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SMOOTHNESS OF THE STEINER SYMMETRIZATION

YOUJIANG LIN

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ABSTRACT. It is proved that for a convex body with C^2 boundary and positive Gauss curvature, its Steiner symmetral is again a convex body with C^2 boundary and positive Gauss curvature.

1. Introduction

Denote n-dimensional Euclidean space by $\mathbb{R}^n$ and let K be a compact convex subset of $\mathbb{R}^n$. Let e_1 be a unit vector in $\mathbb{R}^n$. The Steiner symmetral K_1 of K with respect to the hyperplane $e_1^\perp$ orthogonal to e_1 is the set generated by translating all chords of K parallel to e_1 so that their centers are on $e_1^\perp$. For over 150 years the Steiner symmetrization has been a fundamental geometric method for studying various isoperimetric problems, in particular, affine isoperimetric problems (see, e.g., [1, 2, 4–9, 11, 13–17]). An important property of the Steiner symmetrization is that iterating Steiner symmetrizations of K through a suitable sequence of directions, the sequence of successive Steiner symmetrals of K, converges to a Euclidean ball in the Hausdorff metric (see, e.g., [3, 10]).

In this paper, we study the smoothness of the Steiner symmetrization process. Kiselman [12] showed that $K_1 \cap e_1^\perp$ need not be of class C^2 even if K is of class C^∞. This implies that the Steiner symmetral of a convex body of class C^∞ need not even be of class C^2. Thus, the smoothness problem is not trivial. We prove the following result.

Theorem 1.1. *If $K \subset \mathbb{R}^n$ is a convex body of class C^2_+, i.e., K has C^2 boundary and positive Gauss curvature, then its Steiner symmetral K_1 is also of class C^2_+.*

Let $K|e_1^\perp$ denote the orthogonal projection of K onto the hyperplane $e_1^\perp$. The following corollary follows immediately from Theorem 1.1, since $K_1 \cap e_1^\perp = K|e_1^\perp$.

Corollary 1.2. *If $K \subset \mathbb{R}^n$ is a convex body of class C^2_+, then $K|e_1^\perp$ is a convex body of class C^2_+ in $e_1^\perp$.*

2. Preliminaries

The setting will be Euclidean n-space $\mathbb{R}^n$. We write $e_1, \ldots, e_n$ for the standard orthonormal basis of $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, we will write $|x| = \sqrt{x \cdot x}$. A compact convex

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set with nonempty interior is called a *convex body*. A convex body is *strictly convex* if its boundary does not contain a line segment of positive length. By int$K$ and ∂K we denote, respectively, the interior and boundary of a convex body K.

A convex body K is said to be of class C^k, for some nonnegative integer k, if its boundary hypersurface is a regular submanifold of $\mathbb{R}^n$, in the sense of differential geometry, that is, k-times continuously differentiable. In this paper, smoothness of convex bodies is understood as smoothness of hypersurfaces in the sense of differential geometry. A convex body is of class C^k_+ if it is of class C^k and the Gauss curvature at each point of ∂K is positive.

Let K be a convex body in $\mathbb{R}^n$. For $i = 1, 2, \ldots, n$, the *overgraph and undergraph* functions are defined by

$$\overline{\ell}_i(x) := \max\{t \in \mathbb{R} : x + te_i \in K\}, \quad x \in K|e_i^\perp, \tag{2.1}$$

$$\underline{\ell}_i(x) := \min\{t \in \mathbb{R} : x + te_i \in K\}, \quad x \in K|e_i^\perp, \tag{2.2}$$

where $K|e_i^\perp$ is the orthogonal projection of K onto the hyperplane $e_i^\perp$. Note that $-\overline{\ell}_i$ and $\underline{\ell}_i$ are convex functions.

By (2.1) and (2.2), for any $x \in K|e_i^\perp$, it is easily seen that $(x, \overline{\ell}_i(x)), (x, \underline{\ell}_i(x)) \in \partial K$. Moreover, for $x \in \mathrm{int}(K|e_i^\perp)$, the Gauss curvature H_{n-1} of K at the boundary point $(x, \underline{\ell}_i(x))$ satisfies (see [11, p. 210])

$$H_{n-1}(x, \underline{\ell}_i(x)) = \frac{|\nabla^2 \underline{\ell}_i(x)|}{(1 + |\nabla \underline{\ell}_i(x)|^2)^{\frac{n+1}{2}}}, \tag{2.3}$$

where $|\nabla^2 \underline{\ell}_i|$ denotes the determinant of the Hessian matrix of $\underline{\ell}_i$ and $|\nabla \underline{\ell}_i|$ denotes the Euclidean norm of the gradient of $\underline{\ell}_i$. If $\underline{\ell}_i$ is twice differentiable, then $\underline{\ell}_i$ has positive semi-definite Hessian matrix on $\mathrm{int}(K|e_i^\perp)$ (see Theorem 1.5.13 in [16]). Therefore, by (2.3), if K has C^2 boundary and $x \in \mathrm{int}(K|e_i^\perp)$, then ∂K has positive curvature at $(x, \underline{\ell}_i(x))$ if and only if $\underline{\ell}_i(x)$ has positive definite Hessian matrix.

The *Steiner symmetral* of K with respect to the hyperplane $e_1^\perp$ can be expressed as follows:

$$K_1 := \{x + te_1 : \ x \in K|e_1^\perp, \ |t| \le \frac{\overline{\ell}_1(x) - \underline{\ell}_1(x)}{2}\}. \tag{2.4}$$

By the above definition, the overgraph and undergraph functions of K_1 with respect to e_1, denoted by $\overline{\varrho}_1$ and $\underline{\varrho}_1$, satisfy the following equality:

$$\overline{\varrho}_1(x) = -\underline{\varrho}_1(x) = \frac{\overline{\ell}_1(x) - \underline{\ell}_1(x)}{2}, \quad x \in K|e_1^\perp. \tag{2.5}$$

It is easily checked that K_1 is a convex body symmetric with respect to $e_1^\perp$. Moreover, if K is strictly convex, then $\underline{\ell}_1(x)$ and $-\overline{\ell}_1(x)$ are strictly convex on $x \in K|e_1^\perp$. By (2.5), $-\overline{\varrho}_1(x)$ and $\underline{\varrho}_1(x)$ are also strictly convex on $x \in K|e_1^\perp$. Moreover, it is easily checked that $-\overline{\varrho}_1(x) = \underline{\varrho}_1(x)$ for $x \in \partial(K|e_1^\perp)$. Therefore, K_1 is also strictly convex.

It follows that if K is a convex body of class C^2_+, then K is strictly convex. Moreover, $\underline{\ell}_1(x)$ and $-\overline{\ell}_1(x)$ are C^2 and have positive definite Hessian matrices for $x \in \mathrm{int}(K|e_1^\perp)$. Thus by (2.5), $\underline{\varrho}_1$ and $-\overline{\varrho}_1$ are also C^2 smooth and have positive definite Hessian matrices on $\mathrm{int}(K|e_1^\perp)$, which implies that ∂K_1 is C^2 and has positive curvature at every point $x \in \partial K_1 \backslash e_1^\perp$. Thus we only need to prove the C^2 smoothness and positive curvature for $x \in \partial K_1 \cap e_1^\perp$.

For a fixed $x_o \in \partial K_1 \cap e_1^\perp$, choose a coordinate system so that x_o is the origin, $x_n = 0$ is a support hyperplane of K_1 at x_o and e_n points to the interior of K_1. For simplicity of notation, we let $\varrho_n(x)$, $x \in K_1|e_n^\perp$, denote the undergraph function of K_1 with respect to e_n.

In order to prove that ∂K_1 is C^2 and has positive curvature at x_o, we need to prove that ϱ_n has the following properties:

C^1 **smoothness**: ϱ_n is differentiable at the origin and $\frac{\partial \varrho_n}{\partial x_i}(0) = 0$, $i = 1, 2, \ldots, n-1$;

C^2 **smoothness**: The second partial derivatives $\frac{\partial^2 \varrho_n}{\partial x_i \partial x_j}(x)$, $1 \leq i, j \leq n-1$, exist on a neighborhood of the origin and are continuous at the origin;

Positive Hessian: ϱ_n has positive definite Hessian matrix at the origin.

Let h be a sufficiently small positive number such that

(2.6) $h < \min\{\varrho_n(x) : x \in \partial(K_1|e_n^\perp)\}$ and $h < \min\{\underline{\ell}_n(x) : x \in \partial(K|e_n^\perp)\}$.

For $h > 0$ as in (2.6), let

(2.7)
$$K_{1,h} = K_1 \cap \{(x, x_n) \in \mathbb{R}^n : x_n < h\} \text{ and } K_h = K \cap \{(x, x_n) \in \mathbb{R}^n : x_n < h\}.$$

Let D_1 be the orthogonal projection of $K_{1,h}$ onto $e_n^\perp$. Let D be the orthogonal projection of K_h onto $e_n^\perp$. It is easily checked that for $x \in \partial D_1$ and $y \in \partial D$, $\varrho_n(x) = h = \underline{\ell}_n(y)$. Moreover, D_1 is the Steiner symmetral of D with respect to $e_1^\perp$.

For $x \in D_1$, let $x = (r, z)$, where $r = x_1$ and $z = (x_2, \ldots, x_{n-1})$. Let $r > 0$ and

$$x_n := \varrho_n(r, z). \tag{2.8}$$

By (2.8) and the definition of ϱ_n, we have $(r, z, x_n) \in \partial K_1$. Thus, by the strict convexity of K_1 and the definition of $\overline{\varrho}_1$, we have

$$r = \overline{\varrho}_1(z, x_n). \tag{2.9}$$

Let

$$s := s(r, z) = \underline{\ell}_1(z, x_n) = \underline{\ell}_1(z, \varrho_n(r, z)) \tag{2.10}$$

and

$$t := t(r, z) = \overline{\ell}_1(z, x_n) = \overline{\ell}_1(z, \varrho_n(r, z)). \tag{2.11}$$

By (2.9), (2.5), (2.10) and (2.11), we have

$$r = \overline{\varrho}_1(z, x_n) = \frac{\overline{\ell}_1(z, x_n) - \underline{\ell}_1(z, x_n)}{2} = \frac{t - s}{2}. \tag{2.12}$$

By (2.10), (2.11) and the definitions of $\overline{\ell}_1$ and $\underline{\ell}_1$, we have $(s, z, x_n), (t, z, x_n) \in \partial K$. By $(r, z) \in D_1$, $(s, z), (t, z) \in D$ and (2.8),

$$\underline{\ell}_n(s, z) = \underline{\ell}_n(t, z) = x_n = \varrho_n(r, z). \tag{2.13}$$

If $r = 0$, then $x_n = \varrho_n(0, z)$ and $(0, z, x_n) \in \partial K_1$, so $0 = \overline{\varrho}_1(z, x_n)$. Let

$$s_1 := s_1(z) = \overline{\ell}_1(z, x_n) = \overline{\ell}_1(z, \varrho_n(0, z)). \tag{2.14}$$

By (2.14), we have

$$\underline{\ell}_n(s_1, z) = \varrho_n(0, z). \tag{2.15}$$

In fact, for fixed z, s_1 is the minimum of $\underline{\ell}_n(x_1,z)$ over x_1, so

$$\frac{\partial \underline{\ell}_n}{\partial x_1}(s_1,z)=0. \tag{2.16}$$

Moreover, for fixed z and s,t,s_1 as in (2.10), (2.11) and (2.14), we have $s<s_1<t$ and $s,t\to s_1$ when $r\to 0$.

For fixed $z\in D_1\cap e_1^{\perp}$, let $(-\delta,\delta)=\{x_1\in\mathbb{R}:(x_1,z)\in D_1\}$ and $(\delta_1,\delta_2)=\{x_1\in\mathbb{R}:(x_1,z)\in D\}$. Then $\delta_2-\delta_1=2\delta$. Since K_1 is a strictly convex body and symmetric with respect to $e_1^{\perp}$, $\varrho_n(x_1,z)$ is an even and strictly convex function for $x_1\in(-\delta,\delta)$. Since K is a strictly convex body, $\underline{\ell}_n(x_1,z)$ is a strictly convex function for $x_1\in(\delta_1,\delta_2)$.

Moreover, for fixed z and s_1 as in (2.14), the one-dimensional function $x_n=\underline{\ell}_n(x_1,z)$ for $x_1\in[s_1,\delta_2)$ and the one-dimensional function $x_1=\overline{\ell}_1(z,x_n)$ for $x_n\in[\underline{\ell}_n(s_1,z),h)$ are inverse functions; $x_n=\underline{\ell}_n(x_1,z)$ for $x_1\in(\delta_1,s_1]$ and $x_1=\underline{\ell}_1(z,x_n)$ for $x_n\in[\underline{\ell}_n(s_1,z),h)$ are inverse functions; $x_n=\varrho_n(x_1,z)$ for $x_1\in[0,\delta)$ and $x_1=\overline{\varrho}_1(z,x_n)$ for $x_n\in[\varrho_n(0,z),h)$ are inverse functions. Since inverse functions have reciprocal slopes at reflected points, by (2.13) we have that

$$\frac{\partial \varrho_n}{\partial x_1}(r,z)=\left(\frac{\partial \overline{\varrho}_1}{\partial x_n}(z,x_n)\right)^{-1}, \tag{2.17}$$

$$\frac{\partial \underline{\ell}_n}{\partial x_1}(s,z)=\left(\frac{\partial \underline{\ell}_1}{\partial x_n}(z,x_n)\right)^{-1}, \tag{2.18}$$

and

$$\frac{\partial \underline{\ell}_n}{\partial x_1}(t,z)=\left(\frac{\partial \overline{\ell}_1}{\partial x_n}(z,x_n)\right)^{-1}. \tag{2.19}$$

For fixed $z\in D_1\cap e_1^{\perp}$ and s,t as in (2.10) and (2.11), for simplicity of notation, we let

$$\alpha:=\alpha(r,z)=\frac{\partial \underline{\ell}_n}{\partial x_1}(s,z),\quad \beta:=\beta(r,z)=\frac{\partial \underline{\ell}_n}{\partial x_1}(t,z). \tag{2.20}$$

By (2.17), (2.18), (2.19), (2.12) and (2.20), for $r>0$ we have

$$\frac{\partial \varrho_n}{\partial x_1}(r,z)=\frac{2\alpha\beta}{\alpha-\beta}. \tag{2.21}$$

3. Proof of the main result

Lemma 3.1. *ϱ_n is differentiable at the origin and $\frac{\partial \varrho_n}{\partial x_i}(0)=0$ for $i=1,2,\dots,n-1$.*

Proof. For $r>0$, by $\varrho_n(0)=0$, (2.13), (2.15) and (2.16), we have

$$\begin{aligned}\frac{\partial_+ \varrho_n}{\partial x_1}(0) &= \lim_{r\to 0+}\frac{\varrho_n(r,0)-\varrho_n(0,0)}{r}\\ &= \lim_{r\to 0+}\left(\frac{t-s_1}{2r}\cdot\frac{\underline{\ell}_n(t,0)-\underline{\ell}_n(s_1,0)}{t-s_1}+\frac{s-s_1}{2r}\cdot\frac{\underline{\ell}_n(s,0)-\underline{\ell}_n(s_1,0)}{s-s_1}\right)\\ &= 0. \end{aligned}\tag{3.1}$$

Because $\varrho_n(x_1,0)$ is an even function with respect to x_1, the left derivative of ϱ_n with respect to x_1 at the origin is also zero. Thus $\frac{\partial \varrho_n}{\partial x_1}(0)=0$.

If H is a support hyperplane of K_1 at the origin, by $\frac{\partial \varrho_n}{\partial x_1}(0) = 0$, then H is parallel to e_1. Thus H is also a support hyperplane of K at the point $(s_1, 0)$, where s_1 as in (2.14). Since K is of class C^2_+ and hence of class C^1, K has a unique outer unit normal vector at the boundary point $(s_1, 0)$. Therefore, K_1 has a unique outer unit normal vector at the origin, which implies that ϱ_n is differentiable at the origin (see Lemma 1.5.14 and Theorem 1.5.15 of [16]). Because ϱ_n is a convex function and attains its minimum at the origin, $\frac{\partial \varrho_n}{\partial x_i}(0) = 0$ for $i = 1, 2, \ldots, n-1$. □

By Lemma 3.1 and the arbitrary choice of $x_o \in \partial K_1 \cap e_1^{\perp}$, K_1 is of class C^1.

Lemma 3.2. *For fixed $z \in D_1 \cap e_1^{\perp}$ and α and β as in (2.20), we have*

$$\lim_{r \to 0^+} \frac{\alpha}{\beta} = -1. \tag{3.2}$$

Proof. By (2.16) and $\underline{\ell}_n \in C^2$, for s_1 as in (2.14), we have

$$\underline{\ell}_n(t, z) = \underline{\ell}_n(s_1, z) + 0(t - s_1) + \frac{1}{2}\frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1, z)(t - s_1)^2 + o((t - s_1)^2), \tag{3.3}$$

$$\underline{\ell}_n(s, z) = \underline{\ell}_n(s_1, z) + 0(s - s_1) + \frac{1}{2}\frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1, z)(s - s_1)^2 + o((s - s_1)^2). \tag{3.4}$$

Let $\frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1, z) = c$. Since $\underline{\ell}_n \in C^2$ with positive definite Hessian matrix, we have $c > 0$. By (3.3), (3.4) and $\underline{\ell}_n(t, z) = \underline{\ell}_n(s, z)$, we have

$$\frac{1}{2}c(t - s_1)^2 + o((t - s_1)^2) = \frac{1}{2}c(s - s_1)^2 + o((s - s_1)^2). \tag{3.5}$$

By (3.5) and $s, t \to s_1$ when $r \to 0^+$, we have

$$\lim_{r \to 0^+} \frac{(t - s_1)^2}{(s - s_1)^2} = 1. \tag{3.6}$$

By (2.20), (3.3), (3.4), (3.6) and $s < s_1 < t$, we have

$$\lim_{r \to 0^+} \frac{\alpha}{\beta} = \lim_{r \to 0^+} \frac{c(s - s_1) + o(|s - s_1|)}{c(t - s_1) + o(|t - s_1|)} = -1. \tag{3.7}$$

□

Lemma 3.3. *For fixed $z \in D_1 \cap e_1^{\perp}$, for s, t and s_1 as in (2.10), (2.11) and (2.14), and for $i = 2, \ldots, n-1$, we have*

$$\lim_{r \to 0^+} \frac{\frac{\partial \underline{\ell}_n}{\partial x_i}(s, z) - \frac{\partial \underline{\ell}_n}{\partial x_i}(t, z)}{\frac{\partial \underline{\ell}_n}{\partial x_1}(s, z)} = \frac{2\frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_1}(s_1, z)}{\frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1, z)}, \tag{3.8}$$

where $\frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1, z) > 0$.

Proof. First,

$$\frac{\frac{\partial \underline{\ell}_n}{\partial x_i}(s, z) - \frac{\partial \underline{\ell}_n}{\partial x_i}(t, z)}{\frac{\partial \underline{\ell}_n}{\partial x_1}(s, z)} = \frac{\frac{\partial \underline{\ell}_n}{\partial x_i}(s, z) - \frac{\partial \underline{\ell}_n}{\partial x_i}(s_1, z)}{\frac{\partial \underline{\ell}_n}{\partial x_1}(s, z)} - \frac{\frac{\partial \underline{\ell}_n}{\partial x_1}(t, z)}{\frac{\partial \underline{\ell}_n}{\partial x_1}(s, z)} \cdot \frac{\frac{\partial \underline{\ell}_n}{\partial x_i}(t, z) - \frac{\partial \underline{\ell}_n}{\partial x_i}(s_1, z)}{\frac{\partial \underline{\ell}_n}{\partial x_1}(t, z)}.$$

Since $\underline{\ell}_n \in C^2$ with positive definite Hessian matrix, $\frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1, z) > 0$. By $\frac{\partial \underline{\ell}_n}{\partial x_1}(s_1, z)$ $= 0$ and $s, t \to s_1$ when $r \to 0^+$, we have

$$\lim_{r\to 0^+} \frac{\frac{\partial \underline{\ell}_n}{\partial x_i}(s,z) - \frac{\partial \underline{\ell}_n}{\partial x_i}(s_1,z)}{\frac{\partial \underline{\ell}_n}{\partial x_1}(s,z)} = \lim_{s\to s_1} \frac{\left(\frac{\partial \underline{\ell}_n}{\partial x_i}(s,z) - \frac{\partial \underline{\ell}_n}{\partial x_i}(s_1,z)\right)/(s-s_1)}{\left(\frac{\partial \underline{\ell}_n}{\partial x_1}(s,z) - \frac{\partial \underline{\ell}_n}{\partial x_1}(s_1,z)\right)/(s-s_1)} = \frac{\frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_1}(s_1,z)}{\frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1,z)}$$

and

$$\lim_{r\to 0^+} \frac{\frac{\partial \underline{\ell}_n}{\partial x_i}(t,z) - \frac{\partial \underline{\ell}_n}{\partial x_i}(s_1,z)}{\frac{\partial \underline{\ell}_n}{\partial x_1}(t,z)} = \lim_{t\to s_1} \frac{\left(\frac{\partial \underline{\ell}_n}{\partial x_i}(t,z) - \frac{\partial \underline{\ell}_n}{\partial x_i}(s_1,z)\right)/(t-s_1)}{\left(\frac{\partial \underline{\ell}_n}{\partial x_1}(t,z) - \frac{\partial \underline{\ell}_n}{\partial x_1}(s_1,z)\right)/(t-s_1)} = \frac{\frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_1}(s_1,z)}{\frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1,z)}.$$

By the above three equalities, (2.20) and Lemma 3.2, we have

$$\begin{aligned}\lim_{r\to 0^+} \frac{\frac{\partial \underline{\ell}_n}{\partial x_i}(s,z) - \frac{\partial \underline{\ell}_n}{\partial x_i}(t,z)}{\frac{\partial \underline{\ell}_n}{\partial x_1}(s,z)} &= \lim_{r\to 0^+} \frac{\frac{\partial \underline{\ell}_n}{\partial x_i}(s,z) - \frac{\partial \underline{\ell}_n}{\partial x_i}(s_1,z)}{\frac{\partial \underline{\ell}_n}{\partial x_1}(s,z)} \\ &\quad - \lim_{r\to 0^+} \frac{\beta}{\alpha} \cdot \lim_{r\to 0^+} \frac{\frac{\partial \underline{\ell}_n}{\partial x_i}(t,z) - \frac{\partial \underline{\ell}_n}{\partial x_i}(s_1,z)}{\frac{\partial \underline{\ell}_n}{\partial x_1}(t,z)} \\ &= \frac{2\frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_1}(s_1,z)}{\frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1,z)}.\end{aligned}$$

□

The next three lemmas give the explicit values of the second order partial derivatives of ϱ_n for $x \in D_1 \backslash (D_1 \cap e_1^\perp)$.

Lemma 3.4. *For fixed $z \in D_1 \cap e_1^\perp$, $r > 0$ and s, t, α, β as in* (2.10), (2.11) *and* (2.20), *we have*

$$\frac{\partial^2 \varrho_n}{\partial x_1^2}(r,z) = \frac{4\alpha^3}{(\alpha-\beta)^3} \cdot \frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(t,z) - \frac{4\beta^3}{(\alpha-\beta)^3} \cdot \frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s,z). \tag{3.9}$$

Proof. By $t = \overline{\ell}_1(z, \varrho_n(r,z))$ and (2.19), we have

$$\frac{\partial t}{\partial r} = \frac{\partial \overline{\ell}_1}{\partial x_n}(z, \varrho_n(r,z)) \cdot \frac{\partial \varrho_n}{\partial x_1}(r,z) = \left(\frac{\partial \underline{\ell}_n}{\partial x_1}(t,z)\right)^{-1} \cdot \frac{\partial \varrho_n}{\partial x_1}(r,z). \tag{3.10}$$

By (3.10), (2.21) and (2.20), we have

$$\frac{\partial t}{\partial r} = \frac{2\alpha}{\alpha-\beta}. \tag{3.11}$$

Similarly, by $s = \underline{\ell}_1(z, \varrho_n(r,z))$, (2.18), (2.21) and (2.20), we have

$$\frac{\partial s}{\partial r} = \frac{2\beta}{\alpha-\beta}. \tag{3.12}$$

By partial differentiation of (2.21) with respect to r, (2.20), (3.11) and (3.12), we have

$$\begin{aligned} \frac{\partial^2 \varrho_n}{\partial x_1^2}(r,z) &= 2\frac{\left(\frac{\partial\alpha}{\partial r}\beta + \alpha\frac{\partial\beta}{\partial r}\right)(\alpha-\beta) - \alpha\beta\left(\frac{\partial\alpha}{\partial r} - \frac{\partial\beta}{\partial r}\right)}{(\alpha-\beta)^2} \\ &= 2\frac{\alpha^2\cdot\frac{\partial^2\underline{\ell}_n}{\partial x_1^2}(t,z)\cdot\frac{\partial t}{\partial r} - \beta^2\cdot\frac{\partial^2\underline{\ell}_n}{\partial x_1^2}(s,z)\cdot\frac{\partial s}{\partial r}}{(\alpha-\beta)^2} \\ &= \frac{4\alpha^3}{(\alpha-\beta)^3}\cdot\frac{\partial^2\underline{\ell}_n}{\partial x_1^2}(t,z) - \frac{4\beta^3}{(\alpha-\beta)^3}\cdot\frac{\partial^2\underline{\ell}_n}{\partial x_1^2}(s,z). \end{aligned} \tag{3.13}$$

□

Lemma 3.5. *For fixed $z \in D_1 \cap e_1^\perp$, for $r > 0$ and s, t, α, β as in* (2.10), (2.11) *and* (2.20), *and for $i = 2, 3, \dots, n-1$, we have*

$$\begin{aligned} \frac{\partial^2\varrho_n}{\partial x_1 \partial x_i}(r,z) &= 2\frac{\left(\frac{\partial\underline{\ell}_n}{\partial x_i}(t,z) - \frac{\partial\underline{\ell}_n}{\partial x_i}(s,z)\right)\cdot\left(\alpha^2\frac{\partial^2\underline{\ell}_n}{\partial x_1^2}(t,z) - \beta^2\frac{\partial^2\underline{\ell}_n}{\partial x_1^2}(s,z)\right)}{(\alpha-\beta)^3} \\ &\quad + 2\frac{\alpha^2\frac{\partial^2\underline{\ell}_n}{\partial x_1\partial x_i}(t,z) - \beta^2\frac{\partial^2\underline{\ell}_n}{\partial x_1\partial x_i}(s,z)}{(\alpha-\beta)^2}. \end{aligned} \tag{3.14}$$

Proof. By (2.12),

$$r = \frac{1}{2}(t-s) = \frac{1}{2}\overline{\ell}_1(z,x_n) - \frac{1}{2}\underline{\ell}_1(z,x_n), \tag{3.15}$$

where $x_n = \varrho_n(r,z)$. Partial differentiation of (3.15) with respect to x_i, $i = 2, \dots, n-1$, at (r,z) gives

$$\begin{aligned} 0 = \frac{1}{2}&\left(\frac{\partial\overline{\ell}_1}{\partial x_i}(z,x_n) + \frac{\partial\overline{\ell}_1}{\partial x_n}(z,x_n)\cdot\frac{\partial\varrho_n}{\partial x_i}(r,z)\right) \\ &- \frac{1}{2}\left(\frac{\partial\underline{\ell}_1}{\partial x_i}(z,x_n) + \frac{\partial\underline{\ell}_1}{\partial x_n}(z,x_n)\cdot\frac{\partial\varrho_n}{\partial x_i}(r,z)\right). \end{aligned}$$

By (2.18), (2.19), (2.20) and the above equality, we have

$$\frac{\partial\varrho_n}{\partial x_i}(r,z) = \frac{\alpha\beta}{\alpha-\beta}\left(\frac{\partial\underline{\ell}_1}{\partial x_i}(z,x_n) - \frac{\partial\overline{\ell}_1}{\partial x_i}(z,x_n)\right). \tag{3.16}$$

By the chain rule, $x_n = \underline{\ell}_n(s,z) = \underline{\ell}_n(t,z)$, (2.14), (2.16), (2.18), (2.19) and (2.20),

$$\frac{\partial\overline{\ell}_1}{\partial x_i}(z,x_n) = -\frac{\partial\overline{\ell}_1}{\partial x_n}(z,x_n)\cdot\frac{\partial\underline{\ell}_n}{\partial x_i}(t,z) = -\frac{\partial\underline{\ell}_n}{\partial x_i}(t,z)\Big/\frac{\partial\underline{\ell}_n}{\partial x_1}(t,z) = -\frac{1}{\beta}\frac{\partial\underline{\ell}_n}{\partial x_i}(t,z) \tag{3.17}$$

and

$$\frac{\partial\underline{\ell}_1}{\partial x_i}(z,x_n) = -\frac{\partial\underline{\ell}_1}{\partial x_n}(z,x_n)\cdot\frac{\partial\underline{\ell}_n}{\partial x_i}(s,z) = -\frac{\partial\underline{\ell}_n}{\partial x_i}(s,z)\Big/\frac{\partial\underline{\ell}_n}{\partial x_1}(s,z) = -\frac{1}{\alpha}\frac{\partial\underline{\ell}_n}{\partial x_i}(s,z). \tag{3.18}$$

Putting (3.17) and (3.18) into (3.16), we obtain

$$\frac{\partial\varrho_n}{\partial x_i}(r,z) = \frac{\alpha}{\alpha-\beta}\cdot\frac{\partial\underline{\ell}_n}{\partial x_i}(t,z) - \frac{\beta}{\alpha-\beta}\cdot\frac{\partial\underline{\ell}_n}{\partial x_i}(s,z). \tag{3.19}$$

By $t=\overline{\ell}_1(z,x_n)$, $x_n=\varrho_n(r,z)$, (2.19), (2.20), (3.17) and (3.19), we have

$$\frac{\partial t}{\partial x_i}(z,x_n)=\frac{\partial\overline{\ell}_1}{\partial x_i}(z,x_n)+\frac{\partial\overline{\ell}_1}{\partial x_n}(z,x_n)\cdot\frac{\partial\varrho_n}{\partial x_i}(r,z)=\frac{\frac{\partial\underline{\ell}_n}{\partial x_i}(t,z)-\frac{\partial\underline{\ell}_n}{\partial x_i}(s,z)}{\alpha-\beta}. \tag{3.20}$$

Similarly, we have

$$\frac{\partial s}{\partial x_i}(z,x_n)=\frac{\frac{\partial\underline{\ell}_n}{\partial x_i}(t,z)-\frac{\partial\underline{\ell}_n}{\partial x_i}(s,z)}{\alpha-\beta}. \tag{3.21}$$

Moreover,

$$\frac{\partial\alpha}{\partial x_i}=\frac{\partial(\frac{\partial\underline{\ell}_n}{\partial x_1}(s,z))}{\partial x_i}=\frac{\partial^2\underline{\ell}_n}{\partial x_1^2}(s,z)\cdot\frac{\partial s}{\partial x_i}(z,x_n)+\frac{\partial^2\underline{\ell}_n}{\partial x_1\partial x_i}(s,z) \tag{3.22}$$

and

$$\frac{\partial\beta}{\partial x_i}=\frac{\partial(\frac{\partial\underline{\ell}_n}{\partial x_1}(t,z))}{\partial x_i}=\frac{\partial^2\underline{\ell}_n}{\partial x_1^2}(t,z)\cdot\frac{\partial t}{\partial x_i}(z,x_n)+\frac{\partial^2\underline{\ell}_n}{\partial x_1\partial x_i}(t,z). \tag{3.23}$$

By (2.20), (3.20), (3.21), (3.22) and (3.23), partial differentiation of (2.21) with respect to x_i at (r,z), we have

$$\begin{aligned}\frac{\partial^2\varrho_n}{\partial x_1\partial x_i}(r,z) &= 2\frac{\left(\frac{\partial\alpha}{\partial x_i}\beta+\alpha\frac{\partial\beta}{\partial x_i}\right)(\alpha-\beta)-\alpha\beta\left(\frac{\partial\alpha}{\partial x_i}-\frac{\partial\beta}{\partial x_i}\right)}{(\alpha-\beta)^2}\\ &= 2\frac{(\frac{\partial\underline{\ell}_n}{\partial x_i}(t,z)-\frac{\partial\underline{\ell}_n}{\partial x_i}(s,z))\cdot(\alpha^2\frac{\partial^2\underline{\ell}_n}{\partial x_1^2}(t,z)-\beta^2\frac{\partial^2\underline{\ell}_n}{\partial x_1^2}(s,z))}{(\alpha-\beta)^3}\\ &\quad+2\frac{\alpha^2\frac{\partial^2\underline{\ell}_n}{\partial x_1\partial x_i}(t,z)-\beta^2\frac{\partial^2\underline{\ell}_n}{\partial x_1\partial x_i}(s,z)}{(\alpha-\beta)^2}.\end{aligned}$$

□

Lemma 3.6. *For fixed $z\in D_1\cap e_1^\perp$, for $r>0$ and s, t, α, β as in* (2.10), (2.11) *and* (2.20)*, and for $i,j=2,3,\dots,n-1$, we have*

(3.24)

$$\begin{aligned}&\frac{\partial^2\varrho_n}{\partial x_i\partial x_j}(r,z)\\ &=\frac{(\frac{\partial\underline{\ell}_n}{\partial x_i}(t,z)-\frac{\partial\underline{\ell}_n}{\partial x_i}(s,z))\cdot(\frac{\partial\underline{\ell}_n}{\partial x_j}(t,z)-\frac{\partial\underline{\ell}_n}{\partial x_j}(s,z))\cdot(\alpha\frac{\partial^2\underline{\ell}_n}{\partial x_1^2}(t,z)-\beta\frac{\partial^2\underline{\ell}_n}{\partial x_1^2}(s,z))}{(\alpha-\beta)^3}\\ &\quad+\frac{(\frac{\partial\underline{\ell}_n}{\partial x_j}(t,z)-\frac{\partial\underline{\ell}_n}{\partial x_j}(s,z))\cdot(\alpha\frac{\partial^2\underline{\ell}_n}{\partial x_1\partial x_i}(t,z)-\beta\frac{\partial^2\underline{\ell}_n}{\partial x_1\partial x_i}(s,z))}{(\alpha-\beta)^2}\\ &\quad+\frac{(\frac{\partial\underline{\ell}_n}{\partial x_i}(t,z)-\frac{\partial\underline{\ell}_n}{\partial x_i}(s,z))\cdot(\alpha\frac{\partial^2\underline{\ell}_n}{\partial x_1\partial x_j}(t,z)-\beta\frac{\partial^2\underline{\ell}_n}{\partial x_1\partial x_j}(s,z))}{(\alpha-\beta)^2}\\ &\quad+\frac{\alpha\frac{\partial^2\underline{\ell}_n}{\partial x_i\partial x_j}(t,z)-\beta\frac{\partial^2\underline{\ell}_n}{\partial x_i\partial x_j}(s,z)}{\alpha-\beta}.\end{aligned}$$

Proof. First, we have

$$\frac{\partial(\frac{\partial \underline{\ell}_n}{\partial x_i}(t,z))}{\partial x_j}(r,z) = \frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_1}(t,z)\frac{\partial t}{\partial x_j}(z,x_n) + \frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_j}(t,z) \tag{3.25}$$

and

$$\frac{\partial(\frac{\partial \underline{\ell}_n}{\partial x_i}(s,z))}{\partial x_j}(r,z) = \frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_1}(s,z)\frac{\partial s}{\partial x_j}(z,x_n) + \frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_j}(s,z). \tag{3.26}$$

By (3.25) and (3.26), partial differentiation of (3.19) with respect to x_j at (r,z) gives that

$$\begin{aligned}\frac{\partial^2 \varrho_n}{\partial x_i \partial x_j}(r,z) &= \frac{\partial(\frac{\alpha}{\alpha-\beta})}{\partial x_j}\cdot\frac{\partial \underline{\ell}_n}{\partial x_i}(t,z) + \frac{\alpha}{\alpha-\beta}\left(\frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_j}(t,z) + \frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_1}(t,z)\frac{\partial t}{\partial x_j}\right)\\ &\quad - \frac{\partial(\frac{\beta}{\alpha-\beta})}{\partial x_j}\cdot\frac{\partial \underline{\ell}_n}{\partial x_i}(s,z) - \frac{\beta}{\alpha-\beta}\left(\frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_j}(s,z) + \frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_1}(s,z)\frac{\partial s}{\partial x_j}\right).\end{aligned}$$

By (3.20), (3.21), (3.22), (3.23), the right side of the above equality equals the right side of (3.24). □

The following lemma gives the explicit values of the second order partial derivatives of ϱ_n for $x \in D_1 \cap e_1^\perp$.

Lemma 3.7. *For fixed $z \in D_1 \cap e_1^\perp$, for s_1 as in (2.14) and $i,j = 2,\ldots,n-1$, we have*

$$\frac{\partial^2 \varrho_n}{\partial x_1^2}(0,z) = \frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1,z), \tag{3.27}$$

$$\frac{\partial^2 \varrho_n}{\partial x_1 \partial x_i}(0,z) = 0 = \frac{\partial^2 \varrho_n}{\partial x_i \partial x_1}(0,z), \tag{3.28}$$

and

$$\frac{\partial^2 \varrho_n}{\partial x_i \partial x_j}(0,z) = \frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_j}(s_1,z) - \frac{\frac{\partial^2 \underline{\ell}_n}{\partial x_1 \partial x_i}(s_1,z)\cdot\frac{\partial^2 \underline{\ell}_n}{\partial x_1 \partial x_j}(s_1,z)}{\frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1,z)}. \tag{3.29}$$

Proof. Since $\frac{\partial \varrho_n}{\partial x_1}(0,z) = 0$ and $\frac{\partial \varrho_n}{\partial x_1}(r,z)$ is an odd function with respect to r, by (2.21) we have

$$\frac{\partial^2 \varrho_n}{\partial x_1^2}(0,z) = \lim_{r\to 0^+}\frac{\frac{\partial \varrho_n}{\partial x_1}(r,z) - \frac{\partial \varrho_n}{\partial x_1}(0,z)}{r} = \lim_{r\to 0}\frac{2\frac{\partial \underline{\ell}_n}{\partial x_1}(s,z)\frac{\partial \underline{\ell}_n}{\partial x_1}(t,z)/r^2}{\frac{\partial \underline{\ell}_n}{\partial x_1}(s,z)/r - \frac{\partial \underline{\ell}_n}{\partial x_1}(t,z)/r}. \tag{3.30}$$

By (3.6) and $2r = (t-s_1)+(s_1-s)$, we have

$$\lim_{r\to 0^+}\frac{t-s_1}{r} = \lim_{r\to 0^+}\frac{s_1-s}{r} = 1. \tag{3.31}$$

By (2.16) and (3.31), we have

$$\lim_{r\to 0^+}\frac{\frac{\partial \underline{\ell}_n}{\partial x_1}(t,z)}{r} = \lim_{r\to 0^+}\frac{\frac{\partial \underline{\ell}_n}{\partial x_1}(t,z) - \frac{\partial \underline{\ell}_n}{\partial x_1}(s_1,z)}{t-s_1} = \frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1,z). \tag{3.32}$$

Similarly, we have

$$\lim_{r\to 0^+}\frac{\frac{\partial \underline{\ell}_n}{\partial x_1}(s,z)}{r} = -\frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1,z). \tag{3.33}$$

By (3.30), (3.32) and (3.33), we have

$$\frac{\partial^2 \varrho_n}{\partial x_1^2}(0,z) = \frac{-2\left(\frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1,z)\right)^2}{-2\frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1,z)} = \frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1,z). \tag{3.34}$$

Since $\frac{\partial \varrho_n}{\partial x_1}(0,z) = 0$ for any $z \in D_1 \cap e_1^{\perp}$, $\frac{\partial^2 \varrho_n}{\partial x_1 \partial x_i}(0,z) = 0$ is established.

Since ϱ_n and $\underline{\ell}_n$ are C^1, by (3.19) and (3.2) we have

$$\frac{\partial \varrho_n}{\partial x_i}(0,z) = \lim_{r\to 0^+} \frac{\partial \varrho_n}{\partial x_i}(r,z) = \frac{\partial \underline{\ell}_n}{\partial x_i}(s_1,z). \tag{3.35}$$

By (3.19), (3.35), (3.2), (3.31) and $\underline{\ell}_n \in C^2$, we have

$$\begin{aligned}
&\lim_{r\to 0^+} \frac{\frac{\partial \varrho_n}{\partial x_i}(r,z) - \frac{\partial \varrho_n}{\partial x_i}(0,z)}{r} \\
= &\lim_{r\to 0^+} \frac{\frac{\alpha}{\alpha-\beta}\left(\frac{\partial \underline{\ell}_n}{\partial x_i}(t,z) - \frac{\partial \underline{\ell}_n}{\partial x_i}(s_1,z)\right) - \frac{\beta}{\alpha-\beta}\left(\frac{\partial \underline{\ell}_n}{\partial x_i}(s,z) - \frac{\partial \underline{\ell}_n}{\partial x_i}(s_1,z)\right)}{r} \\
= &\frac{1}{2}\lim_{t\to s_1^+} \frac{\frac{\partial \underline{\ell}_n}{\partial x_i}(t,z) - \frac{\partial \underline{\ell}_n}{\partial x_i}(s_1,z)}{t-s_1} - \frac{1}{2}\lim_{s\to s_1^-} \frac{\frac{\partial \underline{\ell}_n}{\partial x_i}(s,z) - \frac{\partial \underline{\ell}_n}{\partial x_i}(s_1,z)}{s-s_1} \\
= &\frac{1}{2}\frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_1}(s_1,z) - \frac{1}{2}\frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_1}(s_1,z) \\
= &0.
\end{aligned} \tag{3.36}$$

Moreover, since $\frac{\partial \varrho_n}{\partial x_i}(r,z)$ is an even function with respect to r, by (3.36)

$$\lim_{r\to 0^-} \frac{\frac{\partial \varrho_n}{\partial x_i}(r,z) - \frac{\partial \varrho_n}{\partial x_i}(0,z)}{r} = -\lim_{r\to 0^+} \frac{\frac{\partial \varrho_n}{\partial x_i}(r,z) - \frac{\partial \varrho_n}{\partial x_i}(0,z)}{r} = 0. \tag{3.37}$$

By (3.36) and (3.37), we have

$$\frac{\partial^2 \varrho_n}{\partial x_i \partial x_1}(0,z) = \lim_{r\to 0} \frac{\frac{\partial \varrho_n}{\partial x_i}(r,z) - \frac{\partial \varrho_n}{\partial x_i}(0,z)}{r} = 0. \tag{3.38}$$

By (3.35) and $\underline{\ell}_n \in C^2$, we have

$$\begin{aligned}
\frac{\partial^2 \varrho_n}{\partial x_i \partial x_j}(0,z) &= \lim_{\varepsilon\to 0} \frac{\frac{\partial \varrho_n}{\partial x_i}(0,z+\varepsilon e_j) - \frac{\partial \varrho_n}{\partial x_i}(0,z)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\frac{\partial \underline{\ell}_n}{\partial x_i}(s_1(z+\varepsilon e_j), z+\varepsilon e_j) - \frac{\partial \underline{\ell}_n}{\partial x_i}(s_1(z),z)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\frac{\partial \underline{\ell}_n}{\partial x_i}(s_1(z+\varepsilon e_j), z+\varepsilon e_j) - \frac{\partial \underline{\ell}_n}{\partial x_i}(s_1(z+\varepsilon e_j),z)}{\varepsilon} \\
&\quad + \lim_{\varepsilon\to 0} \frac{\frac{\partial \underline{\ell}_n}{\partial x_i}(s_1(z+\varepsilon e_j),z) - \frac{\partial \underline{\ell}_n}{\partial x_i}(s_1(z),z)}{\varepsilon} \\
&= \frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_j}(s_1,z) + \frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_1}(s_1,z) \cdot \lim_{\varepsilon\to 0} \frac{s_1(z+\varepsilon e_j) - s_1(z)}{\varepsilon} \\
&= \frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_j}(s_1,z) - \frac{\frac{\partial^2 \underline{\ell}_n}{\partial x_i \partial x_1}(s_1,z) \cdot \frac{\partial^2 \underline{\ell}_n}{\partial x_1 \partial x_j}(s_1,z)}{\frac{\partial^2 \underline{\ell}_n}{\partial x_1^2}(s_1,z)},
\end{aligned} \tag{3.39}$$

where the last equality is obtained from

(3.40)

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\frac{s_1(z+\varepsilon e_j)-s_1(z)}{\varepsilon}\\ =&-\lim_{\varepsilon\to 0^+}\frac{(\frac{\partial \underline{\ell}_n}{\partial x_1}(s_1(z),z+\varepsilon e_j)-\frac{\partial \underline{\ell}_n}{\partial x_1}(s_1(z),z))/\varepsilon}{(\frac{\partial \underline{\ell}_n}{\partial x_1}(s_1(z+\varepsilon e_j),z+\varepsilon e_j)-\frac{\partial \underline{\ell}_n}{\partial x_1}(s_1(z),z+\varepsilon e_j))/(s_1(z+\varepsilon e_j)-s_1(z))}\\ =&-\frac{\partial^2\underline{\ell}_n}{\partial x_1\partial x_j}(s_1,z)/\frac{\partial^2\underline{\ell}_n}{\partial x_1^2}(s_1,z).\end{aligned}$$

□

Lemmas 3.4-3.7 show the existence of $\frac{\partial^2\varrho_n}{\partial x_i\partial x_j}(x)$, $1\le i,j\le n-1$, for $x\in D_1$. We will prove the continuity of $\frac{\partial^2\varrho_n}{\partial x_i\partial x_j}$ at the origin and prove that ϱ_n has positive definite Hessian matrix at the origin.

Lemma 3.8. *For $i,j=1,2,\dots,n-1$ and any fixed compact set $S\subset D_1\cap e_1^\perp$, $\frac{\partial^2\varrho_n}{\partial x_i\partial x_j}(r,z)$ converges to $\frac{\partial^2\varrho_n}{\partial x_i\partial x_j}(0,z)$ uniformly on S as $r\to 0$.*

Proof. By Lemmas 3.2-3.3 and Lemma 3.7, taking the limit of $r\to 0^+$ in (3.9), (3.14) and (3.24) shows that $\frac{\partial^2\varrho_n}{\partial x_i\partial x_j}(r,z)$ converges to $\frac{\partial^2\varrho_n}{\partial x_i\partial x_j}(0,z)$ pointwise on S as $r\to 0^+$. Moreover, since ϱ_n is symmetric with respect to $e_1^\perp$, for $i,j=2,\dots,n-1$, $\frac{\partial^2\varrho_n}{\partial x_1^2(r,z)}$ and $\frac{\partial^2\varrho_n}{\partial x_i\partial x_j}(r,z)$ are even with respect to r and $\frac{\partial^2\varrho_n}{\partial x_1\partial x_i}(r,z)$ is odd with respect to r. Therefore, $\frac{\partial^2\varrho_n}{\partial x_i\partial x_j}(r,z)$ converges to $\frac{\partial^2\varrho_n}{\partial x_i\partial x_j}(0,z)$ pointwise on S as $r\to 0$.

Since $|s-s_1|+|t-s_1|=2r$ is independent of z and the second partial derivative of $\underline{\ell}_n$ is uniformly continuous on any compact subset of D, the left sides of the equalities (3.2) and (3.8) converge uniformly to their right sides, respectively. By (3.9), (3.14), (3.24) and the uniformly continuity of $\frac{\partial^2\underline{\ell}_n}{\partial x_i\partial x_j}$, we have that $\frac{\partial^2\varrho_n}{\partial x_i\partial x_j}(r,z)$ converges to $\frac{\partial^2\varrho_n}{\partial x_i\partial x_j}(0,z)$ uniformly on S as $r\to 0$. □

Proposition 3.1. *The second partial derivatives of ϱ_n are continuous at the origin.*

Proof. For $z\in D_1\cap e_1^\perp$, if $z\to 0$, then $s_1(z)\to s_1(0)$. By (3.27), (3.28), (3.29) and $\underline{\ell}_n\in C^2$, the second partial derivatives of ϱ_n are continuous at the origin when $z\in D_1\cap e_1^\perp$ and $z\to 0$. By the uniform convergence proved in Lemma 3.8, the second partial derivatives of ϱ_n are continuous at the origin when $x\in D_1$ and $x\to 0$. □

Proposition 3.2. *The Hessian matrix of ϱ_n at the origin is positive definite.*

Proof. Let $A=(a_{ij})_{n-1,n-1}$ denote the Hessian matrix of ϱ_n at the origin and let $B=(b_{ij})_{n-1,n-1}$ denote the Hessian matrix of $\underline{\ell}_n$ at the point $(s_1,0)$. By (3.27), (3.28), (3.29), the kth row ($k=2,\dots,n-1$) of A can be obtained by adding the kth row of B by $-\frac{b_{k1}}{b_{11}}$ times the first row of B. Thus $|A|=|B|$. Since $|B|>0$, $|A|>0$. Moreover, ϱ_n is a convex function, so its Hessian matrix A is semi-positive definite. By $|A|>0$, A is positive definite. □

By Proposition 3.1, Proposition 3.2 and the arbitrary choice of $x_o\in\partial K_1\cap e_1^\perp$, K_1 is of class C_+^2.

4. Open problems

Problem 4.1. For $3 \leq k \leq \infty$, is the Steiner symmetral of a convex body of class C^k_+ again of class C^k_+?

The following problem is provided by the referee.

Problem 4.2. Can Theorem 1.1 be obtained simply from Corollary 1.2?

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References

[1] Wilhelm Blaschke, *Kreis und Kugel* (German), Walter de Gruyter & Co., Berlin, 1956. 2te Aufl. MR0077958
[2] Gabriele Bianchi, Richard J. Gardner, and Paolo Gronchi, *Symmetrization in geometry*, Adv. Math. **306** (2017), 51–88, DOI 10.1016/j.aim.2016.10.003. MR3581298
[3] Gabriele Bianchi and Paolo Gronchi, *Steiner symmetrals and their distance from a ball*, Israel J. Math. **135** (2003), 181–192, DOI 10.1007/BF02776056. MR1997042
[4] Gabriele Bianchi, Daniel A. Klain, Erwin Lutwak, Deane Yang, and Gaoyong Zhang, *A countable set of directions is sufficient for Steiner symmetrization*, Adv. in Appl. Math. **47** (2011), no. 4, 869–873, DOI 10.1016/j.aam.2011.04.005. MR2832382
[5] Károly J. Böröczky, *Stronger versions of the Orlicz-Petty projection inequality*, J. Differential Geom. **95** (2013), no. 2, 215–247. MR3128983
[6] A. Burchard, *Steiner symmetrization is continuous in $W^{1,p}$*, Geom. Funct. Anal. **7** (1997), no. 5, 823–860, DOI 10.1007/s000390050027. MR1475547
[7] R. J. Gardner, *Symmetrals and X-rays of planar convex bodies*, Arch. Math. (Basel) **41** (1983), no. 2, 183–189, DOI 10.1007/BF01196876. MR719423
[8] R. J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), no. 3, 355–405, DOI 10.1090/S0273-0979-02-00941-2. MR1898210
[9] Richard J. Gardner, *Geometric tomography*, 2nd ed., Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, New York, 2006. MR2251886
[10] P. M. Gruber and M. Ludwig, *A Helmholtz-Lie type characterization of ellipsoids. II*, Discrete Comput. Geom. **16** (1996), no. 1, 55–67, DOI 10.1007/BF02711133. MR1397787
[11] Peter M. Gruber, *Convex and discrete geometry*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 336, Springer, Berlin, 2007. MR2335496
[12] Christer O. Kiselman, *How smooth is the shadow of a smooth convex body?*, J. London Math. Soc. (2) **33** (1986), no. 1, 101–109, DOI 10.1112/jlms/s2-33.1.101. MR829391
[13] Erwin Lutwak, Deane Yang, and Gaoyong Zhang, *L_p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), no. 1, 111–132. MR1863023
[14] Erwin Lutwak, Deane Yang, and Gaoyong Zhang, *Orlicz projection bodies*, Adv. Math. **223** (2010), no. 1, 220–242, DOI 10.1016/j.aim.2009.08.002. MR2563216
[15] Erwin Lutwak, Deane Yang, and Gaoyong Zhang, *Orlicz centroid bodies*, J. Differential Geom. **84** (2010), no. 2, 365–387. MR2652465
[16] Rolf Schneider, *Convex bodies: the Brunn-Minkowski theory*, Second expanded edition, Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, 2014. MR3155183
[17] Giorgio Talenti, *The standard isoperimetric theorem*, Handbook of convex geometry, Vol. A, B, North-Holland, Amsterdam, 1993, pp. 73–123, DOI 10.1016/B978-0-444-89596-7.50008-0. MR1242977

School of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, People's Republic of China – and – Department of Mathematics, Tandon School of Engineering, New York University, 6 MetroTech Center, Brooklyn, New York 11201
E-mail address: lxyoujiang@126.com
E-mail address: yjl432@nyu.edu