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REDUCED FUNCTIONS AND JENSEN MEASURES

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ABSTRACT. Let φ be a locally upper bounded Borel measurable function on a Greenian open set Ω in $\mathbb{R}^d$ and, for every $x \in \Omega$, let $v_\varphi(x)$ denote the infimum of the integrals of φ with respect to Jensen measures for x on Ω. Twenty years ago, B.J. Cole and T.J. Ransford proved that v_φ is the supremum of all subharmonic minorants of φ on X and that the sets $\{v_\varphi < t\}$, $t \in \mathbf{R}$, are analytic. In this paper, a different method leading to the inf-sup-result establishes at the same time that, in fact, v_φ is the minimum of φ and a subharmonic function, and hence Borel measurable. This is presented in the generality of harmonic spaces, where semipolar sets are polar, and the key tools are measurability results for reduced functions on balayage spaces which are of independent interest.

1. INTRODUCTION

The motivation for our considerations is a question in connection with Jensen measures which could not be answered in [5]. Let Ω be an open set in $\mathbb{R}^d$, $d \geq 2$ (such that, if $d = 2$, $\mathbb{R}^d \setminus \Omega$ is not polar). We recall that a (Radon) measure μ with compact support in Ω is a *Jensen measure for a point* $x \in \Omega$ if

$$\int v\,d\mu \geq v(x) \quad \text{for every subharmonic function } v \text{ on } \Omega. \tag{1.1}$$

Let φ be a locally upper bounded Borel measurable function on Ω and

$$v_\varphi(x) := \inf\{\int \varphi\,d\mu\colon \mu \text{ a Jensen measure for } x\}, \qquad x \in \Omega.$$

The results [5, Theorem 1.6 and Corollary 1.7] show that

$$v_\varphi = \sup\{v\colon\ v \text{ subharmonic on } \Omega,\, v \leq \varphi\} \tag{1.2}$$

and that the sets $\{v_\varphi < t\}$, $t \in \mathbb{R}$, are analytic (which led the authors B.J. Cole and T.J. Ransford to a definition and the study of quasi-subharmonic functions; cf. also [1]). It remained an open question if the function v_φ is, in fact, Borel measurable (see the lines following [5, Theorem 1.6]).

In this short paper, we shall give a positive answer to this question (even in a much more general setting) using a different method which, at the same time, provides a simpler proof for (1.2).

Our essential tools are measurability properties which we shall prove for reduced functions on balayage spaces $(X, \mathcal{W})$ satisfying the axiom of polarity (section 2) and which are of independent interest.

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In our application to Jensen measures on harmonic spaces (section 3) it is natural to consider superharmonic functions instead of subharmonic functions. Recalling that a function u is superharmonic if and only if $-u$ is subharmonic, this requires us to look upside-down at the definitions, assumptions and statements above.

In both sections, the reader who is not familiar with or not interested in general potential theory may suppose that X is an open subset Ω of $\mathbb{R}^d$ and that $\mathcal{W}$ is the set of all functions $u \geq 0$ on Ω which are hyperharmonic on Ω (that is, which, for each connected component U of Ω, are either superharmonic on U or are identically $+\infty$ on U).

2. Measurability of reduced functions

Let $(X, \mathcal{W})$ be a balayage space (X a locally compact space with countable base and $\mathcal{W}$ the set of all hyperharmonic functions $u \geq 0$ on X; see [4] or [10]). In the following, let u_0 be any strictly positive function in $\mathcal{W} \cap \mathcal{C}(X)$ (say $u_0 = 1$ if $1 \in \mathcal{W}$). We denote by $\mathcal{B}(X)$, $\mathcal{C}(X)$ respectively the set of all numerical Borel measurable functions, real continuous functions on X. As usual, given a set $\mathcal{F}$ of functions, let $\mathcal{F}^+$ be the set of all $f \in \mathcal{F}$ such that $f \geq 0$.

We recall that for every numerical function $\varphi \geq 0$ on X, a *reduced function* R_φ is defined by

$$R_\varphi := \inf\{u \in \mathcal{W} \colon u \geq \varphi\}. \tag{2.1}$$

It is easily seen that the mapping $\varphi \mapsto R_\varphi$ is subadditive, positively homogeneous, and $R_{R_\varphi} = R_\varphi$. In particular, we have $R_v^A := R_{v1_A}$ for $A \subset X$ and $v \in \mathcal{W}$, which leads to reduced measures ε_x^A, $x \in X$, characterized by $\int v \, d\varepsilon_x^A = R_v^A(x)$, $v \in \mathcal{W}$ (by [4, VI.1.1], the mappings $v \mapsto R_v^A$ are additive).

Let $\mathcal{P}(X)$ denote the set of all continuous real potentials on X, that is, of all $p \in \mathcal{W} \cap \mathcal{C}(X)$ satisfying

$$\inf\{R_p^{X \setminus K} \colon K \text{ compact in } X\} = 0.$$

A real function φ on X is called $\mathcal{P}$-bounded if $|\varphi| \leq p$ for some $p \in \mathcal{P}(X)$ (every bounded φ with compact support is $\mathcal{P}$-bounded).

For every numerical function v on X, let $\hat{v}$ denote its lower semicontinuous regularization, that is,

$$\hat{v}(x) := \liminf_{y \to x} v(y), \qquad x \in X.$$

If $\mathcal{V} \subset \mathcal{W}$ and $v := \inf \mathcal{V}$, then $\hat{v} \in \mathcal{W}$. So $\hat{R}_\varphi := \widehat{R_\varphi} \in \mathcal{W}$ for every $\varphi \colon X \to [0, \infty]$. If φ is lower semicontinuous, then $\hat{R}_\varphi \geq \hat{\varphi} = \varphi$, and hence $\hat{R}_\varphi \geq R_\varphi$, $R_\varphi = \hat{R}_\varphi \in \mathcal{W}$. Moreover, R_φ is continuous, upper semicontinuous respectively, if φ is a $\mathcal{P}$-bounded function which is continuous, upper semicontinuous (see [10, Corollary 1.2.2]).

A subset P of X is *polar* if it has no interior points and, for every $x \in X$, there exists a function $u \in \mathcal{W}$ such that $u = \infty$ on $P \setminus \{x\}$ and $u(x) < \infty$. In particular, we know that $R_\varphi = \varphi$ and $\hat{R}_\varphi = 0$ if φ vanishes outside a polar set.

Throughout this paper, let us suppose that $(X, \mathcal{W})$ *satisfies the axiom of polarity* (*Hunt's hypothesis* (H)): *Every* semipolar set, *that is, every set* $\{\hat{v} < v\}$, *where* $v = \inf \mathcal{V}$ *for some* $\mathcal{V} \subset \mathcal{W}$, *is polar.*

Let $\tilde{\mathcal{B}}(X)$ denote the set of all numerical functions φ on X for which there exist functions $\varphi_1, \varphi_2 \in \mathcal{B}(X)$ with $\varphi_1 \leq \varphi \leq \varphi_2$ on X and $\varphi_1 = \varphi_2$ outside a polar set.

Proposition 2.1. *Let* $\varphi\colon X \to [0,\infty]$ *and* P *be a polar set such that* $\hat{R}_\varphi = R_\varphi$ *on* $X \setminus P$. *Then*

$$R_\varphi = (\varphi 1_P) \vee \hat{R}_{\varphi 1_{X\setminus P}} = \varphi \vee \hat{R}_\varphi \quad \text{and} \quad \hat{R}_{\varphi 1_{X\setminus P}} = R_{\varphi 1_{X\setminus P}} = \hat{R}_\varphi. \tag{2.2}$$

In particular, $R_\varphi \in \tilde{\mathcal{B}}(X)$. *Moreover,* $R_\varphi \in \mathcal{B}(X)$ *if* $\varphi \in \mathcal{B}(X)$.

Proof. Let $f := \varphi 1_P$ and $g := \varphi 1_{X\setminus P}$. Trivially,

$$f \vee \hat{R}_g \le \varphi \vee \hat{R}_\varphi \le R_\varphi. \tag{2.3}$$

Moreover, $\hat{R}_\varphi \in \mathcal{W}$ and $\hat{R}_\varphi = R_\varphi \ge g$ on $X \setminus P$. Therefore

$$\hat{R}_\varphi \ge R_g \ge \hat{R}_g. \tag{2.4}$$

Defining $f_0 := 1_{\{R_g<\infty\}}(f - R_g)^+$ we have $\varphi = f \vee g \le f \vee R_g = f_0 + R_g$. Further, $R_{f_0} = f_0$, since $f_0 = 0$ outside the polar set P, and we obtain that

$$f \vee R_g \le R_{f\vee g} = R_\varphi \le R_{f_0+R_g} \le R_{f_0} + R_{R_g} = f_0 + R_g = f \vee R_g.$$

So $R_\varphi = f \vee R_g = f_0 + R_g$. In particular, $\hat{R}_\varphi = \hat{R}_g$ on the complement of the (semi)polar set $P \cup \{\hat{R}_g < R_g\}$, and hence $\hat{R}_\varphi = \hat{R}_g$ (see [4, VI.5.10]). Having (2.4) the second part of (2.2) follows. Its first part is now an immediate consequence of (2.3) and the equality $R_\varphi = f \vee R_g$. □

Combining Proposition 2.1 with the fact (see [4, VI.1.9]) that, for every Borel set A, the function $\hat{R}^A_{u_0}$ is the supremum of all $\hat{R}^K_{u_0}$, K compact in A, we then obtain the following result.

Theorem 2.2. *Let* $\varphi \in \tilde{\mathcal{B}}^+(X)$ *and let* Ψ *denote the set of all bounded upper semicontinuous functions* $\psi \ge 0$ *with compact support in* $\{\varphi > 0\}$. *Then there exists an increasing sequence* (ψ_n) *in* Ψ *such that*

$$\hat{R}_\varphi = \sup\nolimits_{n\in\mathbb{N}} \hat{R}_{\psi_n}. \tag{2.5}$$

In particular,

$$R_\varphi = \varphi \vee \sup\nolimits_{n\in\mathbb{N}} \hat{R}_{\psi_n} = \varphi \vee \sup\nolimits_{n\in\mathbb{N}} R_{\psi_n} = \sup\{R_\psi \colon \psi \in \Psi\}. \tag{2.6}$$

Proof. Let $\varphi_1, \varphi_2 \in \mathcal{B}(X)$ such that $\varphi_1 \le \varphi \le \varphi_2$ and $P_0 := \{\varphi_1 \ne \varphi_2\}$ is polar. For every $t \in \mathbb{Q}^+$, let A_t be the Borel subset $\{\varphi_1 > tu_0\}$ of $\{\varphi > tu_0\}$. The union P of P_0, the set $\{\hat{R}_\varphi < R_\varphi\}$, and the sets $\{\hat{R}^{A_t}_{u_0} < R^{A_t}_{u_0}\}$, $t \in \mathbb{Q}^+$, is polar.

Let $x \in X \setminus P$ and $a < \varphi(x)$. Let us choose $t \in \mathbb{Q}^+$ such that $a < tu_0(x) < \varphi(x)$. Then $x \in A_t$ and $\hat{R}^{A_t}_{u_0}(x) = R^{A_t}_{u_0}(x) = u_0(x) > a/t$. Hence $\hat{R}^K_{u_0}(x) > a/t$ for some compact K in A_t. Obviously, $\psi := tu_0 1_K \in \Psi$ and $\hat{R}_\psi(x) = t\hat{R}^K_{u_0}(x) > a$.

This shows that

$$u := \sup\{\hat{R}_\psi \colon \psi \in \Psi\} \ge \varphi \quad \text{on } X \setminus P.$$

If $\psi_1, \psi_2 \in \Psi$, then $\psi := \psi_1 \vee \psi_2 \in \Psi$ and $\hat{R}_{\psi_1} \vee \hat{R}_{\psi_2} \le \hat{R}_\psi$. So, by [4, I.1.7], there exists an increasing sequence (ψ_n) in Ψ with $u = \sup_{n\in\mathbb{N}} \hat{R}_{\psi_n}$. In particular, $u \in \mathcal{W}$, and hence $u \ge R_{\varphi 1_{X\setminus P}}$. Since trivially $u \le \hat{R}_\varphi$, the proof is completed by Proposition 2.1, monotonicity, and the fact that $\varphi 1_{\{x\}} \in \Psi$ for $x \in \{\varphi > 0\}$. □

3. Application to Jensen measures

From now on, we suppose more restrictively that the balayage space $(X, \mathcal{W})$ satisfying the axiom of polarity is a harmonic space; that is, $\mathcal{W}$ has the following local truncation property: For all open sets U in X and all $u, v \in \mathcal{W}$ such that $u \ge v$ on the boundary ∂U of U, the function w defined by $w := u \wedge v$ on U and v on $X \setminus U$ is contained in $\mathcal{W}$ (see [4, Section III.8]). This means that the reduced measures $\varepsilon_x^{X\setminus V}$ (that is, the harmonic measures μ_x^V) for open sets V and $x \in V$ are supported by ∂V (instead of having supports which could be the entire complement of V).

In probabilistic terms, an associated process will be a diffusion (instead of a process possibly having many jumps). We recall that fairly general linear differential operators L of second order on open subsets X of $\mathbb{R}^d$ (L being the Laplacian in the classical case) lead to harmonic spaces (see, for example, [9, section 7]).

Given an open set U in X, let ${}^*\mathcal{H}(U)$ denote the set of all *hyperharmonic functions* v on U, that is, of all lower semicontinuous $v\colon U \to]-\infty, \infty]$ such that $\int v\, d\mu_x^V \le v(x)$ for every open set V, which is relatively compact in U, and every $x \in V$. If, in addition, the functions $x \mapsto \int v\, d\mu_x^V$ are continuous and finite on V, then such a function v is called *superharmonic* on U. The set of all superharmonic functions on U is denoted by $\mathcal{S}(U)$, and $\mathcal{H}(U) = \mathcal{S}(U) \cap (-\mathcal{S}(U))$ is the set of all harmonic functions on U.

We note that ${}^*\mathcal{H}^+(X) = \mathcal{W}$ and $\mathcal{S}^+(X) \cap \mathcal{C}(X) = \mathcal{W} \cap \mathcal{C}(X)$. In particular, it is compatible with (2.1) to define, for *every* numerical function φ on X,

$$R_\varphi := \inf\{v \in {}^*\mathcal{H}(X) \colon v \ge \varphi\}.$$

In our proofs we shall tacitly use that for every (relatively compact) open set U in X, $(U, {}^*\mathcal{H}^+(U))$ is a harmonic space as well (see [4, V.1.1] in connection with [4, III.2.8 and 6.11]) and that sets $A \subset U$ which are polar (semipolar, respectively) with respect to $(U, {}^*\mathcal{H}^+(U))$ are polar (semipolar, respectively) with respect to $(X, \mathcal{W})$ (see [6, sections 6.2 and 6.3]; the converse is trivial).

Given an open set U in X, we say that a locally lower bounded function v on U is *nearly hyperharmonic* if $\int^* v\, d\mu_x^V \le v(x)$ for every open set V, which is relatively compact in U, and every $x \in V$. As is well-known, $\hat{v} \in {}^*\mathcal{H}(U)$ for every nearly hyperharmonic function on U.

Lemma 3.1. *Let v be a locally lower bounded numerical function on an open set U in X. The following statements are equivalent:*

(i) *v is nearly hyperharmonic on U and the set $\{\hat{v} < v\}$ is polar.*
(ii) *v is the infimum of its hyperharmonic majorants on U.*

Proof. If (i) holds, we may argue as in the proof of (1) ⇒ (2) in [1, Theorem 2]): Let $x \in U$ be such that $v(x) < \infty$, and let $\varepsilon > 0$. There exists $v_x \in {}^*\mathcal{H}^+(U)$ such that $v_x(x) = v(x) - \hat{v}(x) + \varepsilon$ and $v_x = \infty$ on the polar set $\{\hat{v} < v\} \setminus \{x\}$. Then $w := \hat{v} + v_x \in {}^*\mathcal{H}^+(U)$, $w \ge v$ and $w(x) = v(x) + \varepsilon$.

Next suppose that (ii) holds. Then v is obviously nearly hyperharmonic on U. Moreover, the set $\{\hat{v} < v\}$ is semipolar (see [6, Theorem 6.3.2]), and hence polar by the axiom of polarity. □

We shall use the following consequence.

Lemma 3.2. *Let U_n, $n \in \mathbb{N}$, be relatively compact open sets in X with $\overline{U}_n \subset U_{n+1}$ and $\bigcup_{n\in\mathbb{N}} U_n = X$. Moreover, let (v_n) be an increasing sequence of locally lower bounded numerical functions on X such that, for every $n \in \mathbb{N}$,*

$$v_n|_{U_n} = \inf\{w \in {}^*\mathcal{H}(U_n)\colon w \ge v_n|_{U_n}\},$$

and let $v := \lim_{n\to\infty} v_n$. Then $\hat{v} = \lim_{n\to\infty} \hat{v}_n$ and

$$v = \inf\{w \in {}^*\mathcal{H}(X)\colon w \ge v\}. \tag{3.1}$$

Proof. For every $n \in \mathbb{N}$, v_n is nearly hyperharmonic on U_n and $P_n := \{\hat{v}_n < v_n\}$ is polar, by Lemma 3.1. Therefore v is nearly hyperharmonic on X and $\hat{v} = \lim_{n\to\infty} \hat{v}_n$ (see [2, p. 48]). Hence the set $P := \{\hat{v} < v\}$ is contained in the union of all P_n, $n \in \mathbb{N}$. So P is polar, and (3.1) holds, by Lemma 3.1. □

For every open set U in X, let $\mathcal{M}_c(U)$ denote the set of all measures with compact support in U. For every $x \in U$, let $\mathcal{J}_x(U)$ denote the set of all *Jensen measures for x with respect to U*, that is,

$$\mathcal{J}_x(U) := \{\mu \in \mathcal{M}_c(U)\colon \int v\,d\mu \le v(x) \text{ for every } v \in \mathcal{S}(U)\}.$$

If $h \in \mathcal{H}(U)$, then $\pm h \in \mathcal{S}(U)$, and hence

$$\int h\,d\mu = h(x) \quad \text{for all } x \in U \text{ and } \mu \in \mathcal{J}_x(U).$$

Since every function in ${}^*\mathcal{H}(U)$ is an increasing limit of functions in $\mathcal{S}(U)\cap\mathcal{C}(U)$ (see [6, Corollary 2.3.1]), a measure $\mu \in \mathcal{M}_c(U)$ is a Jensen measure for x with respect to U provided $\int u\,d\mu \le u(x)$ for every $u \in \mathcal{S}(U)\cap\mathcal{C}(U)$, and then $\int w\,d\mu \le w(x)$ for every $w \in {}^*\mathcal{H}(U)$.

Of course, $\mathcal{J}_x(U)$ is a convex set containing the Dirac measure ε_x at x and the harmonic measures μ_x^V, V relatively compact open in U and $x \in V$ (see [11] for a detailed discussion).

Let $\varphi \in \tilde{\mathcal{B}}(X)$ be locally lower bounded. If $x \in X$, $\mu \in \mathcal{J}_x(X)$, then $\mu^*(P) = 0$ for every polar set $P \subset X\setminus\{x\}$ (if $u \in \mathcal{W}$ such that $u = \infty$ on $P\setminus\{x\}$ and $u(x) < \infty$, then $\infty \cdot \mu^*(P) \le \int u\,d\mu \le u(x) < \infty$). Hence we may define a function J_φ on X by

$$J_\varphi(x) := \sup\{\int \varphi\,d\mu\colon \mu \in \mathcal{J}_x(X)\}, \qquad x \in X. \tag{3.2}$$

Trivially,

$$J_\varphi \le \inf\{w \in {}^*\mathcal{H}(X)\colon w \ge \varphi\} = R_\varphi. \tag{3.3}$$

Let us begin by proving the reverse inequality for $\varphi \ge 0$ (see Theorem 3.5 for the general case). A first step is the following.

Proposition 3.3. *Let $\psi \ge 0$ be a $\mathcal{P}$-bounded upper semicontinuous function on X. Then $J_\psi = R_\psi$. In particular, J_ψ is upper semicontinuous.*

Proof. Let us fix an exhaustion of X by relatively compact open sets U_n, $n \in \mathbb{N}$, such that $\overline{U}_n \subset U_{n+1}$. For $n \in \mathbb{N}$, we define a function $v_n \ge \psi$ on X by

$$v_n(x) := \inf\{s(x)\colon s \in \mathcal{S}(X)\cap\mathcal{C}(X),\ s \ge \psi \text{ on } \overline{U}_n\}, \qquad x \in U_n,$$

and $v_n := \psi$ on $X \setminus U_n$. Of course, $v_n|_{U_n} = \inf\{w \in {}^*\mathcal{H}(U_n)\colon w \ge v_n|_{U_n}\}$, $n \in \mathbb{N}$, and the sequence (v_n) is increasing. By Lemma 3.2, $v := \lim_{n\to\infty} v_n$ satisfies $v = R_v$. Since $v \ge \psi$, we see that $v \ge R_\psi$.

Let (φ_m) be a sequence of continuous $\mathcal{P}$-bounded functions which is decreasing to ψ. Let us fix $x \in X$ and consider $n \in \mathbb{N}$ with $x \in U_n$. By the theorem of Hahn-Banach, there are measures $\nu_m \in \mathcal{J}_x(X)$, $m \in \mathbb{N}$, which are supported by $\overline{U}_n$ and satisfy

$$\int \varphi_m \, d\nu_m = \inf\{s(x)\colon s \in \mathcal{S}(X) \cap \mathcal{C}(X),\, s \geq \varphi_m \text{ on } \overline{U}_n\}$$

(see, for example, [4, I.2.3]) so that obviously $\int \varphi_m \, d\nu_m \geq v_n(x)$. Having the inequalities $\int u_0 \, d\nu_m \leq u_0(x)$ we know that $\nu_m(\overline{U}_n) \leq u_0(x)/\inf u_0(\overline{U}_n)$ for all $m \in \mathbb{N}$.

Passing to a subsequence we hence may assume without loss of generality that the sequence (ν_m) converges weakly to a measure ν on $\overline{U}_n$ (that is, $\lim_{m\to\infty} \nu_m(f) = \nu(f)$ for every $f \in \mathcal{C}(\overline{U}_n)$). Then, of course, $\nu \in \mathcal{J}_x(X)$ and, for every $k \in \mathbb{N}$,

$$v_n(x) \leq \liminf_{m\to\infty} \int \varphi_m \, d\nu_m \leq \lim_{m\to\infty} \int \varphi_k \, d\nu_m = \int \varphi_k \, d\nu.$$

Letting $k \to \infty$, we see that $v_n(x) \leq \int \psi \, d\nu \leq J_\psi(x)$. We finally let $n \to \infty$ and, using (3.3) and $R_\psi \leq v$, obtain that $v(x) = J_\psi(x) = R_\psi(x)$. □

Having Theorem 2.2, an easy consequence is the following.

Corollary 3.4. *Let $\varphi \in \tilde{\mathcal{B}}(X)$ and $\varphi + h \geq 0$ for some $h \in \mathcal{H}(X)$. Then*

$$J_\varphi = R_\varphi = \varphi \vee \hat{R}_\varphi.$$

Proof. (a) Let us suppose first that $\varphi \geq 0$. By (3.3), $J_\varphi \leq R_\varphi$. On the other hand, by Theorem 2.2, there exist bounded upper semicontinuous functions ψ_n with compact support which satisfy $0 \leq \psi_n \leq \psi_{n+1} \leq \varphi$, $n \in \mathbb{N}$, and

$$R_\varphi = \varphi \vee \sup\nolimits_{n\in\mathbb{N}} R_{\psi_n}.$$

Since $\varepsilon_x \in \mathcal{J}_x(X)$ for every $x \in X$, we know that $\varphi \leq J_\varphi$. By Proposition 3.3, $R_{\psi_n} = J_{\psi_n} \leq J_\varphi$ for all $n \in \mathbb{N}$. Thus also $R_\varphi \leq J_\varphi$. By Theorem 2.1, $R_\varphi = \varphi \vee \hat{R}_\varphi$.

(b) In the general case $\varphi + h \geq 0$ it suffices to observe that $\varphi + h \in \tilde{\mathcal{B}}(X)$, hence $J_{\varphi+h} = R_{\varphi+h}$, by (a), and that obviously $J_\varphi = J_{\varphi+h} - h$ and $R_\varphi = R_{\varphi+h} - h$. □

To obtain the same result for functions $\varphi \in \mathcal{B}(X)$ which are only supposed to be locally lower bounded, we shall apply Corollary 3.4 to relatively compact open subsets U of X assuming that on these sets U there exist strictly positive harmonic functions. This is a rather weak assumption; it is equivalent to $R_{u_0}^{X\setminus U} > 0$. In this process, we have to work with the subset $\mathcal{J}'_x(X)$ of $\mathcal{J}_x(X)$, $x \in X$, defined by

$$\mathcal{J}'_x(X) := \{\mu \in \mathcal{M}_c(X)\colon \mu \in \mathcal{J}_x(U) \text{ for some relatively compact open } U \text{ in } X\},$$

and consider also functions J'_φ defined by

$$J'_\varphi(x) := \sup\{\int \varphi \, d\mu\colon \mu \in \mathcal{J}'_x(X)\}.$$

For the sake of completeness, we recall from [11] that fairly weak assumptions on $(X, \mathcal{W})$ imply that $\mathcal{J}'_x(X) = \mathcal{J}_x(X)$ for every $x \in X$ (see Remark 3.7(2)).

Here is the main result in this section.

Theorem 3.5. *Let $\varphi \in \tilde{\mathcal{B}}(X)$ be locally lower bounded. Then*

$$J_\varphi = J'_\varphi = R_\varphi = \varphi \vee \hat{R}_\varphi.$$

In particular, $J_\varphi \in \tilde{\mathcal{B}}(X)$. Moreover, $J_\varphi \in \mathcal{B}(X)$ if $\varphi \in \mathcal{B}(X)$.

Proof. Since $\mathcal{J}'_x(X) \subset \mathcal{J}_x(X)$, $x \in X$, and (3.3) holds, we have the inequalities

$$R_\varphi \geq J_\varphi \geq J'_\varphi.$$

To prove that $J'_\varphi \geq R_\varphi$ let us choose again relatively compact open sets U_n exhausting X such that $\overline{U}_n \subset U_{n+1}$ for every $n \in \mathbb{N}$. For the moment, let us fix $n \in \mathbb{N}$. By assumption, there is a strictly positive function $h_{n+1} \in \mathcal{H}(U_{n+1})$, and there exists $a_n > 0$ such that the function $h_n := a_n h_{n+1}|_{U_n} \in \mathcal{H}^+(U_n)$ satisfies $\varphi + h_n > 0$ on U_n. By Corollary 3.4 (applied to U_n instead of X),

$$v_n := \inf\{w \in {}^*\mathcal{H}(U_n) \colon w \geq \varphi \text{ on } U_n\} = (\varphi|_{U_n}) \vee \hat{v}_n \tag{3.4}$$

and, for every $x \in U_n$,

$$v_n(x) = \sup\{\int \varphi \, d\mu \colon \mu \in \mathcal{J}_x(U_n)\}. \tag{3.5}$$

Extending the functions v_n to functions on X by $v_n(x) := \varphi(x)$, $x \in X \setminus U_n$, (3.5) implies that the sequence (v_n) is increasing to $v := J'_\varphi$. By Lemma 3.2, we conclude that $v = R_v$ and $\hat{v} = \lim_{n\to\infty} \hat{v}_n$. Since $v \geq \varphi$, we obtain that $J'_\varphi = v \geq R_\varphi$.

Thus $J_\varphi = J'_\varphi = R_\varphi$, and we finally see that $R_\varphi = \varphi \vee \hat{R}_\varphi$, by (3.4). □

Corollary 3.6. *For every locally lower bounded numerical function u on X the following three statements are equivalent:*

(i) *u is the infimum of its hyperharmonic majorants.*
(ii) *$u \in \tilde{\mathcal{B}}(X)$ and $\int u \, d\mu \leq u(x)$ for all $x \in X$ and $\mu \in \mathcal{J}_x(X)$.*
(iii) *$u \in \tilde{\mathcal{B}}(X)$ and $\int u \, d\mu \leq u(x)$ for all $x \in X$ and $\mu \in \mathcal{J}'_x(X)$.*

Proof. Having Theorem 3.5 it suffices to observe that (iii) implies $J'_u = u$. □

Remarks 3.7.

1. An equivalence as in Corollary 3.6 is contained in [1, Theorem 2] under the stronger assumption of having a Brelot space satisfying the axiom of domination.

2. The detailed description of Jensen measures in [11] led to various simple properties implying that (without assuming the axiom of polarity)

$$\mathcal{J}'_x(X) = \mathcal{J}_x(X) \qquad \text{for every } x \in X. \tag{3.6}$$

For example, (3.6) holds if $(X, \mathcal{W})$ has the following approximation property (AP): For every compact K in X, there exists a relatively compact open neighborhood U of K such that, for all $u \in \mathcal{S}(U) \cap \mathcal{C}(U)$ and $\varepsilon > 0$, there exists a function $v \in \mathcal{S}(X) \cap \mathcal{C}(X)$ satisfying $|u - v| < \varepsilon$ on K.

If $(X, \mathcal{W})$ is elliptic, that is, if every superharmonic function $s \geq 0$, $s \neq 0$, on a domain U in X is strictly positive, (AP) follows from [3, Theorem 6.1 and Remark 6.2.1] (cf. also [7, Theorem 6.9] for the classical case and [8, Theorem 1] for the case of a Brelot space satisfying the axiom of domination).

An approach to (3.6), which is much less involved and, by [11, Proposition 3.2], covers the classical case as well, assumes that $(X, \mathcal{W})$ is h_0-transient for some strictly positive $h_0 \in \mathcal{H}(X)$; that is, for every compact K in X, the (closed) set

$\{R^K_{h_0} = h_0\}$ is compact ([11, Theorem 3.3]; see also [11, Corollary 4.4] for several characterizations of 1-transient bounded open sets in the classical case).

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