonal with respect to the L_2 inner product defined by g.

Recall that a Kähler manifold (M, g, J) is a Riemannian manifold (M, g) together with a compatible almost complex structure J, such that $\nabla J = 0$. On $(\mathbb{CP}^n, g_{FS}, J)$, the Kähler form is

$$\Phi = -2\sqrt{-1}\partial\bar{\partial}\ln(z_0\bar{z}_0 + z_1\bar{z}_1 + \cdots + z_n\bar{z}_n);$$

here $\{z_0, z_1, \cdots, z_n\}$ is the natural complex coordinate system of $\mathbb{C}^{n+1}$. Let $\{e_1, \cdots, e_{2n}\}$ be the orthonormal frame. Then its Riemannian curvature tensor can be given by

$$\begin{aligned} R(e_i, e_j, e_k, e_l) =& \frac{1}{2}[g(e_i, e_k)g(e_j, e_l) - g(e_i, e_l)g(e_j, e_k) \\ &+ g(e_i, Je_k)g(e_j, Je_l) - g(e_i, Je_l)g(e_j, Je_k) \\ &+ 2g(e_i, Je_j)g(e_k, Je_l)], \end{aligned} \tag{2.6}$$

and we have

$$Ric = (n+1)g, \ r = 2n(n+1).$$

3. Variational formulas of $\mathcal{F}$ on $\mathbb{CP}^n$

In [8], Kobayashi gave the following variational formula for dimension $n = 4$:

Theorem 3.1 ([8]). *Let M be a compact manifold of dimension 4 for a smooth curve $g = g(t)$ in $\mathcal{M}(M)$. Then*

$$\frac{d}{dt}\mathcal{F}(g) = \int_M \langle X, \frac{d}{dt}g\rangle \, \mathrm{dVol}_g, \tag{3.1}$$

where X is a symmetric 2-tensor defined by $X_{ij} = B_{ijk,k} + C_{mk}W^m_{ijk}$, and B is a Cotten tensor defined by $B_{ijk} = C_{ik,j} - C_{ij,k}$.

From this formula, we can see that Einstein metrics are critical points of $\mathcal{F}$. For general dimension, we get that:

Theorem 3.2 ([5]). *Let M be a compact manifold of dimension n. Then g is a critical point of $\mathcal{F}$ if and only if it satisfies*

$$\begin{aligned} 0 = (\nabla\mathcal{F})_{im} =& - | W |^{\frac{n}{2}-2} W_{ijkl}W_m{}^{jkl} - \frac{2}{n-2} | W |^{\frac{n}{2}-2} W_{ijml}C^{jl} \\ &+ \frac{1}{n} | W |^{\frac{n}{2}} g_{im} + 2(| W |^{\frac{n}{2}-2} W_{ijkm}),^{kj}. \end{aligned} \tag{3.2}$$

With this formula, we prove the following lemma.

Lemma 3.3. *g_{FS} is a critical point of $\mathcal{F}$ on $\mathbb{CP}^n$.*

Proof. In this case, $\nabla Rm = 0$ and g_{FS} is Einstein, and we just need to check that $R_{ijkl}R_{mjkl} = \lambda g_{im}$. Let $\{e_1, \cdots, e_{2n}\}$ be the orthonormal frame. From the expression of Riemannian curvature (2.6), we get

$$\begin{aligned} R_{ijkl}R_{mjkl} &= R_{im} + g(e_i, Je_k)Rm(e_m, Je_l, e_k, e_l) \\ &\quad + g(e_i, Je_j)Rm(e_m, e_j, Je_l, e_l) \\ &= 4R_{im} \\ &= 4(n+1)g_{im}. \end{aligned}$$

In [5], we calculated the second variational formula of $\mathcal{F}$ on torus. Using the same method, we have:

Theorem 3.4. *Let (M, g) be an n-dimensional closed manifold, g a critical point of $\mathcal{F}$ with $\nabla Rm = 0$, g_t a smooth variation of g with $g_0 = g$, and $h = \frac{d}{dt}|_{t=0} g_t \in \mathcal{S}_0(g)$.*

Then

$$
\begin{aligned}
\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{F} = & \, 2(n-4)\int_M |W|^{\frac{n}{2}-4}(W_{ijkm}h_{im,jk})^2\,\mathrm{dVol}_g \\
& + \int_M [\frac{n-3}{n-2}|\triangle h|^2 - \frac{3n-10}{n-2}R_{ijkl}h_{jk}h_{il,ss} \\
& - 2h_{il,jk}R_{mjkl}h_{im} - \frac{r}{n-1}\langle\triangle h, h\rangle \\
& - \frac{4}{n-2}R_{ijkl}h_{jk}R_{imsl}h_{ms} - \frac{2r}{n(n-1)}R_{ijkl}h_{jk}h_{il} \\
& + 2R_{mjsl}R_{ijkl}h_{im}h_{sk} + \frac{2}{(n-1)(n-2)}\langle Ric, h\rangle^2 \\
& - \frac{2(n-3)}{n-2}S^s_{mj}E_{si}(h_{im,j}-h_{jm,i}) - \frac{2}{n-2}E_{jl}h_{im}h_{si}R_{mjsl} \\
& + \frac{2}{n-2}h_{im}(E_{il}S^j_{ml,j} + E_{jk}S^j_{mi,k}) \\
& + \langle E\cdot h, -\triangle h - \frac{1}{n-2}(h\cdot Ric + Ric\cdot h)\rangle]\,\mathrm{dVol}_g\,.
\end{aligned} \tag{3.3}
$$

Here $(Ric\cdot h)_{ij} = \sum_k R_{ik}h_{kj}$, $E = Ric - \frac{r}{n}g$ *is the trace-free Ricci tensor.*

Remark 3.5. When $n = 4$, we know that $W_{ijkl}W_{sjkl} = \frac{|W|^2}{4}g_{is}$. In this case, since $\sum_i W_{ijkl,i} = \frac{n-3}{n-2}B_{jkl}$, $\nabla\mathcal{F}_{ij} = B_{ijk,k} + R_{mk}W_{mijk}$. On higher dimension, we should consider the variation of the tensor $W_{ijkl}W_{sjkl} - \frac{|W|^2}{n}g_{is}$ when calculating the second variational formula of $\mathcal{F}$.

The proof is similar to that of Theorem 4.1 in [5]. For convenience, we sketch the calculation here. Since g is a critical metric, $\nabla\mathcal{F}|_{t=0} = 0$, then

$$
\begin{aligned}
& \frac{d^2}{dt^2}\Big|_{t=0}\mathcal{F} = \int_M \langle \frac{d}{dt}\Big|_{t=0}\nabla\mathcal{F}, h\rangle\,\mathrm{dVol}_g \\
= & \int_M [2(\mid W\mid^{\frac{n}{2}-2} W_{ijkm}),^{kj} + \frac{1}{n}\mid W\mid^{\frac{n}{2}} g_{im} - \mid W\mid^{\frac{n}{2}-2} W_{ijkl}R_m{}^{jkl}]' h^{im}\,\mathrm{dVol}_g\,.
\end{aligned}
$$

With $\nabla Rm = 0$, by a direct computation,

$$
\int_M [(\mid W\mid^{\frac{n}{2}-2}),^k W_{ijkm},^j + (\mid W\mid^{\frac{n}{2}-2}),^j W_{ijkm},^k]' h^{im}\,\mathrm{dVol}_g = 0
$$

and

$$
\int_M [(\mid W\mid^{\frac{n}{2}-2}),^{kj} W_{ijkm}]' h^{im}\,\mathrm{dVol}_g = (n-4)\int_M |W|^{\frac{n}{2}-4}(W_{ijkm}h_{im,jk})^2\,\mathrm{dVol}_g\,.
$$

From the definition of Cotton tensor, we have

$$\begin{aligned}
&\frac{n-2}{n-3}\int_M (W_{ijkm},{}^{kj})' h^{im}\,\mathrm{dVol}_g \\
&= \int_M [B_{mij},{}^{j}]' h^{im}\,\mathrm{dVol}_g \\
&= -\int_M [B_{mij}]' h_{im,j}\,\mathrm{dVol}_g \\
&= \int_M [C_{mi,j}]'(h_{im,j} - h_{jm,i})\,\mathrm{dVol}_g \\
&= \int_M [C'_{mi}(h_{jm,ij} - h_{im,jj}) - S^s_{mj} C_{si}(h_{im,j} - h_{jm,i})]\,\mathrm{dVol}_g \\
&= \int_M [(R'_{mi} - \frac{r}{n} h_{im})(h_{jm,ij} - h_{im,jj}) \\
&\quad - S^s_{mj} E_{si}(h_{im,j} - h_{jm,i})]\,\mathrm{dVol}_g\,.
\end{aligned} \tag{3.4}$$

We also have

$$\begin{aligned}
&\int_M (W_{ijkl} R_m{}^{jkl} - \frac{|W|^2}{n} g_{im})' h^{im}\,\mathrm{dVol}_g \\
&= \int_M [(W_{ijkl})' R_m{}^{jkl} h^{im} + W_{ijkl}(R_m{}^{jkl})' h^{im} - \frac{|W|^2}{n}|h|^2]\,\mathrm{dVol}_g \\
&= \int_M [2(h_{il,jk} - h_{jl,ik}) R_{mjkl} h_{im} - \frac{2}{n-2} h_{im}(C_{il} S^j_{ml,j} + C_{jk} S^j_{mi,k}) \\
&\quad - \frac{2}{n-2}(C^i_k g_{ji} + \delta^i_k C_{ji})' R_{mjkl} h_{im} - 2R_{mjsl} W_{ijkl} h_{im} h_{sk}]\,\mathrm{dVol}_g \\
&= \int_M [2h_{il,jk} R_{mjkl} h_{im} + 2R_{ijkl} h_{jk} R_{imsl} h_{ms} + 2R_{ijkl} h_{jk} h_{is} R_{sl} \\
&\quad - \frac{2}{n-2}(C'_{il}(R_{jk} h_{kl} - R_{ijkl} h_{jk}) - C_{ij} h_{jk} R_{il} h_{lk} - C_{is} h_{sl} R_{ijkl} h_{jk}) \\
&\quad + \frac{r}{n(n-1)}(\langle \triangle h, h\rangle - 2R_{ijkl} h_{jk} R h_{il} - 2R_{ij} h_{jk} h_{ki}) \\
&\quad - 2R_{mjsl} W_{ijkl} h_{im} h_{sk} - \frac{2}{n-2} h_{im}(E_{il} S^j_{ml,j} + E_{jk} S^j_{mi,k})]\,\mathrm{dVol}_g\,.
\end{aligned} \tag{3.5}$$

By combining (3.4) and (3.5), we get (3.3). □

4. Proof of Theorem 1.5

Since g_{FS} is Einstein metric with $Ric = (n+1)g$, from Theorem 3.4, we have

$$\begin{aligned}
\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{F} \geq \int_M |W|^{n-2}\{&\frac{2n-3}{2n-2}|\triangle h|^2 - 2h_{il,jk} h_{im} R_{mjkl} \\
&- \frac{2n(n+1)}{2n-1}\langle \triangle h, h\rangle - \frac{3n-5}{n-1} h_{il,kk} R_{ijml} h_{jm} \\
&- \frac{2}{n-1} R_{ijkl} h_{jk} R_{imsl} h_{ms} + \frac{2(n+1)}{2n-1} R_{ijkl} h_{jk} h_{il} \\
&- 2R_{ijkl} R_{jmsl} h_{im} h_{sk}\}\,\mathrm{dVol}_g\,.
\end{aligned} \tag{4.1}$$

We have from (2.6),

$$R_{ijml}h_{jm} = \frac{1}{2}[-g(e_i, Je_l)g(e_j, Je_m)h(e_j, e_m) - trhg_{il} + h_{il} - 3h(Je_i, Je_l)]. \tag{4.2}$$

For $g(e_j, Je_m)h(e_j, e_m) = 0$ and $trh = 0$, we have

$$R_{ijml}h_{jm} = \frac{1}{2}[h_{il} - 3h(Je_i, Je_l)]. \tag{4.3}$$

By putting (4.3) into (4.1), we get

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{F} \geq \int_M |W|^{n-2}\{&\frac{2n-3}{2n-2}|\triangle h|^2 - 2h_{il,jk}h_{im}R_{mjkl}\\ &-(\frac{2n(n+1)}{2n-1} + \frac{3n-5}{2(n-1)})\langle\triangle h, h\rangle\\ &+\frac{9n-15}{2(n-1)}h_{il,kk}h(Je_i, Je_l) + (\frac{n+1}{2n-1} - \frac{5}{n-1})|h|^2\\ &-(\frac{3(n+1)}{2n-1} - \frac{3}{n-1})h_{il}h(Je_i, Je_l)\\ &-2R_{ijkl}R_{jmsl}h_{im}h_{sk}\}\,\mathrm{dVol}_g\,.\end{aligned} \tag{4.4}$$

We need to compute

$$I := -2R_{ijkl}R_{jmsl}h_{im}h_{sk},$$
$$II := -2h_{il,jk}h_{im}R_{mjkl}.$$

From (2.6), we have

$$\begin{aligned}I = &R_{mj}h_{im}h_{is} - R_{ijkl}h_{jk}h_{il}\\ &+ Ric(e_m, Je_s)h(Je_k, e_m)h(e_k, e_s)\\ &- Rm(e_m, Je_k, e_s, e_l)h(Je_l, e_m)h(e_k, e_s)\\ &+ 2Rm(e_m, e_j, e_s, e_l)h(Je_j, e_m)h(Je_l, e_s).\end{aligned} \tag{4.5}$$

Using the first Bianchi identity, we get

$$\begin{aligned}&2Rm(e_m, e_j, e_s, e_l)h(Je_j, e_m)h(Je_l, e_s)\\ =& -2Rm(e_m, e_s, e_l, e_j)h(Je_j, e_m)h(Je_l, e_s)\\ &- 2Rm(e_m, e_l, e_j, e_s)h(Je_j, e_m)h(Je_l, e_s)\\ =& -2\langle Lh, h\rangle - 2Lh(e_l, e_s)h(Je_l, Je_s)\\ =& 2|h|^2 + 2h(Je_i, Je_j)h(e_i, e_j).\end{aligned} \tag{4.6}$$

Put (4.6) into (4.5); then

$$\begin{aligned}I =&(n+1)|h|^2 + (n+1)h(Je_k, Je_s)h(e_k, e_s)\\ &+ 2Rm(e_m, e_j, e_s, e_l)h(Je_j, e_m)h(Je_l, e_s)\\ &- R_{ijkl}h_{jk}h(Je_i, Je_l) - R_{ijkl}h_{jk}h_{il}\\ =&(n+4)[|h|^2 + h(Je_k, Je_s)h(e_k, e_s)].\end{aligned} \tag{4.7}$$

We have from the expression of Riemannian curvature (2.6),

$$\begin{aligned}II =&h(e_i, Je_l)\nabla^2 h(e_i, e_l, Je_k, e_k) - h(e_i, Je_k)\nabla^2 h(e_i, e_l, Je_l, e_k)\\ &+ \langle\triangle h, h\rangle - 2h(e_i, Je_j)\nabla^2 h(e_i, e_l, e_j, Je_l).\end{aligned} \tag{4.8}$$

From the Ricci identity, we get

$$\begin{aligned}
&\nabla^2 h(e_i, e_l, Je_k, e_k) - \nabla^2 h(e_i, e_l, e_k, Je_k)\\
&= h(e_s, e_l)Rm(e_s, e_i, Je_k, e_k) + h(e_s, e_i)Rm(e_s, e_l, Je_k, e_k)\\
&= 2Ric(e_s, Je_i)h(e_s, e_l) + 2Ric(e_s, Je_l)h(e_s, e_i)\\
&= 2(n+1)[h(Je_i, e_l) + h(Je_l, e_i)].
\end{aligned} \tag{4.9}$$

Since $\nabla^2 h(e_i, e_l, Je_k, e_k) = -\nabla^2 h(e_i, e_l, e_k, Je_k)$, we get

$$\begin{aligned}
&h(e_i, Je_l)\nabla^2 h(e_i, e_l, Je_k, e_k)\\
=&(n+1)[|h|^2 + h(e_i, Je_l)h(Je_i, e_l)]\\
=&(n+1)[|h|^2 - h(e_i, e_l)h(Je_i, Je_l)].
\end{aligned} \tag{4.10}$$

And with (4.3), we have

$$\begin{aligned}
&\nabla^2 h(e_i, Je_l, e_l, e_k) - \nabla^2 h(e_i, Je_l, e_k, e_l)\\
&= h(e_s, Je_l)Rm(e_s, e_i, e_l, e_k) + h(e_i, e_s)Rm(e_s, Je_l, e_l, e_k)\\
&= h(e_s, Je_l)Rm(e_s, e_i, Je_l, Je_k) - h(e_i, e_s)Rm(Je_s, e_l, e_l, e_k)\\
&= -\frac{1}{2}h(e_i, Je_k) - \frac{3}{2}h(Je_i, e_k) - (n+1)h(e_i, Je_k).
\end{aligned} \tag{4.11}$$

So, from (4.8), (4.10) and (4.11), we get

$$\begin{aligned}
II =&\langle \triangle h, h\rangle - 3h(e_i, Je_j)\nabla^2 h(e_i, e_l, e_j, Je_l)\\
&- \frac{1}{2}|h|^2 - (n - \frac{1}{2})h(e_i, e_k)h(Je_i, Je_k).
\end{aligned} \tag{4.12}$$

Putting (4.7) and (4.12) into (4.4), we get

$$\begin{aligned}
\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{F} \geq \int_M |W|^{n-2}\{&\frac{2n-3}{2n-2}|\triangle h|^2 + \frac{9n-15}{2(n-1)}h_{il,kk}h(Je_i, Je_l)\\
&- (\frac{2n^2+1}{2n-1} + \frac{3n-5}{2n-2})\langle \triangle h, h\rangle\\
&- 3h(e_i, Je_j)\nabla^2 h(e_i, e_l, e_j, Je_l)\\
&+ (n + 4 - \frac{5}{n-1} + \frac{3}{2(2n-1)})|h|^2\\
&+ (3 - \frac{9}{2(2n-1)} + \frac{3}{n-1})h(Je_k, Je_s)h(e_k, e_s)\}\, \mathrm{dVol}_g\,.
\end{aligned} \tag{4.13}$$

For $\nabla J = 0$,

$$\begin{aligned}
&|\int_M h(e_i, Je_j)\nabla^2 h(e_i, e_l, e_j, Je_l)\, \mathrm{dVol}_g|\\
=&|\int_M \nabla h(e_i, Je_j, Je_l)\nabla h(e_i, e_l, e_j)\, \mathrm{dVol}_g|\\
\leq&\int_M |\nabla h|^2\, \mathrm{dVol}_g
\end{aligned} \tag{4.14}$$

and

$$\begin{aligned} &| \int_M h_{il,kk} h(Je_i, Je_l)\, \mathrm{dVol}_g \,| \\ (4.15)\qquad &= | - \int_M \nabla h(e_i, e_j, e_k) \nabla h(Je_i, Je_j, e_k)\, \mathrm{dVol}_g \,| \\ &\le \int_M |\nabla h|^2\, \mathrm{dVol}_g\,. \end{aligned}$$

Together with

$$|\sum_{k,s} h(Je_k, Je_s) h(e_k, e_s)| \le |h|^2,$$

we have

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0} \mathcal{F} \ge \int_M &| \, W \, |^{n-2} \, \{ \frac{2n-3}{2n-2} |\triangle h|^2 \\ (4.16)\qquad &+ (\frac{2n^2+1}{2n-1} - \frac{3n-5}{n-1} - 3) |\nabla h|^2 \\ &+ (n+1 - \frac{8}{n-1} + \frac{6}{2n-1}) |h|^2 \} \, \mathrm{dVol}_g\,. \end{aligned}$$

When $n \ge 5$, we can check that $\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{F} \ge 0$, and the equality holds if and only if $h = 0$.

When $n = 3$,

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0} \mathcal{F} \ &\ge \ \int_M | \, W \, | \, \{ \frac{3}{4} |\triangle h|^2 - \frac{6}{5} |\nabla h|^2 + \frac{6}{5} |h|^2 \} \, \mathrm{dVol}_g \\ &= \ \int_M |W| [\frac{3}{4} |\triangle h + \frac{4}{5} h|^2 + \frac{18}{25} |h|^2] \, \mathrm{dVol}_g \ge 0. \end{aligned}$$

When $n = 4$,

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0} \mathcal{F} \ &\ge \ \int_M | \, W \, |^2 \, \{ \frac{5}{6} |\triangle h|^2 - \frac{13}{21} |\nabla h|^2 + \frac{67}{21} |h|^2 \} \, \mathrm{dVol}_g \\ &> \ \int_M |W|^2 [\frac{5}{6} |\triangle h + \frac{13}{35} h|^2 + 3 |h|^2] \, \mathrm{dVol}_g \ge 0, \end{aligned}$$

and the equalities hold if and only if $h = 0$. □

Remark 4.1. For $\mathbb{CP}^2$, we know that $\mathcal{F}(g_{FS}) = \inf\{\mathcal{F}(g), g \in \mathcal{M}\}$, so it must be a stable critical point of $F(g)$. In fact, it is strictly stable.

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