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PATTERN AVOIDANCE SEEN IN MULTIPLICITIES OF MAXIMAL WEIGHTS OF AFFINE LIE ALGEBRA REPRESENTATIONS

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ABSTRACT. We prove that the multiplicities of certain maximal weights of $\mathfrak{g}(A_n^{(1)})$-modules are counted by pattern avoidance on words. This proves and generalizes a conjecture of Jayne-Misra. We also prove similar phenomena in types $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$. Both proofs are applications of Kashiwara's crystal theory.

1. INTRODUCTION

Let $\mathfrak{g} = \mathfrak{g}(A)$ be a Kac-Moody Lie algebra associated with a symmetrizable GCM A. For each dominant integral weight $\Lambda \in \mathcal{P}_A^+$, we have the integrable highest weight module $V(\Lambda)$ and the set of weights $P_A(\Lambda) := \{\mu \in \mathfrak{h}^* \mid V(\Lambda)_\mu \neq 0\}$ with the Weyl group W acting on it. Studies of the multiplicities of weight spaces, i.e., $m_A(\Lambda, \mu) := \dim V(\Lambda)_\mu$ for $\mu \in P_A(\Lambda)$, occupy a central position in combinatorial representation theory. For example, popular algebro-combinatorial ingredients such as Young Tableaux, Kashiwara's crystal, etc., are directly related to such dimension countings.

On the other hand, sometimes information on $P_A(\Lambda)$ or $m_A(\Lambda, \mu)$ gives that of representation theory of seemingly different algebras (and vice versa) via categorification. For example, by virtue of Lascoux-Leclerc-Thibon-Ariki theory and its subsequent developments, we know that $P_{A_{p-1}^{(1)}}(\Lambda)$ parameterizes the blocks of certain cyclotomic Hecke algebras (a.k.a. Ariki-Koike algebras) $\mathcal{H}$ and under this identification it is known that

(a) the orbit space $P_{A_{p-1}^{(1)}}(\Lambda)/W$ enumerates the possible derived equivalence classes of blocks of $\mathcal{H}$ [CR, §7.2],
(b) $m_{A_{p-1}^{(1)}}(\Lambda, \mu)$ tells us the number of irreducible modules of the block [LM, Theorem A].

Similar theorems are expected for other types of "Hecke algebras", such as KLR algebras, Hecke-Clifford algebras, etc., by choosing A suitably.

A rough structure of $P_A(\Lambda)$ is known when A is affine.

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Proposition 1.1 ([Kac, §12.6]). *Let A be an affine GCM. For $\Lambda \in \mathcal{P}^+_A$, we have*

$$P_A(\Lambda) = \bigsqcup_{\lambda \in \max_A(\Lambda)} \{\lambda - n\delta \mid n \geq 0\}$$

where $\max_A(\Lambda)$ *is the set of all maximal weights of* $V(\Lambda)$ *defined as follows:*

$$\max_A(\Lambda) = \{\lambda \in P_A(\Lambda) \mid \lambda + \delta \notin P_A(\Lambda)\}.$$

Clearly, $\max_A(\Lambda)$ is W-invariant and also any $\lambda \in \max_A(\Lambda)$ is W-conjugate to a maximal dominant weight (i.e., $\max_A(\Lambda) = W \cdot (\max_A(\Lambda) \cap \mathcal{P}^+_A)$). It is known that the set of dominant maximal weights $\max_A(\Lambda) \cap \mathcal{P}^+_A$ is finite [Kac, Proposition 12.6].

When Λ is level 1, the Hecke algebras appearing in the the aforementioned correspondence via categorification are Iwahori-Hecke algebras of type A. Note that $\max_X(\Lambda) \cap \mathcal{P}^+_X = \{\Lambda\}$ when Λ is level 1 and X is affine A,D,E type [Kac, Lemma 12.6]. In a course of a study of representation theory of Iwahori-Hecke algebras of type B, the first author studied the set of dominant maximal weights

$$\max_{A^{(1)}_{p-1}}(\Lambda_0 + \Lambda_s) \cap \mathcal{P}^+_{A^{(1)}_{p-1}}$$

for $0 \leq s < p$.

Definition 1.2. Let $p \geq 2$ be an integer (not necessarily a prime). For $\ell \geq 1$ and t, u with $t \geq 0, \ell + t < p - \ell + 1$ and $u \leq p, \ell < u - \ell + 1$, we define two elements of the root lattice Q of $\widehat{\mathfrak{sl}_p} = \mathfrak{g}(A^{(1)}_{p-1})$ as follows:

$$\lambda^p_{\ell,t} = \ell\alpha_0 + \begin{pmatrix} \ell\alpha_1 + \cdots + \ell\alpha_t \\ +(\ell-1)\alpha_{t+1} + (\ell-2)\alpha_{t+2} + \cdots + \alpha_{\ell+t-1} \\ +\alpha_{p-\ell+1} + \cdots + (\ell-2)\alpha_{p-2} + (\ell-1)\alpha_{p-1} \end{pmatrix},$$

$$\mu^p_{\ell,u} = \ell\alpha_0 + \begin{pmatrix} (\ell-1)\alpha_1 + (\ell-2)\alpha_2 + \cdots + \alpha_{\ell-1} \\ +\alpha_{u-\ell+1} + \cdots + (\ell-2)\alpha_{u-2} + (\ell-1)\alpha_{u-1} \\ +\ell\alpha_u + \cdots + \ell\alpha_{p-1} \end{pmatrix}.$$

Recall that $A = A^{(1)}_{p-1} = (2\delta_{ij} - \delta_{i+1,j} - \delta_{i-1,j})_{i,j \in \mathbb{Z}/p\mathbb{Z}}$ and $I = \mathbb{Z}/p\mathbb{Z}$ (see Figure 1). Throughout, we sometimes identify the set $I = \mathbb{Z}/p\mathbb{Z}$ with $\{0, 1, \cdots, p-1\}$.

We note that for $p \geq 2$ and when $t = 0, u = p$, $\lambda^p_{\ell,0}$ is defined exactly when $\mu^p_{\ell,p}$ is defined and in this case we have $\lambda^p_{\ell,0} = \mu^p_{\ell,p}$. For a Dynkin diagram automorpshim (see §3.2) $\omega : I \xrightarrow{\sim} I, i \longmapsto -i$, we have $\omega(\lambda^p_{\ell,t}) = \mu^p_{\ell,p-t}, \omega(\mu^p_{\ell,u}) = \lambda^p_{\ell,p-u}$.

The dominant maximal weights and their multiplicities are given as follows.

Theorem 1.3 ([Ts1, Theorem 1.4]). *Let $p \geq 2$ be an integer (not necessarily a prime) and consider a level 2 weight $\Lambda = \Lambda_0 + \Lambda_s$ of $\widehat{\mathfrak{sl}_p}$ for some $0 \leq s < p$. We have*

(a) $\max_{A^{(1)}_{p-1}}(\Lambda) \cap \mathcal{P}^+_{A^{(1)}_{p-1}} = \{\Lambda\} \sqcup \{\Lambda - \lambda^p_{\ell,s} \mid 1 \leq \ell \leq \lfloor \frac{p-s}{2} \rfloor\}$
$\sqcup \{\Lambda - \mu^p_{\ell,s} \mid 1 \leq \ell \leq \lfloor \frac{s}{2} \rfloor\}$,

(b) $m_{A^{(1)}_{p-1}}(\Lambda, \Lambda - \lambda^p_{\ell,s}) = \mathsf{D}_{\ell,s}$, $m_{A^{(1)}_{p-1}}(\Lambda, \Lambda - \mu^p_{\ell,s}) = \mathsf{D}_{\ell,p-s}$.

Here $\mathsf{D}_{n,m}$ is the number of lattice paths from $(0,0)$ to $(n+m, n)$ with steps $(1,0)$ and $(0,1)$ that do not exceed the diagonal $y = x$. It is not difficult to see $\mathsf{D}_{n,m} = \frac{m+1}{n+m+1}\binom{2n+m}{n}$ [St2, Exercise 6.20.b].

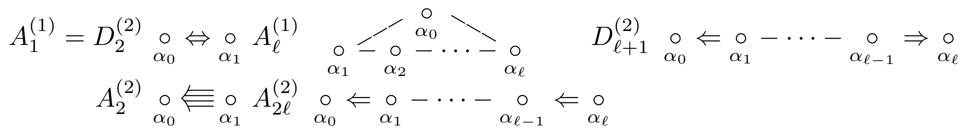
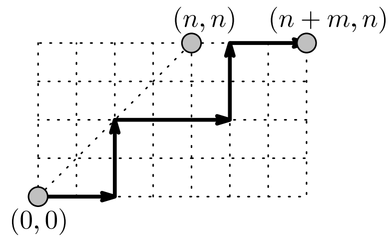


FIGURE 1. Affine Dynkin diagrams of A,D,E.



For a higher level $\Lambda \in \mathcal{P}^+_{A^{(1)}_{p-1}}$, the structure of the set $\max_{A^{(1)}_{p-1}}(\Lambda) \cap \mathcal{P}^+_{A^{(1)}_{p-1}}$ gets complicated, but one can easily see the following whose proof will be recalled in §4.1.

Lemma 1.4. *For $\Lambda = k\Lambda_0 + \Lambda_s$ where $k \geq 1$ and $0 \leq s < p$, we have*

$$\{\Lambda - \lambda^p_{\ell,s} \mid 1 \leq \ell \leq \lfloor \frac{p-s}{2} \rfloor\} \sqcup \{\Lambda - \mu_{\ell,s} \mid 1 \leq \ell \leq \lfloor \frac{s}{2} \rfloor\} \subseteq \max_{A^{(1)}_{p-1}}(\Lambda) \cap \mathcal{P}^+_{A^{(1)}_{p-1}}.$$

Based on an observation that $\mathsf{D}_{n,0}$ is the Catalan number and thus the number of 321-avoiding permutations of n [St2, Exercise 6.19.ee], Jayne-Misra conjectured a link between multiplicities of certain maximal weights of $\widehat{\mathfrak{sl}}_p$-modules and pattern avoidance.

Conjecture 1.5 ([MR1, Conjecture 4.13])**.** *For $1 \leq \ell \leq \lfloor p/2 \rfloor$,*

$$m_{A^{(1)}_{p-1}}((k+1)\Lambda_0, (k+1)\Lambda_0 - \lambda^p_{\ell,0})$$

is equinumerous to $((k+2), (k+1), \cdots, 2, 1)$-avoiding permutations of ℓ.

Our main theorem proves and generalizes it in the following way.

Theorem 1.6. *Let $p \geq 2$ be an integer and consider a level $k+1$ weight of the form $\Lambda = k\Lambda_0 + \Lambda_s$ of $\widehat{\mathfrak{sl}}_p$ for some $0 \leq s < p$ and $k \geq 1$. Then, for $1 \leq \ell \leq \lfloor \frac{p-s}{2} \rfloor$, $m_{A^{(1)}_{p-1}}(\Lambda, \Lambda - \lambda^p_{\ell,s})$ is equinumerous to shuffles of $0^s, 1, 2, \cdots, \ell$ (there are s zeros) that have no strictly decreasing subsequence of length $k+2$.*

By symmetry, for $0 < s < p$ and $1 \leq \ell \leq \lfloor \frac{s}{2} \rfloor$, $m_{A^{(1)}_{p-1}}(\Lambda, \Lambda - \mu^p_{\ell,s})$ is equal to

$$\#\{\text{shuffles of } 0^{p-s}, 1, 2, \cdots, \ell \text{ that have no strictly decreasing subsequence of length } k+2\}.$$

Our proof is based on a result of Ariki-Kreiman-Tsuchioka which characterizes the connected component (known as Kleshchev multipartitions in modular representation theory of Hecke algebras) of $A^{(1)}_{p-1}$-crystal $B(a\Lambda_0 + b\Lambda_s) \subseteq B(\Lambda_0)^{\otimes a} \otimes B(\Lambda_s)^{\otimes b}$ in the tensor product [AKT, Corollary 9.6]. This result is a combinatorial incarnation of Littelmann's result [Lit, Theorem 10.1].

While a link between multiplicities of maximal weights of $\widehat{\mathfrak{sl}_p}$-modules and pattern avoidance was first observed in [MR1], we see similar appearances of pattern avoidance in multiplicities of maximal weights of affine Lie algebras of types $A^{(2)}_{2n}$ and $D^{(2)}_{n+1}$ (for a crystal-theoretic distinction of types $A^{(1)}_n, A^{(2)}_{2n}, D^{(2)}_{n+1}$, see [Ts2, §1]). In the following, Lie theoretic objects associated with $\check{A}$ are written with ˇ attached.

Theorem 1.7. *Let $p \geq 2$ be an integer and consider a level $k+1$ weight of the form $\check{\Lambda} = (k+1)\check{\Lambda}_0$ of $\check{A} = A^{(2)}_{p-1}$ (resp. $D^{(2)}_{1+p/2}$) depending on p being odd (resp. even) where $k \geq 1$ (see Figure 1). For $1 \leq \ell \leq \lfloor p/2 \rfloor$,*

(a) $\gamma_\ell := \check{\Lambda} - \ell\check{\alpha}_0 - (\ell-1)\check{\alpha}_1 - \cdots - \check{\alpha}_{\ell-1} \in \max_{\check{A}}(\check{\Lambda}) \cap \mathcal{P}^+_{\check{A}}$,
(b) *$m_{\check{A}}(\check{\Lambda}, \gamma_\ell)$ is equinumerous to $((k+2), (k+1), k, \cdots, 1)$-avoiding involutions of ℓ.*

Our proof is based on a result of Naito-Sagaki [NS1, Theorem 4.4] on Kashiwara's crystals fixed by a diagram automorphism which is also an application of Littelmann's path model.

When preparing the paper, [MR2] appeared on the arXiv and gives a proof of Conjecture 1.5, i.e., the case of $s = 0$ of Theorem 1.6. Note that in [MR1], Jayne-Misra also give a conjectural formula [MR1, Conjecture 3.9] on the cardinality $\#(\max_{A^{(1)}_{p-1}}(k\Lambda_0) \cap \mathcal{P}^+_{A^{(1)}_{p-1}})$ which we prove in §4 using the q-Lucas theorem dating back to Gauss.

Notation and Conventions. We assume that readers are familiar with Kac-Moody Lie algebras and Kashiwara's crystal theory ([Kac] and [Kas] are standard references).

For integers $a \geq 0$ and $b \geq 1$, we denote by $a\%b$ the remainder of a by b, namely the unique integer $0 \leq c < b$ such that $a - c \in b\mathbb{Z}$.

The set of partitions is denoted by Par and the symbol $\varnothing$ is reserved for the empty partition. For a partition $\lambda = (\lambda_1, \lambda_2, \cdots) \in \mathsf{Par}$, we define $|\lambda| = \sum_{i\geq 1}\lambda_i$ and $\ell(\lambda) = \#\{i \geq 1 \mid \lambda_i \neq 0\} (= ({}^{\mathrm{tr}}\lambda)_1)$. For $n \geq 0$, we put $\mathsf{Par}(n) = \{\lambda \in \mathsf{Par} \mid |\lambda| = n\}$. For $a \geq 0, b \geq 1$, (a^b) is an abbreviation for a partition c such that $c_1 = \cdots = c_b = a$.

The symbol RPar_p (resp. $\mathsf{Par}^{p\text{-core}}$) stands for the set of p-restricted (resp. p-core) partitions for $p \geq 2$. Recall that $\lambda \in \mathsf{Par}$ is p-restricted (resp. p-core) if $\lambda_i - \lambda_{i+1} < p$ for $i \geq 1$ (resp. if there is no removable p-hook). Note that $\mathsf{Par}^{p\text{-core}} \subseteq \mathsf{RPar}_p$.

A semistandard tableaux (SST, for short) is a filling of the Young diagram by integers which are weakly increasing along rows and strictly increasing along columns. A column-strict plane partition (CSPP, for short) is a filling of the Young diagram by positive integers which are weakly decreasing along rows and strictly decreasing along columns. For an SST or a CSPP T, we denote by $\mathsf{sh}(T)$ the underlying Young diagram. The content $\mathsf{cont}(T)$ of T is a multiset of the numbers filled in T.

For $\lambda \in \mathsf{Par}$, we denote by $\mathsf{SST}(\lambda)$ (resp. $\mathsf{CSPP}(\lambda)$) the set of SST (resp. CSPP) of shape λ. As usual, $\mathsf{ST}(\lambda)$ (resp. $\mathsf{RST}(\lambda)$) means the set of standard tableaux T (resp. reverse standard tableaux), i.e., SST (resp. CSPP) such that $\mathsf{cont}(T) = \{1, 2, \cdots, |\lambda|\}$.

Finally, $\mathsf{Mod}(A)$ means the abelian category of finite-dimensional left A-modules and A-homomorphisms between them for a finite-dimensional algebra A over a field $\mathbb{F}$. We denote by $\mathsf{Irr}(\mathsf{Mod}(A))$ the set of isomorphism classes of simple objects in $\mathsf{Mod}(A)$.

2. Proof of Theorem 1.6

In this section, p, k, ℓ, s are as in Theorem 1.6, i.e., $p \geq 2, k \geq 1, 0 \leq s < p, 1 \leq \ell \leq \lfloor (p-s)/2 \rfloor$. We will show that $m_{A^{(1)}_{p-1}}(\Lambda, \Lambda - \lambda^p_{\ell,s}) = \#V$ where $\Lambda = k\Lambda_0 + \Lambda_s$ and

$$V = \{\text{shuffles of } 0^s, 1, 2, \cdots, \ell \text{ that have no strictly decreasing subsequence of length } k+2\}.$$

2.1. **Robinson-Schensted-Knuth correspondence.** Recall the Robinson-Schensted-Knuth correspondence (RSK correspondence, for short) (see [Ful, §4]). Fix a multiset $J = \{w_1, \cdots, w_m\} \subseteq \mathbb{Z}$ with cardinality m (counted with multiplicity). RSK correspondence gives a bijection between the set of shuffles (or words) of $w_1, \cdots, w_m$ and

$$\bigsqcup_{\lambda \in \mathsf{Par}(m)} \{(P, Q) \in \mathsf{SST}(\lambda) \times \mathsf{ST}(\lambda) \mid \mathsf{cont}(P) = J\}.$$

We mean by $(P, Q) = \mathsf{RSK}(w)$ that a shuffle (or a word) w maps to a pair of tableaux (P, Q) of the same shape via RSK correspondence. How RSK correspondence respects ordered subsequences of a shuffle is well known (see [Ful, §3]).

Lemma 2.1. *Let $(P, Q) = \mathsf{RSK}(w)$ with $\lambda = \mathsf{sh}(P) = \mathsf{sh}(Q)$. Then, $\ell(\lambda)$ is the length of the largest strictly decreasing subsequence of w.*

In summary, RSK correspondence gives a bijection between V and V_1 where

$$V_1 = \bigsqcup_{\substack{\lambda \in \mathsf{Par}(\ell+s) \\ \ell(\lambda) \leq k+1}} \{(P, Q) \in \mathsf{SST}(\lambda) \times \mathsf{ST}(\lambda) \mid \mathsf{cont}(P) = \{0^s, 1, \cdots, \ell\}\}.$$

Thus, we know that there is a bijection $V \xrightarrow{\sim} V_2$ where

$$V_2 = \bigsqcup_{\substack{\lambda \in \mathsf{Par}(\ell+s) \\ \ell(\lambda) \leq k+1}} \{(P, Q) \in \mathsf{RST}(\lambda) \times \mathsf{RST}(\lambda) \mid \text{all } \ell+s, \cdots, \ell+1 \text{ appear in the first row of } P\}.$$

For a permutation $w \in \mathfrak{S}_m = \mathsf{Aut}(\{1, \cdots, m\})$, we define $\mathsf{word}(w) = w_1 \cdots w_m$ which is a shuffle of $\{1, 2, \cdots, m\}$ by $w_i = w(i)$ for $1 \leq i \leq m$. We will use the following well-known symmetry in §3 (see [Ful, §4]).

Lemma 2.2. *For $w \in \mathfrak{S}_m$ with $\mathsf{RSK}(\mathsf{word}(w)) = (P, Q)$, we have $\mathsf{RSK}(\mathsf{word}(w^{-1})) = (Q, P)$.*

2.2. **Kleshchev multipartitions.** Crystal theoretically, the number

$$m_{A^{(1)}_{p-1}}(\Lambda, \Lambda - \lambda^p_{\ell,s})$$

is translated as the following counting:

$$\#\{b := \mu \otimes \lambda^{(1)} \otimes \cdots \otimes \lambda^{(k)} \in B(\Lambda_s) \otimes B(\Lambda_0)^{\otimes k} \mid b \in B(\Lambda), \mathrm{wt}(b) = \Lambda - \lambda^p_{\ell,s}\}.$$

Here $B(\Lambda)$ means the naturally embedded one in $B(\Lambda_s) \otimes B(\Lambda_0)^{\otimes k}$.

We adapt the Misra-Miwa realization [MM] for $A^{(1)}_{p-1}$-crystal $B(\Lambda_s)$ for $0 \leq s < p$. We need not know the details of this realization such as the definition of Kashiwara operators. All we need to know is the following basic things and a result [AKT, Corollary 9.6]:

(A) The underlying set of $B(\Lambda_s)$ is RPar_p.
(B) For each $\lambda \in B(\Lambda_s)$ and each box $x = (i,j) \in \lambda$ (this means x is the box inside λ located at the i-th row and the j-th column), x has the quantity $\mathsf{res}(x) = (s-i+j)+p\mathbb{Z} \in \mathbb{Z}/p\mathbb{Z}$, called the residue of x.
(C) For each $\lambda \in B(\Lambda_s)$,

$$\mathrm{wt}(x) = \mathrm{wt}_s(x) := \Lambda_s - \sum_{i\in\mathbb{Z}/p\mathbb{Z}} \#\{x \in \lambda \mid \mathsf{res}(x) = i\} \cdot \alpha_i. \tag{2.1}$$

Theorem 2.3 ([AKT, Corollary 9.6]). *Let $b := \mu \otimes \lambda^{(1)} \otimes \cdots \otimes \lambda^{(k)} \in B(\Lambda_s) \otimes B(\Lambda_0)^{\otimes k}$. Then $b \in B(k\Lambda_0 + \Lambda_s)$ (i.e., b is a Kleshchev multipartition) if and only if $\tau_{(p-s)\%p}(\mathsf{base}(\mu)) \supseteq \mathsf{roof}(\lambda^{(1)})$ and $\mathsf{base}(\lambda^{(i)}) \supseteq \mathsf{roof}(\lambda^{(i+1)})$ for all $1 \le i < k$.*

Here base, τ_m [AKT] where $0 \le m < p$ and roof [KLMW] are explicit maps

$$\begin{cases} \mathsf{base}, \mathsf{roof} : \mathsf{RPar}_p \longrightarrow \mathsf{Par}^{p\text{-core}} \\ \tau_m : \mathsf{Par}^{p\text{-core}} \longrightarrow \mathsf{Par}^{p\text{-core}} \end{cases}$$

and $\lambda' \supseteq \mu'$ means that λ' contains μ' as Young diagrams. We need not know the precise definitions of maps $\mathsf{base}, \mathsf{roof}$ and τ_m, however we need the following:
(a) For a p-core partition λ, we have $\lambda = \mathsf{base}(\lambda) = \mathsf{roof}(\lambda)$ [AKT, Definition 2.5,2.8].
(b) For a p-core partition $\lambda = (\lambda_1, \cdots, \lambda_a)$, we have $\tau_m(\lambda) = (\nu_1, \cdots, \nu_{a+m})$ [AKT, Proposition 9.4] where

$$\nu_i = \begin{cases} \lambda_i + ((p-m)\%p), & (1 \le i \le m), \\ \min\{\lambda_i + ((p-m)\%p), \lambda_{i-m}\}, & (m < i \le a), \\ \min\{(p-m)\%p, \lambda_{i-m}\}, & (a < i \le a+m). \end{cases}$$

Note that $\tau_0 = \mathsf{id}_{\mathsf{Par}^{p\text{-core}}}$ and $\tau_m(\lambda) = \mathsf{shift}_{(p-m)\%p}(\lambda) \cap (\infty^m, \lambda)$ where $\mathsf{shift}_t(\lambda) = (\lambda_i + t)_{i \ge 1}$ for $\lambda \in \mathsf{Par}$ and $t \ge 0$. Of course, $\mathsf{shift}_t(\lambda)$ and (∞^m, λ) are not Young diagrams in the usual sense. But in this section, an infinite Young diagram ν of these forms only appears as the form $\nu \supseteq \mu$ for a usual finite Young diagram $\mu \in \mathsf{Par}$.

Proposition 2.4. *As subsets of RPar_p^{k+1}, we define*

$$X = \{(\mu, \lambda^{(1)}, \cdots, \lambda^{(k)}) \mid (*) \text{ and } \tau_{(p-s)\%p}(\mathsf{base}(\mu)) \supseteq \mathsf{roof}(\lambda^{(1)}),$$
$$1 \le \forall i < k, \mathsf{base}(\lambda^{(i)}) \supseteq \mathsf{roof}(\lambda^{(i+1)})\},$$
$$Y = \{(\mu, \lambda^{(1)}, \cdots, \lambda^{(k)}) \in (\mathsf{Par}^{p\text{-core}})^{k+1} \mid (*) \text{ and } \tau_{(p-s)\%p}(\mu) \supseteq \lambda^{(1)} \supseteq \cdots \supseteq \lambda^{(k)}\},$$
$$Z = \{(\mu, \lambda^{(1)}, \cdots, \lambda^{(k)}) \mid (*) \text{ and } \mathsf{shift}_s(\mu) \supseteq \lambda^{(1)} \supseteq \cdots \supseteq \lambda^{(k)}\},$$
$$Z' = \{(\mu, \lambda^{(1)}, \cdots, \lambda^{(k)}) \in (\mathsf{Par}^{p\text{-core}})^{k+1} \mid (*) \text{ and } \mathsf{shift}_s(\mu) \supseteq \lambda^{(1)} \supseteq \cdots \supseteq \lambda^{(k)}\},$$

where $(*)$ means the condition $\mathrm{wt}_s(\mu) + \sum_{i=1}^k \mathrm{wt}_0(\lambda^{(i)}) = \Lambda - \lambda^p_{\ell,s}$. Then, we have $X = Y = Z = Z'$.

Proof. First, observe that $(*)$ implies $\mu \subseteq (\ell^{\ell+s})$ and $\lambda^{(i)} \subseteq ((\ell+s)^\ell)$ for $1 \le i \le k$. Especially, $(*)$ implies $\mu, \lambda^{(1)}, \cdots, \lambda^{(k)} \in \mathsf{Par}^{p\text{-core}}$. By (a) above, $X = Y$ and $Z = Z'$.

When $s = 0$, it is clear that $Y = Z'$. Assume $0 < s < p$. Note that $\tau_{p-s}(\mu) \supseteq \lambda^{(1)}$ if and only if $\mathsf{shift}_s(\mu) \supseteq \lambda^{(1)}$ and $(\infty^{p-s}, \mu) \supseteq \lambda^{(1)}$. By $\lambda^{(1)} \subseteq ((\ell+s)^\ell)$, the latter condition is automatically satisfied. Thus, we get $Y = Z'$. □

In summary, we now know that $m_{A^{(1)}_{p-1}}(\Lambda, \Lambda - \lambda^p_{\ell,s}) = \#Z$.

Definition 2.5. Let $\{\beta_b \mid b \in \mathbb{Z}\}$ be formal linearly independent elements over $\mathbb{Z}$.

(a) for $p \geq 2$, we define a map (where $\bigoplus_{i\in\mathbb{Z}/p\mathbb{Z}} \mathbb{Z}\alpha_i$ is a root lattice of $\widehat{\mathfrak{sl}_p} = \mathfrak{g}(A^{(1)}_{p-1})$) by

$$T_p : \bigoplus_{b\in\mathbb{Z}} \mathbb{Z}\beta_b \longrightarrow \bigoplus_{i\in\mathbb{Z}/p\mathbb{Z}} \mathbb{Z}\alpha_i, \quad \beta_b \longmapsto \alpha_{b+p\mathbb{Z}},$$

(b) for $\ell \geq 1$ and $s \geq 0$, we define

$$\begin{aligned}\nu^p_{\ell,s} &= \beta_{-\ell+1} + \cdots + (\ell-2)\beta_{-2} + (\ell-1)\beta_{-1} \\ &\quad + \ell\beta_0 + \cdots + \ell\beta_s + (\ell-1)\beta_{s+1} + (\ell-2)\beta_{s+2} + \cdots + \beta_{\ell+s-1}.\end{aligned}$$

Corollary 2.6. *We have $Z = Z''$ where as subsets of RPar^{k+1}_p we define*

$$Z'' = \{(\mu, \lambda^{(1)}, \cdots, \lambda^{(k)}) \in (\mathsf{Par}^{p\text{-core}})^{k+1} \mid (**) \text{ and } \mathsf{shift}_s(\mu) \supseteq \lambda^{(1)} \supseteq \cdots \supseteq \lambda^{(k)}\}$$

where $(**)$ means the condition $\sum_{(i,j)\in\mu} \beta_{s-i+j} + \sum_{a=1}^{k} \sum_{(i,j)\in\lambda^{(a)}} \beta_{-i+j} = \nu^p_{\ell,s}$.

Proof. The conditions $0 \leq s < p$ and $1 \leq \ell \leq \lfloor \frac{p-s}{2} \rfloor$ imply $T_p(\nu^p_{\ell,s}) = \lambda^p_{\ell,s}$. Thus, $Z'' \subseteq Z$. The reverse inclusion follows from the fact that for $(\mu, \lambda^{(1)}, \cdots, \lambda^{(k)}) \in Z$ we have $\mu \subseteq (\ell^{\ell+s})$ and $\lambda^{(i)} \subseteq ((\ell+s)^{\ell})$ for $1 \leq i \leq k$ as in the proof of Proposition 2.4. □

2.3. **Plane partitions.** Recall that a 2-dimensional array of non-negative integers $\pi = (\pi_{ij})_{i,j\geq1}$ is a plane partition if $\pi_{ij} \geq \pi_{i+1,j}, \pi_{i,j+1}$ for $i, j \geq 1$ and the support $\{(i,j) \in \mathbb{Z}_{\geq1} \times \mathbb{Z}_{\geq1} \mid \pi_{ij} > 0\}$ is a finite set. We denote by PP the set of plane partitions.

Definition 2.7. For a plane partition π, we define

$$\mathrm{wt}(\pi) = \sum_{a\geq1} \sum_{(i,j)\in\pi_{*,a}} \beta_{j-i}$$

as an element of $\bigoplus_{b\in\mathbb{Z}} \mathbb{Z}\beta_b$ where $\pi_{*,j} = (\pi_{1j}, \pi_{2,j}, \cdots) \in \mathsf{Par}$.

Clearly, we have (see (2.1))

$$T_p(\mathrm{wt}(\pi)) = \sum_{a\geq1} (\Lambda_0 - \mathrm{wt}_0(\pi_{*,a})). \tag{2.2}$$

Recall a famous bijection (that appears most frequently in proving MacMahon plane partition generating functions (see [St2, Corollary 7.20.3]))

$$\Pi : \bigsqcup_{\lambda\in\mathsf{Par}} \mathsf{CSPP}(\lambda) \times \mathsf{CSPP}(\lambda) \xrightarrow{\sim} \mathsf{PP}. \tag{2.3}$$

The correspondence $(P, Q) \mapsto \Pi(P, Q)$ is briefly described as follows (for a detailed explanation including an example, see [St2, §7.20]):

> Let $p^a, q^a \in \mathsf{Par}$ be the a-th columns of P and Q. Then, the a-th column of $\Pi(P, Q)$ is a partition given by the Frobenius notation $\rho(p^a, q^a)$.

We get Lemma 2.8 because we have

$$\sum_{(i,j)\in\rho(p^a,q^a)} \beta_{j-i} = \sum_{i\geq 0} \#p^a_{>i}\cdot\beta_i + \sum_{i<0} \#q^a_{>-i}\cdot\beta_i$$

where $\#r_{>b}$ is the number of parts of r that are larger than b for $r\in\{p^a,q^a\}$ and $b\in\mathbb{Z}$.

Lemma 2.8. *Let $P,Q\in\mathsf{CSPP}(\lambda)$ for some $\lambda\in\mathsf{Par}$. For $\pi=\Pi(P,Q)$, we have*

$$\lambda_1 = \ell({}^{\mathrm{tr}}\lambda) = \max\{j\geq 1 \mid \pi_{*,j}\neq\varnothing\},$$
$$\mathrm{wt}(\pi) = \sum_{i\geq 0}\#P_{>i}\cdot\beta_i + \sum_{i<0}\#Q_{>-i}\cdot\beta_i$$

where $\#R_{>i}$ is the number of boxes of R whose number is larger than i for $R\in\{P,Q\}$.

Note that in the setting of Lemma 2.8, we have

(a) the coefficient of β_0 in $\mathrm{wt}(\pi)$ is $|\lambda|$,
(b) $P,Q\in\mathsf{RST}(\lambda)$ if and only if $\mathrm{wt}(\pi)=\beta_{-|\lambda|+1}+2\beta_{-|\lambda|+2}+\cdots+|\lambda|\beta_0+\cdots+2\beta_{|\lambda|-2}+\beta_{|\lambda|-1}$.

Proposition 2.9. *The bijection Π (see (2.3)) gives a bijection*

$$\Pi\circ\mathsf{swap}\circ({}^{\mathrm{tr}}(\cdot)\times{}^{\mathrm{tr}}(\cdot))|_{V_2} : V_2 \xrightarrow{\sim} V_3,\quad (P,Q)\longmapsto\Pi({}^{\mathrm{tr}}Q,{}^{\mathrm{tr}}P),$$

where $\beta=\beta_{-\ell-s+1}+2\beta_{-\ell-s+2}+\cdots+(\ell+s)\beta_0+\cdots+2\beta_{\ell+s-2}+\beta_{\ell+s-1}$ and

$$V_3=\{\pi\in\mathsf{PP}\mid \pi_{*,1}\supseteq(s^{\ell+s}),\mathrm{wt}(\pi)=\beta,\pi_{*,k+2}=\varnothing\}.$$

Proof. Take $(P,Q)\in V_2$. Because the first column of ${}^{\mathrm{tr}}P$ contains $\ell+s,\cdots,\ell+1$, we see $\Pi({}^{\mathrm{tr}}Q,{}^{\mathrm{tr}}P)_{*,1}\supseteq(s^{\ell+s})$ by the construction of Π. Thus, $\Pi({}^{\mathrm{tr}}Q,{}^{\mathrm{tr}}P)\in V_3$ by Lemma 2.8.

Conversely, take $\pi\in V_3$. Since Π is a bijection, there are unique $\lambda\in\mathsf{Par}$ and $P,Q\in\mathsf{CSPP}(\lambda)$ such that $\pi=\Pi({}^{\mathrm{tr}}Q,{}^{\mathrm{tr}}P)$. By Lemma 2.8, $|\lambda|=\ell+s$, $\ell(\lambda)\leq k+1$ and $P,Q\in\mathsf{RST}(\lambda)$. Observe that $\mathrm{wt}(\pi)=\beta$ implies $\pi_{*,1}\subseteq((\ell+s)^{\ell+s})$. Thus, $(s^{\ell+s})\subseteq\pi_{*,1}\subseteq((\ell+s)^{\ell+s})$. From this, we easily see that all $\ell+s,\cdots,\ell+1$ must appear in the first row of P. In other words, $(P,Q)\in V_2$. □

In §3, we will use a symmetry that obviously follows from the construction of Π.

Lemma 2.10. *For $\lambda\in\mathsf{Par}$ and $P,Q\in\mathsf{CSPP}(\lambda)$, put $\pi=\Pi(P,Q),\pi'=\Pi(Q,P)$. Then, $\pi'_{*,i}={}^{\mathrm{tr}}(\pi_{*,i})$ for $i\geq 1$.*

2.4. **Proof of Theorem 1.6.** Let us define maps Φ and Ψ by

$$\Phi : V_3\longrightarrow Z,\quad \pi\longmapsto(\mu,\pi_{*,2},\pi_{*,3},\cdots,\pi_{*,k+1}),\tag{2.4}$$

$$\Psi : Z\longrightarrow V_3,\quad (\mu',\lambda'^{(1)},\cdots,\lambda'^{(k)})\longmapsto\pi',\tag{2.5}$$

where (note that $(s^{\ell+s})\subseteq\pi_{*,1}\subseteq((\ell+s)^{\ell+s})$ as in the proof of Proposition 2.9 and $\mu'\subseteq(\ell^{\ell+s}),\lambda'^{(a)}\subseteq((\ell+s)^{\ell})$ for $1\leq a\leq k$ as in the proof of Corollary 2.6)

(a) $\mu=(\nu_1-s,\nu_2-s,\cdots,\nu_{\ell+s}-s)$ for $\nu=\pi_{*,1}$,
(b) $\pi'_{*,a+1}=\lambda'^{(a)}$ for $1\leq a\leq k$ and $\pi'_{*,1}=(\mu_1+s,\cdots,\mu_{\ell+s}+s)$.

In §2.5, we show that both Φ and Ψ are well defined. This completes the proof because by construction Φ and Ψ are mutually inverse of each other.

2.5. **Well-definedness of maps Φ and Ψ.** As a preparation, a direct calculation shows

$$\beta - \beta_{\square} = \nu^p_{\ell,s} \tag{2.6}$$

where $\beta = \sum_{(i,j)\in(\ell+s)^{\ell+s}} \beta_{j-i}$ and $\beta_{\square} = \sum_{(i,j)\in(s^{\ell+s})} \beta_{j-i}$ for $\ell \geq 1, s \geq 0$ (see Definition 2.5 and Proposition 2.9).

To prove the well-definedness of Φ (resp. Ψ), it is enough to show

$$\mathrm{wt}_s(\mu) + \sum_{a=1}^{k} \mathrm{wt}_0(\pi_{*,a+1}) = \Lambda - \lambda^p_{\ell,s} \quad (\text{resp. } \mathrm{wt}(\pi') = \beta)$$

in the situation of (2.4) (resp. (2.5)). A check for it is shown in §2.5.1 (resp. §2.5.2).

2.5.1. By $\Lambda_0 - \mathrm{wt}_0(\nu) = (\Lambda_s - \mathrm{wt}_s(\mu)) + \sum_{(i,j)\in(s^{\ell+s})} \alpha_{(j-i)+p\mathbb{Z}}$, (2.2) and $T_p(\nu^p_{\ell,s}) = \lambda^p_{\ell,s}$,

$$\begin{aligned}\mathrm{wt}_s(\mu) + \sum_{a=1}^{k} \mathrm{wt}_0(\pi_{*,a+1}) &= \Lambda - T_p(\beta) + \sum_{(i,j)\in(s^{\ell+s})} \alpha_{(j-i)+p\mathbb{Z}} \\ &= \Lambda - T_p(\beta - \beta_{\square}) = \Lambda - \lambda^p_{\ell,s}.\end{aligned}$$

2.5.2. By Corollary 2.6 and (2.6),

$$\mathrm{wt}(\pi') = \beta_{\square} + \sum_{(i,j)\in\mu'} \beta_{(s+j)-i} + \sum_{a=1}^{k} \sum_{(i,j)\in\lambda'^{(a)}} \beta_{j-i} = \beta.$$

3. Proof of Theorem 1.7

In this section, p, k, ℓ are as in Theorem 1.7, i.e., $p \geq 2, k \geq 1, 1 \leq \ell \leq \lfloor p/2 \rfloor$. As in §2, we keep identifying RPar_p with $B(\Lambda_0)$ as $A^{(1)}_{p-1}$-crystal through Misra-Miwa realization and use results in §2 substituting $s = 0$.

3.1. Mullineux involution.

Definition 3.1 (see [Mat, 6.42]). For each $b \in B(\Lambda_0) = \mathsf{RPar}_p$ of the form $b = \tilde{f}_{i_j} \cdots \tilde{f}_{i_1}\varnothing$ for some $i_1, \cdots, i_j \in \mathbb{Z}/p\mathbb{Z}$, $\mathsf{M}(b) = \tilde{f}_{-i_j} \cdots \tilde{f}_{-i_1}\varnothing$ is well defined.

As in [AKT, Proposition 5.12], there is a crystal morphism $S_h : B(\Lambda_0) \to B(h\Lambda_0)$ for $h \geq 1$ with certain properties. Let us briefly recall what will be needed. Under the canonical embedding $B(h\Lambda_0) \hookrightarrow B(\Lambda_0)^{\otimes h}$, we can write $S_h(\lambda)$ of the form

$$S_h(\lambda) = \lambda^{(1)} \otimes \cdots \otimes \lambda^{(h)}. \tag{3.1}$$

Denoting (3.1) as

$$S_h(\lambda)^{1/h} = (\lambda^{(1)})^{\otimes 1/h} \otimes \cdots \otimes (\lambda^{(h)})^{\otimes 1/h}$$

and replacing an occurrence of $(\mu^{\otimes 1/h})^{\otimes k}$ with $\mu^{\otimes k/h}$, we can write

$$S_h(\lambda)^{1/h} = \nu_1^{\otimes a_1} \otimes \nu_2^{\otimes a_2 - a_1} \otimes \cdots \otimes \nu_s^{\otimes 1 - a_{s-1}}. \tag{3.2}$$

Here $0 < a_1 < \cdots < a_{s-1} < 1$ in $\mathbb{Q}$ and $\nu_1, \cdots, \nu_s \in \mathsf{RPar}_p$ are pairwise distinct.

As in [AKT, Theorem 5.13], for any $\lambda \in \mathsf{RPar}_p$, the right hand side of (3.2) is stable for any sufficiently divisible $h \geq 1$. Furthermore,

(a) $\nu_1, \nu_2, \cdots, \nu_s \in \mathsf{Par}^{p\text{-core}}$ [AKT, Theorem 5.13.(1)],
(b) $\nu_1 \supsetneq \nu_2 \supsetneq \cdots \supsetneq \nu_s$ [AKT, Theorem 5.14],

(c) $\nu_1 = \mathsf{roof}(\lambda), \nu_s = \mathsf{base}(\lambda)$ [AKT, Definition 5.17, Corollary 6.4, Corollary 8.5],
(d) for any sufficiently divisible h, we have (see [AKT, Proof of Proposition 5.21])

$$S_h(\mathsf{M}(\lambda))^{1/h} = ({}^{\mathrm{tr}}\nu_1)^{\otimes a_1} \otimes ({}^{\mathrm{tr}}\nu_2)^{\otimes (a_2-a_1)} \otimes \cdots \otimes ({}^{\mathrm{tr}}\nu_s)^{\otimes (1-a_{s-1})}. \tag{3.3}$$

Corollary 3.2 ([AKT, Proposition 5.21])**.** *For any* $\lambda \in \mathsf{RPar}_p$*, we have* $\mathsf{base}(\mathsf{M}(\lambda)) = {}^{\mathrm{tr}}\mathsf{base}(\lambda)$ *and* $\mathsf{roof}(\mathsf{M}(\lambda)) = {}^{\mathrm{tr}}\mathsf{roof}(\lambda)$*.*

Corollary 3.3. *For any* $\lambda \in \mathsf{Par}^{p\text{-core}}$*, we have* $\mathsf{M}(\lambda) = {}^{\mathrm{tr}}\lambda$*.*

Remark 3.4. The involution $\mathsf{M} : \mathsf{RPar}_p \xrightarrow{\sim} \mathsf{RPar}_p$ is known as Mullineux involution in modular representation theory of symmetric groups and Hecke algebras (see [LLT, §7]). Under the identification via cellular algebra structure (see [Mat, 3.43])

$$\mathsf{RPar}_p \xrightarrow{\sim} \bigsqcup_{n\geq 0} \mathsf{Irr}(\mathsf{Mod}(\mathbb{F}_p\mathfrak{S}_n)), \quad \lambda \longmapsto D^\lambda_{\mathbb{F}_p},$$

Ford-Kleshchev showed $D^\lambda_{\mathbb{F}_p} \otimes \mathsf{sign}_{\mathbb{F}_p} \cong D^{\mathsf{M}(\lambda)}_{\mathbb{F}_p}$ that was known as the Mullineux conjecture. Here $\mathsf{sign}_{\mathbb{F}}$ is the sign representation for a field $\mathbb{F}$. It is a classical result that $S^\lambda_{\mathbb{Q}} \otimes \mathsf{sign}_{\mathbb{Q}} \cong S^{{}^{\mathrm{tr}}\lambda}_{\mathbb{Q}}$ where $\{S^\lambda_{\mathbb{Q}} \mid \lambda \in \mathsf{Par}(n)\} = \mathsf{Irr}(\mathsf{Mod}(\mathbb{Q}\mathfrak{S}_n))$ are the classical Specht modules. (3.3) says that choosing an appropriate model of "Young diagram", $\text{-} \otimes \mathsf{sign}_{\mathbb{F}}$ is always given by transposition of Young diagram even over positive characteristics (for Hecke algebras, see [AKT, §5]).

3.2. **Diagram automorphisms and orbit Lie algebras.** Let $A = (a_{ij})_{i,j\in I}$ be a symmetrizable GCM with a corresponding Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$. A diagram automorphism $\omega : I \xrightarrow{\sim} I$ is a bijection such that $a_{\omega(i),\omega(j)} = a_{ij}$ for $i, j \in I$. For a symmetrizable GCM with a diagram automorphism, the orbit Lie algebra $\breve{\mathfrak{g}} = \mathfrak{g}(\breve{A})$, which is again a Kac-Moody Lie algebra, is defined as follows (see [FSS, §2.2]):

(i) put $c_{ij} = \sum_{k=0}^{N_j-1} a_{i,\omega^k(j)}$ for $i, j \in I$ where $N_i = \#\{\omega^k(i) \mid k \in \mathbb{Z}\}$,
(ii) set $\breve{I} = \{i \in I/\omega \mid c_{ii} > 0\}$ and $\breve{A} = (\breve{a}_{ij} := 2c_{ij}/c_{jj})_{i,j\in\breve{I}}$.

In our case of $A = A^{(1)}_{p-1} = (2\delta_{ij} - \delta_{i+1,j} - \delta_{i-1,j})_{i,j\in\mathbb{Z}/p\mathbb{Z}}$ and $I = \mathbb{Z}/p\mathbb{Z}$, we adapt

$$\omega : \mathbb{Z}/p\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}, \quad i \longmapsto -i,$$

as a diagram automorphism. Then, the orbit Lie algebra is $\breve{\mathfrak{g}} = \mathfrak{g}(\breve{A})$ where $\breve{A} = A^{(2)}_{p-1}$ (resp. $D^{(2)}_{1+p/2}$) depending on p being odd (resp. even). Recall that

$$\breve{\delta} = \begin{cases} 2\breve{\alpha}_0 + \cdots + 2\breve{\alpha}_{(p-3)/2} + \breve{\alpha}_{(p-1)/2}, & (\breve{A} = A^{(2)}_{p-1}), \\ \breve{\alpha}_0 + \cdots + \breve{\alpha}_{p/2}, & (\breve{A} = D^{(2)}_{1+p/2}). \end{cases} \tag{3.4}$$

We identify the set $\{i \in I/\omega \mid c_{ii} > 0\}$ above with

$$\breve{I} = \begin{cases} \{0, 1, \cdots, (p-1)/2\}, & (\breve{A} = A^{(2)}_{p-1}), \\ \{0, 1, \cdots, p/2\}, & (\breve{A} = D^{(2)}_{1+p/2}). \end{cases}$$

For $i \in \breve{I}$, a direct calculation shows

$$c_{ii} = \begin{cases} 1 & (p \equiv 1 \pmod 2 \text{ and } i = (p-1)/2), \\ 2 & (\text{otherwise}). \end{cases}$$

As in Theorem 1.7, Lie theoretic objects associated with $\check{\mathfrak{g}}$ are written with $\check{}$ attached.

3.3. **Naito-Sagaki's fixed points crystals.** Let B_n be the connected component in $\mathsf{RPar}_p^n \cong B(\Lambda_0)^{\otimes n}$ that is isomorphic to $B(n\Lambda_0)$ as a $\mathfrak{g}$-crystal for $n \geq 1$. By Theorem 2.3,

$$B_n = \{(\lambda^{(1)}, \cdots, \lambda^{(n)}) \in \mathsf{RPar}_p^n \mid 1 \leq \forall i < n, \mathsf{base}(\lambda^{(i)}) \supseteq \mathsf{roof}(\lambda^{(i+1)})\}.$$

By virtue of Naito-Sagaki [NS1, Theorem 4.4], the set of fixed points $B_n^{\mathsf{M}^n}$ has a $\check{\mathfrak{g}}$-crystal structure that is isomorphic to $B(n\check{\Lambda}_0)$. All we need is the correspndence on weights:

the weight $\check{\mathrm{wt}}(b)$ of $b = (x_1, \cdots, x_n) \in B_n^{\mathsf{M}^n}$ as a $\check{\mathfrak{g}}$-crystal is given by

$$\check{\mathrm{wt}}(b) = n\check{\Lambda}_0 - \sum_{i \in \check{I}} m_i \check{\alpha}_i \Longleftrightarrow \sum_{i=1}^{n} \mathrm{wt}_0(x_i) = n\Lambda_0 - \sum_{i \in \check{I}} \frac{2m_i}{c_{ii}} \sum_{r=1}^{N_i - 1} \alpha_{\iota(\omega^r(i))} \tag{3.5}$$

where $\iota : \check{I} \hookrightarrow I, i \mapsto i + p\mathbb{Z}$ is an injection (see also [NS2, (1.2.2)]).

Since $\ell - 1 < (p-1)/2$ (resp. $\ell - 1 < p/2$) for odd p (resp. even p), the right hand side of (3.5) is equal to $(k+1)\Lambda_0 - \lambda_{\ell,0}^p$ whenever the left hand side of (3.5) is given by

$$\gamma_\ell = (k+1)\check{\Lambda}_0 - \ell\check{\alpha}_0 - (\ell-1)\check{\alpha}_1 - \cdots - \check{\alpha}_{\ell-1}$$

for $n = k+1$. Thus, we have $m_{\check{A}}((k+1)\check{\Lambda}_0, \gamma_\ell) = \#(Z'^{\mathsf{M}^{k+1}})$ where (see Proposition 2.4)

$$Z' = \{(\lambda^{(1)} \supseteq \cdots \supseteq \lambda^{(k+1)}) \in (\mathsf{Par}^{p\text{-core}})^{k+1} \mid \sum_{i=1}^{k+1} \mathrm{wt}_0(\lambda^{(i)}) = (k+1)\Lambda_0 - \lambda_{\ell,0}^p\}.$$

3.4. **Proof of Theorem 1.7.** In §2.4, we presented bijections

$$V_2 \xrightarrow{\sim} V_3 \xrightarrow{\sim} Z = Z', \quad (P, Q) \longmapsto \pi := \Pi({}^{\mathrm{tr}}Q, {}^{\mathrm{tr}}P) \longmapsto (\pi_{*,1}, \cdots, \pi_{*,k+1}),$$

where $V_2 = \bigsqcup_{\substack{\lambda \in \mathsf{Par}(\ell) \\ \ell(\lambda) \leq k+1}} \mathsf{RST}(\lambda)^2$ and $V_3 = \{\pi \in \mathsf{PP} \mid \mathrm{wt}(\pi) = \sum_{(i,j) \in \ell^\ell} \beta_{j-i}, \pi_{*,k+2} = \varnothing\}$.

By Corollary 3.3 and Lemma 2.10, we have

$$\#(Z'^{\mathsf{M}^{k+1}}) = \sum_{\substack{\lambda \in \mathsf{Par}(\ell) \\ \ell(\lambda) \leq k+1}} \# \mathsf{RST}(\lambda).$$

This is equal to $\sum_{\lambda \in \mathsf{Par}(\ell), \ell(\lambda) \leq k+1} \# \mathsf{RST}(\lambda)$ and it is equinumerous to $((k+2), (k+1), k, \cdots, 1)$-avoiding involution of ℓ by Lemma 2.1 and Lemma 2.2. This completes the proof of Theorem 1.7 (b).

We now know that $m_{\check{A}}((k+1)\check{\Lambda}_0, \gamma_\ell) > 0$. Thus, to prove Theorem 1.7 (a), it is enough to show that $m_{\check{A}}((k+1)\check{\Lambda}_0, \gamma_\ell + \check{\delta}) = 0$ (see Proposition 1.1). This follows from Proposition 3.5 and the condition $\ell - 1 < (p-1)/2$ (resp. $\ell - 1 < p/2$) when p is odd (resp. p is even).

Proposition 3.5 ([Kac, Proposition 12.5.(a)]). *Let A be an affine GCM. For $\Lambda \in \mathcal{P}^+_A$,*

$$P_A(\Lambda) = W \cdot \{\lambda \in \mathcal{P}^+_A \mid \lambda \leq \Lambda\}.$$

4. Appendix: on the number of $\max_{A^{(1)}_{p-1}}(k\Lambda_0) \cap \mathcal{P}^+_{A^{(1)}_{p-1}}$

We prove a conjecture of Jayne-Misra on the number $\#(\max_{A^{(1)}_{p-1}}(k\Lambda_0) \cap \mathcal{P}^+_{A^{(1)}_{p-1}})$.

Proposition 4.1 ([MR1, Conjecture 3.9]). *For $k \geq 1$ and $p \geq 2$,*

$$\#(\max_{A^{(1)}_{p-1}}(k\Lambda_0) \cap \mathcal{P}^+_{A^{(1)}_{p-1}}) = \frac{1}{p+k} \sum_{d\mathbb{Z} \supseteq k\mathbb{Z}, p\mathbb{Z}} \phi(d) \binom{(p+k)/d}{k/d}$$

where $\phi(d) = \#(\mathbb{Z}/d\mathbb{Z})^\times$ is Euler's totient function.

4.1. **Proof of Lemma 1.4.** Recall that $p \geq 2, k \geq 1, 0 \leq s < p$ and $\Lambda = k\Lambda_0 + \Lambda_s$. Depending on $s \neq 0$ or not, we define the set $S^{(p,s)}_k$ as follows:

$$\begin{aligned} S^{(p,0)}_k = \{(x_i)^p_{i=0} \in \mathbb{Z}^{p+1} \mid x_0 = x_p = 0, x_1 + x_{p-1} \leq k, \\ 0 < \forall i < p, -x_{i-1} + 2x_i - x_{i+1} \geq 0\}, \end{aligned}$$

$$\begin{aligned} S^{(p,s)}_k = \{(x_i)^p_{i=0} \in \mathbb{Z}^{p+1} \mid x_0 = x_p = 0, x_1 + x_{p-1} \leq k-1, \\ 0 < \forall i < p, \delta_{i,s} - x_{i-1} + 2x_i - x_{i+1} \geq 0\}. \end{aligned}$$

As in [Ts1, §3.1,§3.2], the following gives a bijection:

$$S^{(p,s)}_k \xrightarrow{\sim} \max_{A^{(1)}_{p-1}}(\Lambda) \cap \mathcal{P}^+_{A^{(1)}_{p-1}}, \quad (x_0, \cdots, x_p) \longmapsto \Lambda + \sum_{i=0}^{p-1} (x_i + q_0)\alpha_i,$$

where $q_0 = \max\{q \leq 0 \mid 1 \leq \forall i < p, x_i + q \leq 0 \text{ and } 1 \leq \exists i < p, x_i + q = 0\}$. Clearly $S^{(p,s)}_k \subseteq S^{(p,s)}_{k+1}$ and q_0 does not depend on k, thus we deduce Lemma 1.4.

4.2. **q-binomial coefficients and q-Lucas theorem.** Let $\begin{bmatrix} a \\ b \end{bmatrix} = [a]!/([b]![a-b]!)$ be a q-binomial coefficient for $0 \leq b \leq a$ and $[c]! = \prod_{n=1}^{c}(q^n - 1)/(q-1)$.

Proposition 4.2 ([St1, pp.66]). *For any $j, k \geq 0$, we have $\sum_{\substack{\lambda \in \mathsf{Par} \\ \ell(\lambda) \leq j, \lambda_1 \leq k}} q^{|\lambda|} = \begin{bmatrix} k+j \\ j \end{bmatrix}$.*

The following congruent property for q-binomial coefficients is known as q-Lucas theorem (see also [St1, Exercise 14 of Chapter 1] for Lucas theorem for binomial coefficients).

Proposition 4.3 ([Sag, Theorem 2.2]). *Let ζ be a primitive d-th root of unity where $d \geq 1$. For any $n, j \geq 0$,*

$$\begin{bmatrix} n \\ j \end{bmatrix}\Big|_{q=\zeta} = \binom{\lfloor n/d \rfloor}{\lfloor j/d \rfloor} \cdot \begin{bmatrix} n\%d \\ j\%d \end{bmatrix}\Big|_{q=\zeta}.$$

4.3. **Proof of Proposition 4.1.** As in §4.1, $\#(\max_{A^{(1)}_{p-1}}(k\Lambda_0)\cap\mathcal{P}^+_{A^{(1)}_{p-1}}) = \#S^{(p,0)}_k$.

Let us define sets T and U as follows:

$$T = \{(y_1,\cdots,y_p)\in\mathbb{Z}^p \mid y_1\geq\cdots\geq y_p, y_1+\cdots+y_p=0, y_1-y_p\leq k\},$$
$$U = \{(\lambda_1,\cdots,\lambda_{p-1})\in\mathbb{Z}^{p-1} \mid k\geq\lambda_1\geq\cdots\lambda_{p-1}\geq 0, \lambda_1+\cdots+\lambda_{p-1}\in p\mathbb{Z}\}.$$

The following maps are bijections:

$$S^{(p,0)}_k \xrightarrow{\sim} T, \quad (x_0,\cdots,x_p)\longmapsto(x_1-x_0,\cdots,x_p-x_{p-1}),$$
$$T \xrightarrow{\sim} U, \quad (y_1,\cdots,y_p)\longmapsto(y_1-y_p,\cdots,y_{p-1}-y_p).$$

By Proposition 4.2, we have

$$\#U = \frac{1}{p}\sum_{\zeta^p=1}\begin{bmatrix}k+p-1\\p-1\end{bmatrix}\Big|_{q=\zeta}.$$

Let ζ be a primitive d-th root of unity for some $1\leq d\leq p$ with $d\mathbb{Z}\ni p$. Then,

$$\begin{bmatrix}k+p-1\\p-1\end{bmatrix}\Big|_{q=\zeta} = \binom{\lfloor(k+p-1)/d\rfloor}{\lfloor(p-1)/d\rfloor}\begin{bmatrix}(k+p-1)\%d\\d-1\end{bmatrix}\Big|_{q=\zeta} \tag{4.1}$$

by Proposition 4.3. The right hand side of (4.1) vanishes unless $k+p-1\equiv d-1 \pmod d \Leftrightarrow d\mathbb{Z}\ni k$. When $d\mathbb{Z}\ni k$, the right hand side of (4.1) becomes $\binom{\lfloor(k+p-1)/d\rfloor}{\lfloor(p-1)/d\rfloor} = \binom{(k+p)/d-1}{p/d-1}$. Thus, we know that $\#(\max_{A^{(1)}_{p-1}}(k\Lambda_0)\cap\mathcal{P}^+_{A^{(1)}_{p-1}}) = \#S^{(p,0)}_k = \#U$ is equal to

$$\frac{1}{p}\sum_{d\mathbb{Z}\supseteq k\mathbb{Z},p\mathbb{Z}}\phi(d)\binom{(p+k)/d-1}{p/d-1} = \frac{1}{p+k}\sum_{d\mathbb{Z}\supseteq k\mathbb{Z},p\mathbb{Z}}\phi(d)\binom{(p+k)/d}{k/d}.$$

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References

[AKT] Susumu Ariki, Victor Kreiman, and Shunsuke Tsuchioka, *On the tensor product of two basic representations of $U_v(\widehat{\mathfrak{sl}}_e)$*, Adv. Math. **218** (2008), no. 1, 28–86, DOI 10.1016/j.aim.2007.11.018. MR2409408

[CR] Joseph Chuang and Raphaël Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification*, Ann. of Math. (2) **167** (2008), no. 1, 245–298, DOI 10.4007/annals.2008.167.245. MR2373155

[FSS] Jürgen Fuchs, Bert Schellekens, and Christoph Schweigert, *From Dynkin diagram symmetries to fixed point structures*, Comm. Math. Phys. **180** (1996), no. 1, 39–97. MR1403859

[Ful] William Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry. MR1464693

[Kac] Victor G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990. MR1104219

[Kas] Masaki Kashiwara, *Bases cristallines des groupes quantiques* (French), Cours Spécialisés [Specialized Courses], vol. 9, Société Mathématique de France, Paris, 2002. Edited by Charles Cochet. MR1997677

[KLMW] V. Kreiman, V. Lakshmibai, P. Magyar, and J. Weyman, *Standard bases for affine* SL(n)*-modules*, Int. Math. Res. Not. **21** (2005), 1251–1276, DOI 10.1155/IMRN.2005.1251. MR2146340

[Lit] Peter Littelmann, *A plactic algebra for semisimple Lie algebras*, Adv. Math. **124** (1996), no. 2, 312–331, DOI 10.1006/aima.1996.0085. MR1424313

[LLT] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon, *Hecke algebras at roots of unity and crystal bases of quantum affine algebras*, Comm. Math. Phys. **181** (1996), no. 1, 205–263. MR1410572

[LM] Sinéad Lyle and Andrew Mathas, *Blocks of cyclotomic Hecke algebras*, Adv. Math. **216** (2007), no. 2, 854–878, DOI 10.1016/j.aim.2007.06.008. MR2351381

[MM] Kailash Misra and Tetsuji Miwa, *Crystal base for the basic representation of $U_q(\mathfrak{sl}(n))$*, Comm. Math. Phys. **134** (1990), no. 1, 79–88. MR1079801

[MR1] Rebecca L. Jayne and Kailash C. Misra, *On multiplicities of maximal weights of $\widehat{sl}(n)$-modules*, Algebr. Represent. Theory **17** (2014), no. 4, 1303–1321, DOI 10.1007/s10468-014-9470-2. MR3228490

[MR2] R. Jayne and K. Misra, *Lattice Paths, Young Tableaux, and Weight Multiplicities*, arXiv:1508.06930

[Mat] Andrew Mathas, *Iwahori-Hecke algebras and Schur algebras of the symmetric group*, University Lecture Series, vol. 15, American Mathematical Society, Providence, RI, 1999. MR1711316

[NS1] Satoshi Naito and Daisuke Sagaki, *Standard paths and standard monomials fixed by a diagram automorphism*, J. Algebra **251** (2002), no. 1, 461–474, DOI 10.1006/jabr.2001.9135. MR1900295

[NS2] Satoshi Naito and Daisuke Sagaki, *Crystal bases and diagram automorphisms*, Representation theory of algebraic groups and quantum groups, Adv. Stud. Pure Math., vol. 40, Math. Soc. Japan, Tokyo, 2004, pp. 321–341. MR2074598

[Sag] Bruce E. Sagan, *Congruence properties of q-analogs*, Adv. Math. **95** (1992), no. 1, 127–143, DOI 10.1016/0001-8708(92)90046-N. MR1176155

[St1] Richard P. Stanley, *Enumerative combinatorics. Volume 1*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012. MR2868112

[St2] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. MR1676282

[Ts1] Shunsuke Tsuchioka, *Catalan numbers and level 2 weight structures of $A^{(1)}_{p-1}$*, New trends in combinatorial representation theory, RIMS Kôkyûroku Bessatsu, B11, Res. Inst. Math. Sci. (RIMS), Kyoto, 2009, pp. 145–154. MR2562491

[Ts2] Shunsuke Tsuchioka, *Hecke-Clifford superalgebras and crystals of type $D^{(2)}_l$*, Publ. Res. Inst. Math. Sci. **46** (2010), no. 2, 423–471, DOI 10.2977/PRIMS/13. MR2722783

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