ON σ-COUNTABLY TIGHT SPACES

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ABSTRACT. Extending a result of R. de la Vega, we prove that an infinite homogeneous compactum has cardinality $\mathfrak{c}$ if it is the union of either countably many *dense* or finitely many arbitrary countably tight subspaces. The question if every infinite homogeneous and σ-countably tight compactum has cardinality $\mathfrak{c}$ remains open.

We also show that if an arbitrary product is σ-countably tight, then all but finitely many of its factors must be countably tight.

1. INTRODUCTION

Let us start by recalling that a topological space X is *countably tight* (in short: CT) if the closure of any subset A of X is equal to the union of the closures of all countable subsets of A. Moreover, the space X is *homogeneous* if for any two points $x, y \in X$ there is an autohomeomorphism $h : X \to X$ such that $h(x) = y$. We refer the reader to [5] and [6] concerning our notation and terminology regarding set-theoretic topology.

In [4] R. de la Vega verified an old conjecture of Arhangel'skii by proving that every infinite countably tight homogeneous compactum has cardinality $\mathfrak{c}$. (A short proof of de la Vega's theorem is given in [3].) The aim of this paper is to see whether in this result CT could be weakened to σ-CT, i.e. if de la Vega's result remains valid when the homogeneous compactum is only assumed to be the union of countably many CT subspaces. We conjecture that the answer to this question is affirmative and provide results that, at least to us, convincingly point in this direction.

In fact, we shall prove below that an infinite homogeneous compactum has cardinality $\mathfrak{c}$ if it is either the union of *finitely many* CT subspaces or the union of countably many *dense* CT subspaces.

Just to see that the assumption of compactness is really essential in these types of results, we mention here the following example. Consider the Cantor cube $\mathbb{C}_\kappa = \{0,1\}^\kappa$ and in it the subspaces

$$\sigma_i = \left\{ x \in \{0,1\}^\kappa : |\{\alpha < \kappa : x(\alpha) \neq i\}| < \omega \right\}$$

for $i \in \{0,1\}$. Then $\sigma_0 \cup \sigma_1$ is a σ-compact subgroup of $\mathbb{C}_\kappa$ that is the union of two CT, even Frèchet, subspaces but, as is easily seen, has tightness and cardinality κ, for any cardinal κ.

Of course, it is natural to raise the following question: Is there at all a homogeneous σ-CT compactum that is not CT? Now, if our above conjecture is valid and

$\mathfrak{c} < 2^{\omega_1}$, then, by the Čech–Pospišil theorem, every homogeneous σ-CT compactum is even first countable; hence any such example can only exist in a model in which $\mathfrak{c} = 2^{\omega_1}$. On the other hand, we know that the answer to this question is trivially negative if homogeneity is dropped: The compact ordinal space $\omega_1 + 1$ is the union of two CT subspaces but is not CT.

2. Subseparable G_δ-sets exist in σ-CT compacta

In the proof of de la Vega's result on the size of homogeneous CT compacta a crucial role was played by Arhangel'skii's observation that every CT compactum admits (non-empty) subseparable G_δ-sets. Naturally, we call a set subseparable if it is included in the closure of some countable set. The main aim of this section is to show that this statement is also valid for σ-CT compacta. To achieve this aim, we formulate and prove several auxiliary lemmas.

For any space X we shall denote by $\mathcal{G}(X)$ the family of all non-empty closed G_δ subsets of X. Clearly, if X is regular, then for every G_δ subset H of X we have $H = \bigcup \{G \in \mathcal{G}(X) : G \subset H\}$.

Lemma 2.1. *Let D be a countably compact dense subset of the normal space X. Then for every countable cover $\{A_n : n < \omega\}$ of X there are $H \in \mathcal{G}(X)$ and $n < \omega$ such that $H \cap D \cap A_n$ is G_δ-dense in H.*

Proof. Let us note first that D is actually G_δ-dense in X. Indeed, the normality of X implies that any $H \in \mathcal{G}(X)$ is of the form $H = \bigcap \{U_n : n < \omega\}$ where U_n is open and $\overline{U_{n+1}} \subset U_n$ for all $n < \omega$. Thus, if we pick $x_n \in D \cap U_n$, then any accumulation point of the sequence $\{x_n : n < \omega\}$ is in H; hence $D \cap H \neq \emptyset$ because D is countably compact.

Assume next that the conclusion of our lemma is false. Then we can find a decreasing sequence $\{H_n : n < \omega\} \subset \mathcal{G}(X)$ such that $H_n \cap D \cap A_n = \emptyset$ for all $n < \omega$. But then $\bigcap \{H_n \cap D : n < \omega\} \neq \emptyset$ as D is countably compact, contradicting that $\{A_n : n < \omega\}$ covers X. $\qquad\square$

Lemma 2.2. *Let X be a countably compact regular space and $\{A_n : n < \omega\}$ be any countable cover of X. Then there is $H \in \mathcal{G}(X)$ such that, for every $n < \omega$, if $A_n \cap H \neq \emptyset$, then $A_n \cap H$ is G_δ-dense in H.*

Proof. Starting with $H_0 = X$, we may define by straightforward recursion sets $H_n \in \mathcal{G}(X)$ for all $n < \omega$ such that if $A_n \cap H_n$ is not G_δ-dense in H_n, then $H_{n+1} \subset H_n$ and $A_n \cap H_{n+1} = \emptyset$. Clearly, then $H = \bigcap \{H_n : n < \omega\} \in \mathcal{G}(X)$ is as required. $\qquad\square$

Lemma 2.3. *If X is a σ-CT compactum, then any pairwise disjoint collection of dense ω-bounded subspaces of X is countable.*

Proof. We have a countable cover $\{A_n : n < \omega\}$ of X where each A_n is CT. Assume now that $\{D_\alpha : \alpha < \omega_1\}$ are dense ω-bounded subspaces of X. Applying Lemma 2.1, we may then define by transfinite recursion on $\alpha < \omega_1$ sets $H_\alpha \in \mathcal{G}(X)$ and natural numbers $n_\alpha < \omega$ such that

(1) if $\beta < \alpha$, then $H_\alpha \subset H_\beta$,
(2) $H_\alpha \cap D_\alpha \cap A_{n_\alpha}$ is G_δ-dense in H_α.

Now, pick distinct $\beta < \alpha < \omega_1$ such that $n_\beta = n_\alpha = n$. Then, since A_n is CT and $H_\alpha \cap D_\alpha$ is ω-bounded, we have $\emptyset \neq A_n \cap H_\alpha \subset D_\alpha$, and similarly $A_n \cap H_\beta \subset D_\beta$. But $\emptyset \neq A_n \cap H_\alpha \subset A_n \cap H_\beta$, and this clearly implies that $D_\alpha \cap D_\beta \neq \emptyset$. $\qquad \square$

It is easy to see that in the Cantor cube $\mathbb{C}_{\omega_1} = \{0,1\}^{\omega_1}$ there are uncountably many (in fact, 2^{ω_1} many) pairwise disjoint dense ω-bounded subspaces of the form $f + \Sigma$, where Σ is the subgroup of $\mathbb{C}_{\omega_1}$ consisting of all its members having countable support. Also, it is obvious that if $\pi : X \to Y$ is an irreducible continuous map between compacta, then for every dense ω-bounded subspace D of Y its inverse image $\pi^{-1}[D]$ is a dense ω-bounded subspace of X. Using Shapirovskii's well-known result (see e.g. 3.18 of [6]), we immediately obtain the following corollary of Lemma 2.3.

Lemma 2.4. *If X is a σ-CT compactum, then no closed subspace of X can be mapped onto $\mathbb{C}_{\omega_1}$. In particular, then every non-empty closed subspace Y has a point $y \in Y$ with $\pi\chi(y, Y) \leq \omega$.*

For a CT compactum X, again by Shapirovskii, we have the stronger result that $\pi\chi(X) \leq \omega$; see 3.14 of [6]. This, however, fails for σ-CT compacta, as is shown by the very simple example of the ordinal space $\omega_1 + 1$ that is the union of two CT compacta.

We are now ready to present the main result of this section.

Theorem 2.5. *Every σ-CT compactum X has a non-empty subseparable G_δ subset.*

Proof. If X has a G_δ point, i.e. a point of first countability, then we are done. So, we may assume that X is nowhere first countable, which clearly implies that $\chi(x, H) > \omega$ whenever $x \in H \in \mathcal{G}(X)$. Then $|H| \geq 2^{\omega_1}$ by the Čech–Pospišil theorem.

We have $X = \bigcup\{A_n : n < \omega\}$ where every A_n is CT. By Lemma 2.2 we may also assume without any loss of generality that every A_n is G_δ-dense in X.

Now, assume that no member of $\mathcal{G}(X)$ is subseparable and by transfinite recursion on $\alpha < \omega_1$ define $S_\alpha, T_\alpha \in \mathcal{G}(X)$, points $x_\alpha \in S_\alpha$, and countable sets B_α^n for $n < \omega$ such that the following inductive hypotheses hold:

(1) $S_\alpha \cap T_\alpha = \emptyset$,
(2) $T_\alpha \subset \bigcap\{T_\beta : \beta < \alpha\}$,
(3) $B_\beta^n \subset S_\alpha$ for any $\beta < \alpha$ and $n < \omega$,
(4) $B_n^\alpha \subset A_n \cap S_\alpha \cap \bigcap\{T_\beta : \beta < \alpha\}$ and $x_\alpha \in \overline{B_n^\alpha}$ for all $n < \omega$.

S_0, T_0 are any two disjoint members of $\mathcal{G}(X)$, and $x_0 \in S_0$ is chosen to satisfy $\pi\chi(x_0, S_0) = \omega$. This is possible by Lemma 2.4. This implies the existence of a countable set $B_0^n \subset A_n \cap S_0$ with $x_0 \in \overline{B_0^n}$ for each $n < \omega$ because A_n is dense in S_0. If $0 < \alpha < \omega_1$ and the construction has been completed for all $\beta < \alpha$, put $T = \bigcap\{T_\beta : \beta < \alpha\}$ and $B = \bigcup\{B_n^\beta : \beta < \alpha, n < \omega\}$. Then B is countable; hence $T \setminus \overline{B}$ is a non-empty G_δ by our indirect assumption. Consequently, there are disjoint $H, K \in \mathcal{G}(X)$ such that $H \subset T$ and $\overline{B} \subset K$. Next we may choose disjoint sets $H_0, H_1 \in \mathcal{G}(H) \subset \mathcal{G}(X)$ and the point $x_\alpha \in H_0$ with $\pi\chi(x_\alpha, H_0) = \omega$. Then again we have a countable set $B_\alpha^n \subset A_n \cap H_0$ with $x_\alpha \in \overline{B_n^\alpha}$ for each $n < \omega$. Then, putting $S_\alpha = K \cup H_0$ and $T_\alpha = H_1$, it is easy to see that the inductive hypotheses remain valid, completing the recursive construction.

Let x be a complete accumulation point of the set $\{x_\alpha : \alpha < \omega_1\}$. Then there is $n < \omega$ for which $x \in A_n$; moreover (4) implies both $x \in \bigcap_{\alpha < \omega_1} T_\alpha$ and

$x \in \overline{\bigcup_{\alpha<\omega_1} B_\alpha^n}$. Consequently, as A_n is CT, there is some $\alpha < \omega_1$ such that $x \in \overline{\bigcup_{\beta<\alpha} B_\beta^n}$; hence $x \in S_\alpha$ by (3). But this would imply $x \in S_\alpha \cap T_\alpha$, contradicting (1). □

3. A "TWO COVER" THEOREM

A subseparable subspace of a regular space has weight $\leq \mathfrak{c}$, so in view of the previous section any σ-CT compactum has many G_δ sets of weight $\leq \mathfrak{c}$. The result we prove in this section, however, needs more: having a cover of the space by G_δ sets of weight $\leq \mathfrak{c}$. Of course, if the space in question is also homogeneous, then the existence of a non-empty G_δ set of weight $\leq \mathfrak{c}$ implies the existence of such a cover. Also, being σ-CT just means that our space has a countable cover by CT sets. Thus we have the two covers referred to in the title of this section.

Theorem 3.1. *Let X be a Lindelöf regular space with two covers $\mathcal{Y}$ and $\mathcal{H}$ such that*

(1) $|\mathcal{Y}| \leq \mathfrak{c}$; moreover every $Y \in \mathcal{Y}$ is CT and satisfies

$$X = \bigcup \{\overline{A} : A \in [Y]^{\leq \mathfrak{c}}\};$$

(2) $\mathcal{H} \subset \mathcal{G}(X)$ and $w(H) \leq \mathfrak{c}$ for every $H \in \mathcal{H}$;
(3) for every set $D \in [X]^{\leq \mathfrak{c}}$ we have $w(\overline{D}) \leq \mathfrak{c}$.
Then $w(X) \leq \mathfrak{c}$.

The proof of this theorem will be based on the following two rather general lemmas. The first one deals with a cover $\mathcal{Y}$ as in (1) and the second with a cover like $\mathcal{H}$ in (2).

Lemma 3.2. *Let X be any space with a cover $\mathcal{Y}$ exactly as in (1) above. Moreover assume that the closure $\overline{D}$ of every set $D \in [X]^{\leq \mathfrak{c}}$ is Lindelöf and has pseudo-character $\psi(\overline{D}, X) \leq \mathfrak{c}$. Then $d(X) \leq \mathfrak{c}$.*

Proof. We shall say that a set $S \subset X$ is $\mathcal{Y}$-saturated if $Y \cap S$ is dense in S for every $Y \in \mathcal{Y}$. Obviously, any union of $\mathcal{Y}$-saturated sets is $\mathcal{Y}$-saturated. A simple ω-length iteration of property (1) yields that for every point $x \in X$ we may fix a $\mathcal{Y}$-saturated set $S(x) \in [X]^\mathfrak{c}$ with $x \in S(x)$.

We may also fix for every set $D \in [X]^{\leq \mathfrak{c}}$ a collection $\mathcal{U}(D)$ of open sets with $|\mathcal{U}(D)| \leq \mathfrak{c}$ such that $\cap \mathcal{U}(D) = \overline{D}$.

Next, by transfinite recursion on $\alpha < \omega_1$ we define $\mathcal{Y}$-saturated sets $D_\alpha \in [X]^\mathfrak{c}$ as follows. We start by choosing D_0 as an arbitrary $\mathcal{Y}$-saturated set of size $\mathfrak{c}$. (If $|X| \leq \mathfrak{c}$, then we are done.) Also, if α is a limit, then we simply put $D_\alpha = \bigcup_{\beta<\alpha} D_\beta$.

If D_α has been defined, then in the successor case $\alpha + 1$ we first consider the collection $\mathcal{V}_\alpha = \bigcup \{\mathcal{U}(D_\beta) : \beta \leq \alpha\}$ and then put

$$\mathcal{W}_\alpha = \{\bigcup \mathcal{V} : \mathcal{V} \in [\mathcal{V}_\alpha]^{\leq \omega} \text{ and } X \setminus \bigcup \mathcal{V} \neq \emptyset\}.$$

Clearly, we have $|\mathcal{W}_\alpha| \leq \mathfrak{c}$. For each $W \in \mathcal{W}_\alpha$ we may then fix a point $x_W \in X \setminus W$ and then put

$$D_{\alpha+1} = D_\alpha \cup \bigcup \{S(x_W) : W \in \mathcal{W}_\alpha\}.$$

Finally, we put $D = \bigcup_{\alpha<\omega_1} D_\alpha$; then we clearly have $|D| = \mathfrak{c}$. We shall now show that D is dense in X.

Claim 1. $\overline{D} = \bigcup_{\alpha<\omega_1} \overline{D_\alpha}$.

Indeed, for any $x \in \overline{D}$ there is $Y \in \mathcal{Y}$ with $x \in Y$, and, since D is $\mathcal{Y}$-saturated, this implies that $x \in \overline{Y \cap D}$. This, in turn, implies that there is a countable subset $A \subset Y \cap D$ with $x \in \overline{A}$ because Y is CT. But then there is some $\alpha < \omega_1$ for which $A \subset D_\alpha$, hence $x \in \overline{D_\alpha}$.

The following claim then finishes the proof.

Claim 2. $X = \overline{D}$.

Assume that $x \in X \setminus \overline{D}$. Then for each $\alpha < \omega_1$ there is $U_\alpha \in \mathcal{U}(D_\alpha)$ with $x \notin U_\alpha$. But then $\{U_\alpha : \alpha < \omega_1\}$ is an open cover of the Lindelöf subspace $\overline{D}$; hence there is a countable ordinal $\gamma < \omega_1$ such that $W = \bigcup_{\alpha < \gamma} \supset \overline{D}$ as well. But then we also have $W \in \mathcal{W}_\gamma$, hence $x_W \in D_{\gamma+1} \subset D$, contradicting that $x_W \notin W \supset D$. $\square$

Lemma 3.3. *Let X be a regular space and assume that $Z \subset X$ is a Lindelöf subspace of weight $w(Z) \leq \mathfrak{c}$. Moreover Z admits a cover $\mathcal{H} \subset \mathcal{G}(X)$ with $w(H) \leq \mathfrak{c}$ for every $H \in \mathcal{H}$. Then $\psi(Z, X) \leq \mathfrak{c}$.*

Proof. We first show that $w(Z) \leq \mathfrak{c}$ implies $|\mathcal{G}(Z)| \leq \mathfrak{c}$. So we fix an open base $\mathcal{B}$ of Z with $|\mathcal{B}| \leq \mathfrak{c}$. Every set $S \in \mathcal{G}(Z)$ is then the intersection of a countable family $\mathcal{U}$ of sets open in Z. Now S, being closed in Z, is also Lindelöf. Hence for every $U \in \mathcal{U}$ there is a countable subfamily $\mathcal{B}_U$ of $\mathcal{B}$ such that $S \subset B_U = \bigcup \mathcal{B}_U \subset U$. Consequently we have $S = \bigcap\{B_U : U \in \mathcal{U}\}$ as well. Thus we conclude that

$$|\mathcal{G}(Z)| \leq \left|\left[[\mathcal{B}]^\omega\right]^\omega\right| = \mathfrak{c}\,.$$

Of course, this means that we may assume without any loss of generality that $|\mathcal{H}| \leq \mathfrak{c}$ as well.

Clearly, the regularity of X and $w(H) \leq \mathfrak{c}$ imply that $\psi(H \cap Z, H) \leq \mathfrak{c}$ for each $H \in \mathcal{H}$, but then $\psi(H \cap Z, X) \leq \mathfrak{c}$ as well, since H is a G_δ. So we may fix, for every $H \in \mathcal{H}$, a family $\mathcal{V}_H$ of open sets in X with $|\mathcal{V}_H| \leq \mathfrak{c}$ such that $\bigcap \mathcal{V}_H = H \cap Z$. Then $|\mathcal{H}| \leq \mathfrak{c}$ implies that $\mathcal{V} = \bigcup\{\mathcal{V}_H : H \in \mathcal{H}\}$ has cardinality $\leq \mathfrak{c}$ as well. Finally, we put

$$\mathcal{W} = \{\bigcup \mathcal{V}' : \mathcal{V}' \in [\mathcal{V}]^{\leq \omega} \text{ and } Z \subset \bigcup \mathcal{V}'\}\,.$$

Clearly, we have $|\mathcal{W}| \leq \mathfrak{c}$ as well.

We claim that $Z = \bigcap \mathcal{W}$; hence $\psi(Z, X) \leq \mathfrak{c}$. To see this, pick any point $x \in X \setminus Z$. Then for each $H \in \mathcal{H}$ there is a member $V_H \in \mathcal{V}_H$ such that $x \notin V_H$. The Lindelöf property of Z implies that $\mathcal{H}$ has a countable subfamily $\mathcal{H}'$ such that $\mathcal{V}' = \{V_H : H \in \mathcal{H}'\}$ covers Z. But then $W = \bigcup \mathcal{V}' \in \mathcal{W}$ and clearly $x \notin W$. $\square$

We are now ready to give the proof of Theorem 3.1. First observe that conditions (2) and (3) of the theorem together with Lemma 3.3 imply that $\psi(\overline{D}, X) \leq \mathfrak{c}$ whenever $D \in [X]^{\leq \mathfrak{c}}$. This, however, means that X satisfies (with $\mathcal{Y}$) all the conditions of Lemma 3.2; hence we have a dense set D in X of size $\leq \mathfrak{c}$. But this, in turn, implies that $w(X) = w(\overline{D}) \leq \mathfrak{c}$, completing the proof of Theorem 3.1.

For further use in the next section, we present one more lemma.

Lemma 3.4. *Assume that X is a regular space and $\mathcal{Y}$ is a cover of X as in (1) of Theorem 3.1. Then for every $D \in [X]^{\leq \mathfrak{c}}$ we have $nw(\overline{D}) \leq \mathfrak{c}$.*

Proof. Since every subset of X of size $\leq \mathfrak{c}$ is included in a $\mathcal{Y}$-saturated subset of size $\leq \mathfrak{c}$, we may assume without loss of generality that D is $\mathcal{Y}$-saturated. Now, we claim that the family $\mathcal{N} = \{\overline{A} : A \in [D]^\omega\}$ is a network for $\overline{D}$.

Indeed, assume that $x \in \overline{D}$ and U is any open set containing x. Choose an open V such that $x \in V \subset \overline{V} \subset U$. There is some $Y \in \mathcal{Y}$ with $x \in Y$ and the $\mathcal{Y}$-saturatedness of D then implies $x \in \overline{V \cap D \cap Y}$. But then, as Y is CT, there is a countable set $A \subset V \cap D \cap Y$ such that $x \in \overline{A}$. This clearly implies that $x \in \overline{A} \subset \overline{V} \subset U$, which shows that $\mathcal{N}$ is a network for $\overline{D}$. $\square$

Since for every compactum X we have $nw(X) = w(X)$, this allows us to obtain the following simplified form of Theorem 3.1 for compact spaces.

Corollary 3.5. *Let X be a compactum with covers $\mathcal{Y}$ and $\mathcal{H}$ such that*

(1) $|\mathcal{Y}| \leq \mathfrak{c}$ and every $Y \in \mathcal{Y}$ is CT and dense in X;
(2) $\mathcal{H} \subset \mathcal{G}(X)$ and $w(H) \leq \mathfrak{c}$ for every $H \in \mathcal{H}$.
  Then $w(X) \leq \mathfrak{c}$.

Proof. Let us start by noting that if $x \in H \in \mathcal{H}$, then, by compactness, we have $\chi(x, X) = \psi(x, X) \leq \psi(x, H) \cdot \omega \leq \mathfrak{c}$; hence $\chi(X) \leq \mathfrak{c}$. But this clearly implies that $X = \bigcup \{\overline{A} : A \in [Y]^{\leq \mathfrak{c}}\}$ for every dense $Y \subset X$. Consequently, $\mathcal{Y}$ satisfies all requirements of (1) from Theorem 3.1. Hence by Lemma 3.4 we have $nw(\overline{D}) = w(\overline{D}) \leq \mathfrak{c}$ for every $D \in [X]^{\leq \mathfrak{c}}$. But this is just condition (3) of Theorem 3.1 which implies that $w(X) \leq \mathfrak{c}$. $\square$

We close this section by acknowledging that the method we used to prove the results of this section was motivated by the proof of Theorem 6.4 in [1]. There, in turn, the authors give credit to the approach that R. Buzyakova used in [2].

4. Adding homogeneity

Although we could not prove that all infinite homogeneous σ-CT compacta are of size $\mathfrak{c}$, the results of this section provide significant steps in that direction.

Theorem 4.1. *Assume that the compactum X is the union of countably many dense CT subspaces; moreover X^ω is homogeneous. Then $|X| \leq \mathfrak{c}$.*

Proof. Let $\mathcal{Y}$ be a countable family of dense CT subspaces of X that covers X. Then, by Lemma 2.4, there is a point $x \in X$ with $\pi\chi(x, X) \leq \omega$. Consequently X^ω also has a point of countable π-character, namely the point all of whose co-ordinates are equal to x. But X^ω is homogeneous; hence we actually have $\pi\chi(X^\omega) = \omega$.

Next, applying Theorem 2.5 we obtain the existence of some $G \in \mathcal{G}(X)$ that is subseparable and hence has weight $w(G) \leq \mathfrak{c}$. But then we also have $G^\omega \in \mathcal{G}(X^\omega)$; moreover $w(G^\omega) \leq \mathfrak{c}$ as well. Now, the homogeneity of X^ω then implies that actually X^ω can be covered by closed G_δ-sets of weight $\leq \mathfrak{c}$. Consequently, this is also true for X; i.e. there is a cover $\mathcal{H} \subset \mathcal{G}(X)$ of X such that $w(H) \leq \mathfrak{c}$ for all $H \in \mathcal{H}$.

Thus the two covers $\mathcal{Y}$ and $\mathcal{H}$ of X satisfy both conditions (1) and (2) of Corollary 3.5. Hence we can apply it to conclude that $w(X) \leq \mathfrak{c}$, which, in turn, implies that $w(X^\omega) \leq \mathfrak{c}$ as well. But it was shown in [7] that any homogeneous compactum Z satisfies the inequality $|Z| \leq w(Z)^{\pi\chi(Z)}$; consequently, we conclude that $|X| \leq |X^\omega| \leq \mathfrak{c}$. (Of course, we have $|X^\omega| = \mathfrak{c}$, unless X is a singleton.) $\square$

In our next result we can get rid of the annoying condition of density for the members of $\mathcal{Y}$; however we have to pay a price: the cover of X by CT subspaces needs to be *finite*. Also, we need to assume that X itself, and not just X^ω, is homogeneous.

Theorem 4.2. *If X is an infinite homogeneous compactum that is the union of finitely many CT subspaces, then $|X| = \mathfrak{c}$.*

Proof. We start with the trivial remark that $|X| \geq \mathfrak{c}$ for any infinite homogeneous compactum.

Following the arguments in the previous proof of Theorem 4.1, but using now the homogeneity of X, we may conclude that $\pi\chi(X) = \omega$. Moreover there is a cover $\mathcal{H} \subset \mathcal{G}(X)$ of X such that $w(H) \leq \mathfrak{c}$ for all $H \in \mathcal{H}$.

Of course, we also have the finite cover $\mathcal{Y}$ of X by CT subspaces that may not be dense. But it is straightforward then to find a non-empty open subset U of X such that for every $Y \in \mathcal{Y}$ we have $Y \cap U$ is dense in U whenever $Y \cap U \neq \emptyset$. We then put $\mathcal{Z} = \{Y \in \mathcal{Y} : Y \cap U \neq \emptyset\}$. Consider any non-empty open subset V of U such that $\overline{V} \subset U$. But then Corollary 3.5 can be applied to $\overline{V}$ and the covers $\mathcal{Z}$ and $\mathcal{H}$ restricted to $\overline{V}$ to conclude that $w(V) \leq w(\overline{V}) \leq \mathfrak{c}$.

Now, using the homogeneity and the compactness of X, we can cover X by finitely many open sets each homeomorphic to V, which clearly implies that $w(X) = w(V) \leq \mathfrak{c}$. Thus we are done because we have $|X| \leq w(X)^{\pi\chi(X)} = \mathfrak{c}$, again by [7]. $\square$

As is well-known, $\mathfrak{c} = 2^\omega < 2^{\omega_1}$ implies that any homogeneous compactum of size $\mathfrak{c}$ is first countable. Consequently, under this assumption, the compacta figuring in Theorems 4.1 and 4.2 all turn out to be first countable, hence CT. This fact makes the following natural problem even more interesting.

Problem 4.1. Is it consistent to have a homogeneous compactum that is σ-CT but not CT?

5. σ-CT PRODUCTS

Since in Theorem 4.1 one requires the homogeneity of X^ω instead of X, it is natural to raise the question: What if, similarly, we require the σ-CT property from X^ω rather than from X?

Now, the main aim of this section is to prove that if a product of (T_1) spaces is σ-CT, then all but finitely many of its factors are actually CT. This clearly implies that if X^ω is σ-CT, then X is actually CT.

We shall in fact prove a stronger result for which we need the following lemma. We recall that a non-empty subset $S \subset X$ of a space X is called a *weak P set* if for every countable subset T of its complement $X \setminus S$ we have $\overline{T} \cap S = \emptyset$.

Lemma 5.1. *Consider the product $X \times Y$ where X has a nowhere dense weak P subset S. Then no dense CT subset A of $X \times Y$ intersects the subproduct $S \times Y$.*

Proof. Indeed, then $S \times Y$ is clearly a nowhere dense weak P subset of $X \times Y$. Consequently, $B = A \cap (X \times Y \setminus S \times Y)$ is also dense in $X \times Y$, and no point of $S \times Y$ is in the closure of a countable subset of B. But this clearly implies that no point of $S \times Y$ can be in A because it is CT. $\square$

Theorem 5.2. *Assume that $\{X_i : i < \omega\}$ is a sequence of spaces such that each X_i has a nowhere dense weak P subset S_i. Then their product $X = \prod\{X_i : i < \omega\}$ is not σ-CT.*

Proof. Let $\{A_n : n < \omega\}$ be any countable collection of CT subspaces of X. We shall show that $\{A_n : n < \omega\}$ does not cover X. To do that, we are going to define

a strictly increasing sequence $\{k_n : n < \omega\}$ of natural numbers and, for each $n < \omega$, points $x_i \in X_i$ for $i < k_n$ such that if we put $Y_i = \{x_i\}$ for $i < k_n$ and $Y_i = X_i$ for $i \geq k_n$, then we have

$$Z_n = \prod \{Y_i : i < \omega\} \subset X \setminus \bigcup_{m < n} A_m \,.$$

To start with, we simply put $k_0 = 0$. Next, if k_n and $x_i \in X_i$ for $i < k_n$ have been chosen for some $n < \omega$, then we have to define $k_{n+1} > k_n$ and the points $x_i \in X_i$ for $k_n \leq i < k_{n+1}$. To do that, we distinguish two cases.

Case 1. $A_n \cap Z_n$ is not dense in Z_n. Then we can find $k_{n+1} > k_n$ and a non-empty open set $U_i \subset X_i$ for all $k_n \leq i < k_{n+1}$ such that A_n is disjoint from the subproduct of Z_n that is obtained by shrinking X_i to U_i for $k_n \leq i < k_{n+1}$ and leaving all other factors unchanged. Thus if we pick $x_i \in U_i$ for $k_n \leq i < k_{n+1}$, then the inductive hypothesis will clearly remain valid for $n + 1$.

Case 2. $A_n \cap Z_n$ is dense in Z_n. In this case we put $k_{n+1} = k_n + 1$ and pick $x_{k_n} \in S_{k_n}$. Then Lemma 5.1 implies that $Z_{n+1} \cap A_n = \emptyset$ and, as $Z_{n+1} \subset Z_n$, we are again done.

Having completed the induction it is obvious that the point of the product X whose ith co-ordinate is x_i for all $i < \omega$ does not belong to $\bigcup_{n < \omega} A_n$. Hence, indeed, $\{A_n : n < \omega\}$ does not cover X. $\qquad\square$

Corollary 5.3. *If the product $X = \prod \{X_i : i \in I\}$ is σ-CT, then only finitely many of its factors X_i can have a nowhere dense weak P subset. In particular, all but finitely many of its factors X_i are CT.*

Proof. Indeed, if infinitely many factors X_i would have a nowhere dense weak P subset, then X would contain a subspace homeomorphic to a countably infinite product as in Theorem 5.2, which is clearly impossible.

The second part follows because it is clear that any space Y that is not CT has a subspace Z containing a point $z \in Z$ such that the singleton $\{z\}$ is a nowhere dense weak P subset of Z. $\qquad\square$

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References

[1] A. V. Arhangel'skii and J. van Mill, *Topological homogeneity*, Recent progress in general topology. III, Atlantis Press, Paris, 2014, pp. 1–68, DOI 10.2991/978-94-6239-024-9_1. MR3204728

[2] Raushan Z. Buzyakova, *Cardinalities of some Lindelöf and ω_1-Lindelöf T_1/T_2-spaces*, Topology Appl. **143** (2004), no. 1-3, 209–216, DOI 10.1016/j.topol.2004.02.017. MR2080291

[3] N. A. Carlson and G. J. Ridderbos, *On several cardinality bounds on power homogeneous spaces*, Houston J. Math. **38** (2012), no. 1, 319–332. MR2917289

[4] Ramiro de la Vega, *A new bound on the cardinality of homogeneous compacta*, Topology Appl. **153** (2006), no. 12, 2118–2123, DOI 10.1016/j.topol.2005.08.005. MR2239075

[5] Ryszard Engelking, *General topology,* translated from the Polish by the author, 2nd ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989. MR1039321

[6] István Juhász, *Cardinal functions in topology—ten years later*, 2nd ed., Mathematical Centre Tracts, vol. 123, Mathematisch Centrum, Amsterdam, 1980. MR576927

[7] Jan van Mill, *On the cardinality of power homogeneous compacta*, Topology Appl. **146/147** (2005), 421–428, DOI 10.1016/j.topol.2003.07.020. MR2107161

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