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AN ELEMENTARY PROOF OF THE POSITIVITY OF THE INTERTWINING OPERATOR IN ONE-DIMENSIONAL TRIGONOMETRIC DUNKL THEORY

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ABSTRACT. This note is devoted to the intertwining operator in the one-dimensional trigonometric Dunkl setting. We obtain a simple integral expression of this operator and deduce its positivity.

1. INTRODUCTION

We use the lecture notes [6] as a general reference about trigonometric Dunkl theory. In dimension 1, this special function theory is a deformation of Fourier analysis on $\mathbb{R}$, depending on two complex parameters k_1 and k_2, where the classical derivative is replaced by the Cherednik operator

$$\begin{aligned} Df(x) &= \big(\tfrac{d}{dx}\big)f(x) + \big\{\tfrac{k_1}{1-e^{-x}} + \tfrac{2k_2}{1-e^{-2x}}\big\}\big\{f(x)-f(-x)\big\} - \big(\tfrac{k_1}{2}+k_2\big)f(x) \\ &= \big(\tfrac{d}{dx}\big)f(x) + \big\{\tfrac{k_1+k_2}{2}\coth\tfrac{x}{2} + \tfrac{k_2}{2}\tanh\tfrac{x}{2}\big\}\big\{f(x)-f(-x)\big\} - \big(\tfrac{k_1}{2}+k_2\big)f(-x)\,, \end{aligned}$$

the Lebesgue measure by $A(x)dx$, where

$$A(x) = |2\sinh\tfrac{x}{2}|^{2k_1}|2\sinh x|^{2k_2}\,,$$

and the exponential function $e^{i\lambda x}$ by the Opdam hypergeometric function

$$\begin{aligned} G_{i\lambda}(x) &= \overbrace{\varphi_{2\lambda}^{k_1+k_2-\frac{1}{2},k_2-\frac{1}{2}}\big(\tfrac{x}{2}\big)}^{{}_2\mathrm{F}_1(\frac{k_1}{2}+k_2+i\lambda,\frac{k_1}{2}+k_2-i\lambda;k_1+k_2+\frac{1}{2};-\sinh^2\frac{x}{2})} \\ &\quad + \frac{\frac{k_1}{2}+k_2+i\lambda}{2k_1+2k_2+1}\,(\sinh x)\,\underbrace{\varphi_{2\lambda}^{k_1+k_2+\frac{1}{2},k_2+\frac{1}{2}}\big(\tfrac{x}{2}\big)}_{{}_2\mathrm{F}_1(\frac{k_1}{2}+k_2+1+i\lambda,\frac{k_1}{2}+k_2+1-i\lambda;k_1+k_2+\frac{3}{2};-\sinh^2\frac{x}{2})}\,. \end{aligned}$$

Here $\varphi_\lambda^{\alpha,\beta}(x)$ denotes the Jacobi function and ${}_2\mathrm{F}_1(a,b;c;Z)$ the classical hypergeometric function.

In a series of papers ([2], [5], [7], [3], [8], [9], [10], [11], ...), Trimèche and his collaborators studied an intertwining operator $\mathcal{V}: C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$, which is characterized by

$$\mathcal{V}\circ\big(\tfrac{d}{dx}\big) = D\circ\mathcal{V} \quad\text{and}\quad \delta_0\circ\mathcal{V} = \delta_0\,,$$

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and the dual operator $\mathcal{V}^t\colon C_c^\infty(\mathbb{R}) \longrightarrow C_c^\infty(\mathbb{R})$, which satisfies

$$\int_{-\infty}^{+\infty} \mathcal{V}f(x)\, g(x)\, A(x)\, dx = \int_{-\infty}^{+\infty} f(y)\, \mathcal{V}^t g(y)\, dy\,.$$

Let us mention in particular the following facts.

• *Eigenfunctions*. For every $\lambda \in \mathbb{C}$,

$$\mathcal{V}(x \longmapsto e^{i\lambda x}) = G_{i\lambda}\,.$$

• *Explicit expression*. An integral representation of $\mathcal{V}$ was computed in [2] (and independently in [1]) under the assumption that $k_1 \geq 0$, $k_2 \geq 0$ with $k_1 + k_2 > 0$.

• *Analytic continuation*. It was shown in [3] that the intertwining operator $\mathcal{V}$ extends meromorphically with respect to $k \in \mathbb{C}^2$, with singularities in

$$\{\, k \in \mathbb{C}^2 \,|\, k_1 + k_2 + \tfrac{1}{2} \in -\mathbb{N} \}.$$

• *Positivity*. On the one hand, the positivity of $\mathcal{V}$ was disproved in [2] by using the above-mentioned expression of $\mathcal{V}$ in the case $k_1 > 0$, $k_2 > 0$. On the other hand, the positivity of $\mathcal{V}$ was investigated in [8], [9], [10], [11] by using the positivity of a heat type kernel in the case $k_1 \geq 0$, $k_2 \geq 0$.

In Section 2, we obtain an integral representation of $\mathcal{V}$ and $\mathcal{V}^t$ when $\operatorname{Re} k_1 > 0$ and $\operatorname{Re} k_2 > 0$. The expression is simpler and the proof is quicker than the previous ones in [2] or [1]. In Section 3, we deduce the positivity of $\mathcal{V}$ and $\mathcal{V}^t$ when $k_1 > 0$, $k_2 > 0$, and comment on the positivity issue.

2. Integral representation of the intertwining operator

In this section, we resume the computations in [2, Section 2] and prove the following result.

Theorem 2.1. *Let* $k = (k_1, k_2) \in \mathbb{C}^2$ *with* $\operatorname{Re} k_1 > 0$ *and* $\operatorname{Re} k_2 > 0$. *Then*

$$\mathcal{V}f(x) = \int_{|y|<|x|} \mathcal{K}(x,y)\, f(y)\, dy \qquad \forall\, x \in \mathbb{R}^*$$

and

$$\mathcal{V}^t g(y) = \int_{|x|>|y|} \mathcal{K}(x,y)\, g(x)\, A(x)\, dx\,,$$

where

$$\mathcal{K}(x,y) = \tfrac{c}{4} A(x)^{-1} \int_{|y|}^{|x|} \sigma(x,y,z)\, (\cosh \tfrac{z}{2} - \cosh \tfrac{y}{2})^{k_1 - 1} \times (\cosh x - \cosh z)^{k_2 - 1} (\sinh \tfrac{z}{2})\, dz\,, \tag{2.1}$$

with

$$c = 2^{3k_1 + 3k_2}\, \frac{\Gamma(k_1 + k_2 + \frac{1}{2})}{\sqrt{\pi}\, \Gamma(k_1)\, \Gamma(k_2)} \tag{2.2}$$

and

$$\sigma(x,y,z) = (\operatorname{sign} x) \left\{ e^{\frac{x}{2}} (2 \cosh \tfrac{x}{2}) - e^{-\frac{y}{2}} (2 \cosh \tfrac{z}{2}) \right\}. \tag{2.3}$$

Proof. As observed in [2] and [5],

$$\mathcal{V}f(x) = \int_{-|x|}^{+|x|} \mathcal{K}(x,y)\, f(y)\, dy$$

is an integral operator whose kernel

$$\mathcal{K}(x,y)=\tfrac{1}{4}\,K\big(\tfrac{x}{2},\tfrac{y}{2}\big)+(\operatorname{sign}x)\big(\tfrac{k_1}{4}+\tfrac{k_2}{2}\big)A(x)^{-1}\widetilde{K}\big(\tfrac{x}{2},\tfrac{y}{2}\big)\\ -(\operatorname{sign}x)\,\tfrac{1}{2}A(x)^{-1}\tfrac{\partial}{\partial y}\,\widetilde{K}\big(\tfrac{x}{2},\tfrac{y}{2}\big) \tag{2.4}$$

can be expressed in terms of the kernel

$$K(x,y)=2\,c\,A(2x)^{-1}|\sinh 2x|\\ \times\int_{|y|}^{|x|}(\cosh z-\cosh y)^{k_1-1}(\cosh 2x-\cosh 2z)^{k_2-1}(\sinh z)\,dz \tag{2.5}$$

of the intertwining operator in the Jacobi setting (see [4, Subsection 5.3]) and of its integral

$$\widetilde{K}(x,y)=\int_{|y|}^{|x|}K(w,y)\,A(2w)\,dw\\ =\tfrac{c}{k_2}\int_{|y|}^{|x|}(\cosh z-\cosh y)^{k_1-1}(\cosh 2x-\cosh 2z)^{k_2}(\sinh z)\,dz\,. \tag{2.6}$$

Let us integrate by parts (2.6) and differentiate the resulting expression with respect to y. This way, we obtain

$$\widetilde{K}(x,y)=\tfrac{4c}{k_1}\int_{|y|}^{|x|}(\cosh z-\cosh y)^{k_1}(\cosh 2x-\cosh 2z)^{k_2-1}\\ \times(\cosh z)\,(\sinh z)\,dz \tag{2.7}$$

and

$$\tfrac{\partial}{\partial y}\widetilde{K}(x,y)=-\,4\,c\,(\sinh y)\int_{|y|}^{|x|}(\cosh z-\cosh y)^{k_1-1}(\cosh 2x-\cosh 2z)^{k_2-1}\\ \times(\cosh z)\,(\sinh z)\,dz\,. \tag{2.8}$$

We conclude by substituting (2.5), (2.6), (2.7), (2.8) in (2.4) and more precisely (2.6), respectively (2.7), in

$$(\operatorname{sign}x)\,\tfrac{k_2}{2}A(x)^{-1}\widetilde{K}\big(\tfrac{x}{2},\tfrac{y}{2}\big)\,,\quad\text{respectively}\quad(\operatorname{sign}x)\,\tfrac{k_1}{4}A(x)^{-1}\widetilde{K}\big(\tfrac{x}{2},\tfrac{y}{2}\big)\,.$$

□

Remark 2.2. Let $x,y\in\mathbb{R}$ such that $|x|>|y|$. The expression (2.1) extends meromorphically with respect to $k\in\mathbb{C}^2$, with singularities in $\{k\in\mathbb{C}^2\,|\,k_1+k_2+\frac{1}{2}\in-\mathbb{N}\}$, produced by the factor $\Gamma(k_1+k_2+\frac{1}{2})$ in (2.2). In the limit cases where either k_1 or k_2 vanishes, (2.1) reduces to the following expressions, already obtained in [2] and [1]:

- Assume that $k_1=0$ and $\operatorname{Re}k_2>0$. Then

$$\mathcal{K}(x,y)=2^{k_2-1}\frac{\Gamma(k_2+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(k_2)}\,|\sinh x|^{-2k_2}\\ \times(\cosh x-\cosh y)^{k_2-1}(\operatorname{sign}x)\,(e^{x}-e^{-y})\,. \tag{2.9}$$

- Assume that $k_2=0$ and $\operatorname{Re}k_1>0$. Then

$$\mathcal{K}(x,y)=2^{k_1-2}\frac{\Gamma(k_1+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(k_1)}\,|\sinh\tfrac{x}{2}|^{-2k_1}\\ \times(\cosh\tfrac{x}{2}-\cosh\tfrac{y}{2})^{k_1-1}(\operatorname{sign}x)\,(e^{\frac{x}{2}}-e^{-\frac{y}{2}})\,. \tag{2.10}$$

3. Positivity of the intertwining operator

Corollary 3.1. *Assume that* $k_1 > 0$ *and* $k_2 > 0$. *Then the kernel* (2.1) *is strictly positive for every* $x, y \in \mathbb{R}$ *such that* $|x| > |y|$. *Hence the intertwining operator* $\mathcal{V}$ *and its dual* $\mathcal{V}^t$ *are positive.*

Proof. Let us check the positivity of (2.3) when $x, y, z \in \mathbb{R}$ satisfy $|x| > z > |y|$. On the one hand, if $x > 0$, then

$$\begin{aligned}\sigma(x,y,z) &= e^{\frac{x}{2}}\big(2\cosh\tfrac{x}{2}\big) - e^{-\frac{y}{2}}\big(2\cosh\tfrac{z}{2}\big)\\ &> \big(e^{\frac{x}{2}} - e^{-\frac{y}{2}}\big)\big(2\cosh\tfrac{x}{2}\big) > 0\,.\end{aligned}$$

On the other hand, if $x < 0$, then

$$\begin{aligned}\sigma(x,y,z) &= e^{-\frac{y}{2}}\big(2\cosh\tfrac{z}{2}\big) - e^{\frac{x}{2}}\big(2\cosh\tfrac{x}{2}\big)\\ &> e^{-\frac{y}{2}}\big(2\cosh\tfrac{y}{2}\big) - e^{\frac{x}{2}}\big(2\cosh\tfrac{x}{2}\big) = e^{-y} - e^{x} > 0\,.\end{aligned}$$

□

Remark 3.2. As already observed in [2], the positivity of (2.9), respectively (2.10), is immediate in the limit case where $k_1 = 0$ and $k_2 > 0$, respectively $k_2 = 0$ and $k_1 > 0$.

Remark 3.3. The positivity of $\mathcal{V}$ was mistakenly disproved in [2, Theorem 2.11] when $k_1 > 0$ and $k_2 > 0$. More precisely, by using a more complicated formula than (2.1), the density $\mathcal{K}(x,y)$ was shown to be negative when $x > 0$ and $y \searrow -x$. The error in the proof lies in the expression A_1, which is equal to $\frac{k}{k'}\frac{\sinh(2x)-\sinh(2|y|)}{E}$ and which tends to $+2\frac{k}{k'}\frac{\cosh(2x)}{\sinh(2x)} > 0$.

Remark 3.4. A different approach, based on the positivity of a heat type kernel, was used in [8], [9], [10] and [11] in order to tackle the positivity of $\mathcal{V}$. While [8] may be right, the same flaw occurs in [9], [10], [11], namely the cut–off $\mathbf{1}_{Y_\ell}$ breaks down the differential-difference equations, which are not local.

In conclusion, this note settles in a simple way the positivity issue in dimension 1 and hence in the product case. Otherwise, the positivity of the interwining operator $\mathcal{V}$ and its dual $\mathcal{V}^t$, when the multiplicity function k is ≥ 0, remains an open problem in higher dimensions.

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