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A CHAIN TRANSITIVE ACCESSIBLE PARTIALLY HYPERBOLIC DIFFEOMORPHISM WHICH IS NON-TRANSITIVE

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ABSTRACT. In this paper, we construct a partially hyperbolic skew-product diffeomorphism on $\mathbb{T}^3$, which is accessible and chain transitive, but not transitive.

1. Introduction

Let M be a closed Riemannian manifold, and $f : M \to M$ a diffeomorphism. We say f is transitive, if for any two non-empty open sets $U, V \subset M$, there exists $n > 0$, such that $f^n(U) \cap V \neq \emptyset$. The transitivity of f is equivalent to the existence of a point x whose positive orbit $\{f^n(x) : n > 0\}$ is dense in M.

We call a point $x \in M$ a non-wandering point of f, if for any neighborhood U_x of x, there exists $n > 0$, such that $f^n(U_x) \cap U_x \neq \emptyset$. The non-wandering set $\Omega(f)$ is the set of all non-wandering points of f.

For two points $x, y \in M$, we say y is chain attainable from x, if for any $\epsilon > 0$, there exists a finite sequence $\{x_i\}_{i=0}^n$ with $x_0 = x$ and $x_n = y$, such that $d(f(x_i), x_{i+1}) < \epsilon$ for any $0 \le i \le n-1$. A point $x \in M$ is called a chain recurrent point, if it is chain attainable from itself. The set of chain recurrent points is called a chain recurrent set of f, denoted by $\mathrm{CR}(f)$. If every point is chain recurrent, we say f is chain transitive.

It is clear that every non-wandering point is chain recurrent and if f is transitive, then it is chain transitive, but not vice versa. However, from the powerful chain connecting lemma [3], there exists a residual subset $\mathcal{R} \subset \mathrm{Diff}^1(M)$, such that for any $f \in \mathcal{R}$, we have $\Omega(f) = \mathrm{CR}(f)$ and if f is chain transitive, then f is transitive.

A diffeomorphism $f : M \to M$ is partially hyperbolic, if the tangent bundle TM splits into three continuous non-trivial Df-invariant bundles $TM = E^{ss} \oplus E^c \oplus E^{uu}$, such that $Df|_{E^{ss}}$ is uniformly contracting, $Df|_{E^{uu}}$ is uniformly expanding, and $Df|_{E^c}$ lies between them:

$$\| Df|_{E^{ss}(x)} \| < \| Df^{-1}|_{E^c(f(x))} \|^{-1},$$
$$\| Df|_{E^c(x)} \| < \| Df^{-1}|_{E^{uu}(f(x))} \|^{-1}, \quad \text{for all } x \in M.$$

It is known ([12, (4.1) Theorem]) that there is a unique f-invariant foliation $\mathcal{W}^{ss}$ (resp. $\mathcal{W}^{uu}$) tangent to E^{ss} (resp. E^{uu}).

An important geometric property of partially hyperbolic diffeomorphisms is accessibility. A partially hyperbolic diffeomorphism f is accessible, if any two points

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in M can be joined by an arc consisting of finitely many segments contained in the leaves of foliations $\mathcal{W}^{ss}$ and $\mathcal{W}^{uu}$. Accessibility plays a key role for proving the ergodicity of partially hyperbolic diffeomorphisms ([7, 11]). Moreover, it has been observed ([6, 8, 11]) that most of partially hyperbolic diffeomorphisms are accessible.

M. Brin [5] has proved that for a partially hyperbolic diffeomorphism $f : M \to M$, if f is accessible and $\Omega(f) = M$, then f is transitive. See also [1]. So it is natural to ask the following question: *if a partially hyperbolic diffeomorphism f is accessible and* $\mathrm{CR}(f) = M$*, is f transitive?* In this paper, we construct an example which gives a negative answer to this question. This implies Brin's result could not be generalized to the case where $\mathrm{CR}(f) = M$.

Let $A : \mathbb{T}^2 \to \mathbb{T}^2$ be a hyperbolic automorphism over $\mathbb{T}^2$. We say $f : \mathbb{T}^3 \to \mathbb{T}^3$ is a partially hyperbolic skew-product over A, if for every $(x, t) \in \mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{S}^1$, we have

$$f(x, t) = (Ax, \varphi_x(t)) \qquad \text{and} \qquad \|A^{-1}\|^{-1} < \|\varphi'_x(t)\| < \|A\|.$$

We will consider $\mathbb{S}^1 = \mathbb{R}/2\mathbb{Z}$, and usually use the coordinate $\mathbb{S}^1 = [-1, 1]/\{-1, 1\}$.

Our main result is the following theorem.

Theorem 1. *There exists a partially hyperbolic skew-product C^∞ diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$, such that f is accessible and chain transitive, but not transitive.*

2. Construction of diffeomorphism

We will first construct a chain transitive partially hyperbolic skew-product diffeomorphism on $\mathbb{T}^3$, such that its non-wandering set is not the whole $\mathbb{T}^3$ and not transitive. Then a small perturbation will achieve the accessibility, and still preserve the dynamical properties.

First we need a diffeomorphism on $\mathbb{S}^1$ that is chain transitive but the non-wandering set is not the whole circle.

Let $\theta : \mathbb{S}^1 \to \mathbb{S}^1$ be defined as

$$\theta(t) = -\cos(2\pi t) + 1, \qquad t \in \mathbb{R}/2\mathbb{Z}.$$

It is a C^∞ function on $\mathbb{S}^1$. We can see that $\theta \geq 0$ on $\mathbb{S}^1$, and it has two zero points 0 and $-1 = 1$. The vector field $\{\theta(t) \cdot \frac{\partial}{\partial t}\}$ is a C^∞ vector field on $\mathbb{S}^1$, and its time-r map for $0 < r \ll 1$ is the diffeomorphism we need on the circle (see Figure 1), i.e., the time-r map of $\theta(t) \cdot \frac{\partial}{\partial t}$ is chain transitive, and its non-wandering set consists of only two fixed points 0 and $-1 = 1$. Using the product structure, we can define a vector field X on $\mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{S}^1$.

Lemma 2.1.

$$X(x, t) = \theta(t) \cdot \frac{\partial}{\partial t}, \qquad \forall (x, t) \in \mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{S}^1,$$

is a C^∞ vector field on $\mathbb{T}^3$. Moreover, for every $r > 0$, the time-r map X_r of the flow generated by X satisfies the following properties:

- *$X_r(x, t) = (x, \varphi(t))$ for every $(x, t) \in \mathbb{T}^3$.*
- *For $i = 0, 1$, $X_r(x, i) = (x, i)$ for every $x \in \mathbb{T}^2$.*
- *For every $\delta \in (0, 1/2)$, for every $(x, t) \in \mathbb{T}^3$ with $t \notin \{0, 1\}$, we have $\varphi(t) > t$. In particular, there exists* $0 < \tau = \tau(r, \delta) < \delta/2$*, such that*

$$\varphi(t) > t + \tau, \qquad \forall (x, t) \in \mathbb{T}^2 \times \{-\delta, 1 - \delta\}.$$

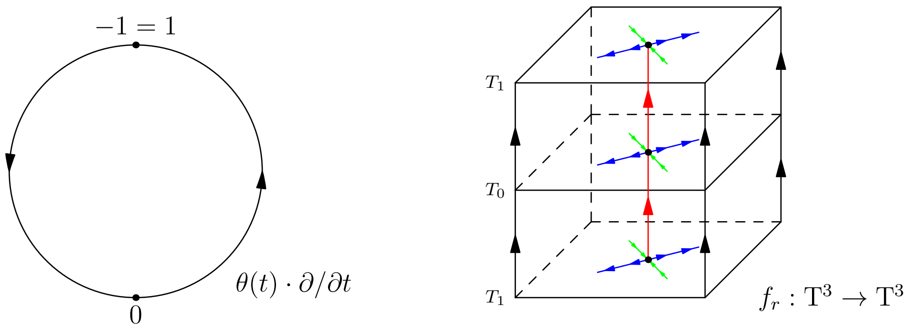

FIGURE 1. Chain transitive systems with non-empty wandering sets.

For $r > 0$, define $f_r = X_r \circ (A \times \mathrm{id}) : \mathbb{T}^3 \to \mathbb{T}^3$:

$$f_r(x,t) = (Ax, \varphi(t)),$$

where $A : \mathbb{T}^2 \to \mathbb{T}^2$ is a hyperbolic automorphism, and $\varphi(t)$ is the function in the last lemma. Then for every $\delta \in (0, 1/2)$ and $\tau = \tau(\delta, r)$ in the last lemma, if r is small enough, f_r satisfies the following properties (Figure 1):

- f_r is a partially hyperbolic skew-product diffeomorphism on $\mathbb{T}^3$. Let the partially hyperbolic splitting be:

$$T\mathbb{T}^3 = E^{ss} \oplus E^c \oplus E^{uu},$$

  and denote by $W^{ss/uu}$ the stable/unstable manifolds generated by $E^{ss/uu}$.
- Let $p \in \mathbb{T}^2$ be a fixed point of A. Then in the fixed center fiber $S_p = \{p\} \times \mathbb{S}^1$, $f_r|_{S_p}$ is chain transitive and has two fixed points $P_i = (p, i) \in \mathbb{T}^2 \times \mathbb{S}^1$ for $i = 0, 1$.
- For $i = 0, 1$, f_r preserves $\mathbb{T}_i = \mathbb{T}^2 \times \{i\}$ invariant, and $f_r|_{\mathbb{T}_i} = A|_{\mathbb{T}_i}$. Moreover,

$$\mathbb{T}_i = \overline{W^{ss}(P_i, f_r)} = \overline{W^{uu}(P_i, f_r)}.$$

- For every $(x, t) \in \mathbb{T}^2 \times \{-\delta, 1 - \delta\}$, we have $\varphi(t) > t + \tau$.

Now f_r is a chain transitive but non-transitive partially hyperbolic diffeomorphism on $\mathbb{T}^3$. However, f_r is not accessible, since the sum of stable and unstable bundles of f_r is integrable. We will make another perturbation to achieve the accessibility, and preserve other dynamical properties.

Let $p \in \mathbb{T}^2$ be a fixed point of the hyperbolic automorphism A. Take a small enough neighborhood $U(p)$ of p in $\mathbb{T}^2$, such that

- for every $x \in U(p) \backslash W^s_{loc}(p)$, there exists some $n > 0$, such that $A^n x \notin U(p)$;
- for every $x \in U(p) \backslash W^u_{loc}(p)$, there exists some $n < 0$, such that $A^n x \notin U(p)$.

Now take a local coordinate $\{(x_s, x_u)\}$ in $U(p)$ with $p = (0, 0)$, so that

$$A(x_s, x_u) = (\lambda \cdot x_s, \lambda^{-1} \cdot x_u),$$

for every $(x_s, x_u) \in [-10, 10]_s \times [-10, 10]_u \subset U(p)$. Here λ is the eigenvalue of A with $0 < |\lambda| < 1$, and we assume $1/10 < \lambda < 1$ for simplicity. In the rest of this paper, the local coordinate of $(U(p); (x_s, x_u))$ is the only coordinate we will use in $\mathbb{T}^2$.

Now we define a C^∞ function $\alpha : \mathbb{T}^2 \to [0,1]$, such that

$$\alpha(x) = \begin{cases} 0, & x \in [-1,1]_s \times [-1,1]_u \subset U(p), \\ 1, & x \in \mathbb{T}^2 \setminus [-3,3]_s \times [-3,3]_u, \\ \in (0,1), & \text{otherwise.} \end{cases}$$

The function α prescribes the perturbation region on $\mathbb{T}^2$. And the next function γ shows the way of perturbations along fibers.

Let $\gamma : \mathbb{S}^1 = [-1,1]/\{-1=1\} \to \mathbb{R}$ be a C^∞ function, such that

$$\gamma(t) : \begin{cases} > 0, & t \in [-1,-1+\tau) \cup (-\tau,\tau) \cup (1-\tau,1], \\ = 0, & t \in [-1+\tau,-\tau] \cup [\tau,1-\tau]. \end{cases}$$

Here, $\tau = \tau(\delta, r) < \delta/2$ is determined by Lemma 2.1.

We define a C^∞ vector field Y on $\mathbb{T}^3$ by

$$Y(x,t) = -\alpha(x)\gamma(t) \cdot \frac{\partial}{\partial t}, \qquad \forall (x,t) \in \mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{S}^1.$$

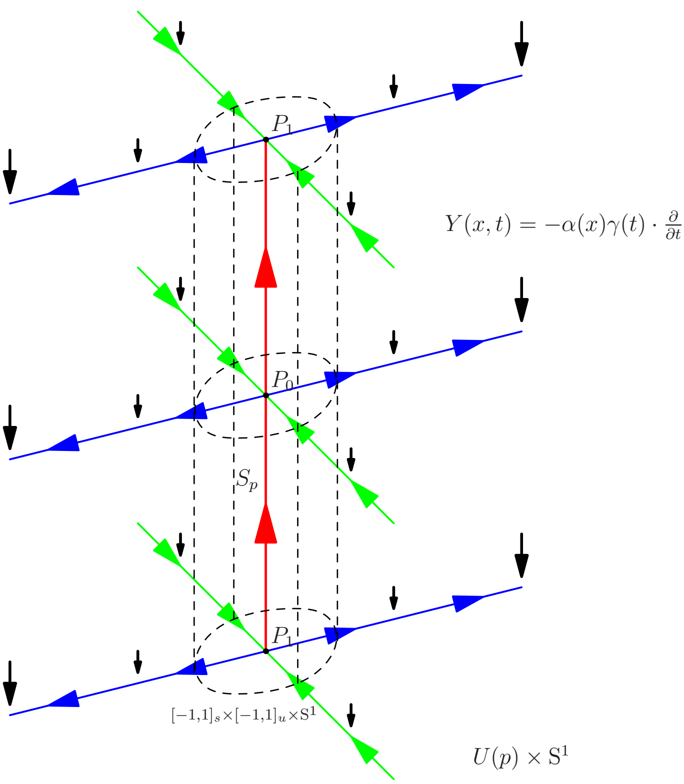

FIGURE 2. The perturbation made by Y_ρ.

For $\rho > 0$, the time-ρ map Y_ρ satisfies the following properties (see Figure 2):

- $Y_\rho(x,t) = (x, \psi_x(t))$, and $Y_\rho(x,t) = (x,t)$ for every $(x,t) \in [-1,1]_s \times [-1,1]_u \times \mathbb{S}^1$.

- For $i = 0, 1$, $\psi_x(i) \le i$ for every $x \in \mathbb{T}^2$. Precisely, for $i = 0, 1$,
  - $\psi_x(i) = i$, for every $x \in [-1, 1]_s \times [-1, 1]_u$;
  - $\psi_x(i) < i$, for every $x \in \mathbb{T}^2 \setminus [-1, 1]_s \times [-1, 1]_u$.
- For every $(x, t) \in \mathbb{T}^2 \times ([-1 + \tau, -\tau] \cup [\tau, 1 - \tau])$, we have $Y_\rho(x, t) = (x, t)$.

Now the composition diffeomorphism $f = Y_\rho \circ f_r : \mathbb{T}^3 \to \mathbb{T}^3$ is the diffeomorphism we promised in our main theorem.

Proposition 2.2. *If ρ and r are small enough, then the diffeomorphism $f = Y_\rho \circ f_r$: $\mathbb{T}^3 \to \mathbb{T}^3$ satisfies the following properties:*

(1) f is a partially hyperbolic skew-product diffeomorphism:

$$f(x, t) = (Ax, \psi_{Ax} \circ \varphi(t)), \qquad \forall (x, t) \in \mathbb{T}^3.$$

(2) When restricted in the fixed fiber S_p, $f|_{S_p}$ has two fixed points P_0, P_1, and is chain transitive.

(3) For $i = 0, 1$, $f(\mathbb{T}^2 \times [i - \delta, i]) \subset \mathbb{T}^2 \times [i - \delta + \tau, i]$.

(4) For $i = 0, 1$, $W^{uu}(P_i, f) \subset [i - \delta + \tau, i]$; $W^{ss}(P_0, f) \subset [0, 1 - \delta]$ and $W^{ss}(P_1, f) \subset [-1, -\delta]$.

(5) For $i = 0, 1$, $W^{uu}(P_i, f) \cap W^{ss}(P_i, f) = \{P_i\}$.

Proof. We prove these five properties one by one:

(1) From the definition of f, we have that for every $(x, t) \in \mathbb{T}^3$,

$$f(x, t) = Y_\rho \circ X_r \circ (A \times \mathrm{Id}) = Y_\rho \circ f_r(x, t) = Y_\rho(Ax, \varphi(t)) = (Ax, \psi_{Ax} \circ \varphi(t)).$$

Moreover, if ρ and r are small enough, f is a C^∞ small perturbation of the partially hyperbolic diffeomorphism $A \times \mathrm{Id}$. Thus f is a partially hyperbolic skew-product diffeomorphism on $\mathbb{T}^3$.

(2) From its definition, the vector field Y is zero in a neighborhood of S_p, and hence Y_ρ is the identity map in a neighborhood of S_p. This implies $f|_{S_p} \equiv f_r|_{S_p}$, which is chain transitive and has two fix points $P_i = (p, i) \in \mathbb{T}^2 \times \mathbb{S}^1$ for $i = 0, 1$.

(3) Since we have $\varphi(i) = i$, for $i = 0, 1$, then $\psi_{Ax} \circ \varphi(i) = \psi_{Ax}(i) \le i$ for every $x \in \mathbb{T}^2$.

For $t = -\delta$ or $t = 1 - \delta$, we have $\varphi(t) > t + \tau$. Since $\tau < \delta/2$, and $Y_\rho(x, t) = (x, t)$ for every $(x, t) \in \mathbb{T}^2 \times ([-1 + \tau, -\tau] \cup [\tau, 1 - \tau])$, we have

$$Y_\rho(x, t) = (x, \psi_x(t)) = (x, t), \qquad \forall (x, t) \in \mathbb{T}^2 \times \{-\delta + \tau, 1 - \delta + \tau\}.$$

Since ψ_x preserves the orientation, $\psi_{Ax} \circ \varphi(t) > \psi_{Ax}(t + \tau) = t + \tau$, for every $(x, t) \in \mathbb{T}^2 \times \{-\delta, 1 - \delta\}$.

Since both ψ_x and ϕ preserve the orientation, the conclusion follows.

(4) From the construction of ψ, we have that for i=0, 1,

$$\psi_{Ax} \circ \varphi(i) = i, \ \forall x \in [-\lambda^{-1}, \lambda^{-1}]_s \times [-\lambda, \lambda]_u.$$

This implies

$$W^{uu}(P_i, f) \cap (\{0_s\} \times [-1, 1]_u \times \mathbb{S}^1) = \{0_s\} \times [-1, 1]_u \times \{i\} \triangleq W^{uu}_{loc}(P_i, f).$$

Since $W^{uu}(P_i, f) = \bigcup_{n>0} f^n(W^{uu}_{loc}(P_i, f))$, by item 3, we have $W^{uu}(P_i, f) \subset [i - \delta + \tau, i]$.

Similarly,

$$W^{ss}(P_i, f) \cap ([-1, 1]_s \times \{0_u\} \times \mathbb{S}^1) = [-1, 1]_s \times \{0_u\} \times \{i\} \triangleq W^{ss}_{loc}(P_i),$$

and $W^{ss}(P_i, f) = \bigcup_{n>0} f^{-n}(W^{ss}_{loc}(P_i, f))$. From item 3, we have

$$f^{-1}(\mathbb{T}^2 \times [0, 1-\delta]) \subset [0, 1-\delta]$$

and $f^{-1}(\mathbb{T}^2 \times [-1, -\delta]) \subset [-1, -\delta]$. Hence $W^{ss}(P_0, f) \subset [0, 1-\delta]$ and $W^{ss}(P_1, f) \subset [-1, -\delta]$.

(5) By the construction of ψ,

$$\psi_{Ax} \circ \varphi(i) < i, \ \forall x \in \mathbb{T}^2 \setminus [-\lambda^{-1}, \lambda^{-1}]_s \times [-\lambda, \lambda]_u.$$

We claim that

$$W^{uu}(P_i, f) \cap (\mathbb{T}^2 \times i) = \{0_s\} \times [-1, 1]_u \times \{i\}.$$

In fact, the right hand side is clearly contained in the left hand side. On the other hand, take any point $(x, i) \in W^{uu}(P_i, f)$. Denote $f^{-n}(x, i) = (x_n, t_n)$. Then for n large enough, $t_n = i$. Hence $t_n = i$ for all n. But this implies that $x_n \in [-\lambda^{-1}, \lambda^{-1}]_s \times [-\lambda, \lambda]_u$ for $n \geq 1$. Thus $x_n \in \{0_s\} \times [-\lambda, \lambda]_u$ for $n \geq 1$. So, (x, i) is in the right hand side.

Similarly, we can show that

$$W^{ss}(P_i, f) \cap (\mathbb{T}^2 \times i) = [-\lambda^{-1}, \lambda^{-1}]_s \times \{0_u\} \times \{i\}.$$

Item 4 implies that $W^{uu}(P_i, f) \cap W^{ss}(P_i, f) \subset \mathbb{T}^2 \times i$ and hence

$$W^{uu}(P_i, f) \cap W^{ss}(P_i, f) = \{P_i\}.$$

□

3. Dynamical and geometrical properties of f

Now we can prove the main theorem from the following three lemmas (Lemmas 3.1, 3.2, 3.5).

Lemma 3.1. *The diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ is chain transitive.*

Proof. From the first and second properties of f in Proposition 2.2, we know that f is a partially hyperbolic skew-product diffeomorphism on $\mathbb{T}^3$, and thus the stable and unstable manifolds of the fixed fiber S_p are dense on $\mathbb{T}^3$. The density of $W^u(S_p)$ implies that every point in $\mathbb{T}^3$ is chain attainable from some point in S_p; the density of $W^s(S_p)$ implies that every point in $\mathbb{T}^3$ is chain attainable to a point in S_p. Since $f|_{S_p}$ is chain transitive, every point in $\mathbb{T}^3$ is chain attainable from itself, i.e., $\mathrm{CR}(f) = \mathbb{T}^3$, and f is chain transitive on $\mathbb{T}^3$. □

Lemma 3.2. *The diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ is accessible.*

Proof. Since f is a partially hyperbolic skew-product diffeomorphism on $\mathbb{T}^3$, if f is not accessible, then from theorem 1.6 of [9], f has a compact us-leaf. Here a us-leaf is a complete 2-dimensional submanifold which is tangent to $E^{ss} \oplus E^{uu}$ of f. It is a torus transverse to the $\mathbb{S}^1$-fiber of $\mathbb{T}^3$. Since the compact us-leaf is saturated by $\mathcal{W}^{ss}$ and $\mathcal{W}^{uu}$, it intersects every $\mathbb{S}^1$-fiber of $\mathbb{T}^3$. Moreover, this us-leaf must intersect every $\mathbb{S}^1$-fiber with only finitely many points since it is a compact and complete submanifold.

If f does not have any periodic us-torus, then theorem 1.9 of [9] shows that f is semi-conjugate to A times an irrational rotation on $\mathbb{S}^1$, which implies f has no periodic points. This contradicts that P_0 and P_1 are two fixed points of f, and thus f must have a periodic compact us-leaf $\mathbb{T}_{us}$.

From the periodicity of $\mathbb{T}_{us}$, we know that $\mathbb{T}_{us} \cap S_p$ only contains P_0 or P_1, and $f(\mathbb{T}_{us}) = \mathbb{T}_{us}$. Assuming $P_0 \in \mathbb{T}_{us}$, then from Theorem 1.7 of [9], we have

$$\mathbb{T}_{us} = \overline{W^{ss}(P_0, f)} = \overline{W^{uu}(P_0, f)}.$$

In particular, $W^{ss}(P_0, f)$ and $W^{uu}(P_0, f)$ have strong homoclinic intersections, which contradicts item 5 of Proposition 2.2. The same argument works for $P_1 \in \mathbb{T}_{us}$, and thus f must be accessible. □

Proposition 3.3. *Let $g : \mathbb{T}^3 \to \mathbb{T}^3$ be a partially hyperbolic skew-product diffeomorphism. If g preserves the orientation of center foliation, and has two disjoint g-invariant compact u-saturated sets, then g is not transitive. In particular, if g is transitive, it has only one g-invariant minimal u-saturated set.*

Proof. Let Λ_1 and Λ_2 be two disjoint g-invariant compact u-saturated sets. Then for every point $x \in \mathbb{T}^2$ and every center fiber $S_x = \{x\} \times \mathbb{S}^1 \subset \mathbb{T}^3$, $S_x \cap \Lambda_i \neq \emptyset$, for $i = 1, 2$.

Now we define a function $\Phi : \mathbb{T}^3 \to \mathbb{R}$. For every $(x, t) \in \mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{S}^1$, the value $\Phi(x, t)$ is defined as follows:

- If $(x, t) \in \Lambda_1 \cup \Lambda_2$, then $\Phi(x, t) = 0$.
- If $(x, t) \notin \Lambda_1 \cup \Lambda_2$, following the natural orientations "$<$" in $\mathbb{S}^1$-fibers, let $t_1 < t < t_2$, such that $(\{x\} \times (t_1, t_2)) \cap (\Lambda_1 \cup \Lambda_2) = \emptyset$, and $(x, t_i) \in \Lambda_1 \cup \Lambda_2$, for $i = 1, 2$.
  - If $(x, t_1) \in \Lambda_1$, and $(x, t_2) \in \Lambda_2$, then
  $$\Phi(x, t) = (t - t_1) \cdot (t_2 - t) > 0.$$
  - If $(x, t_1) \in \Lambda_2$, and $(x, t_2) \in \Lambda_1$, then
  $$\Phi(x, t) = -(t - t_1) \cdot (t_2 - t) < 0.$$
  - If $(x, t_1), (x, t_2) \in \Lambda_1$ or $(x, t_1), (x, t_2) \in \Lambda_2$, then
  $$\Phi(x, t) = 0.$$

The function Φ is well defined, since $\Lambda_1 \cap \Lambda_2 = \emptyset$, and $\Lambda_i \cap S_x \neq \emptyset$ for $i = 1, 2$ and every $x \in \mathbb{T}^2$. Moreover, given a point $x \in \mathbb{T}^2$, Φ is continuous in S_x. Denote two sets

$$U^+ = \{(x, t) \in \mathbb{T}^3 : \Phi(x, t) > 0\} \qquad \text{and} \qquad U^- = \{(x, t) \in \mathbb{T}^3 : \Phi(x, t) < 0\}.$$

Since both Λ_1 and Λ_2 are g-invariant, and g preserves the orientation of center fibers, we can see that both U^+ and U^- are g-invariant. So we only need to show that they are open sets, which will imply that g is not transitive.

From the definition of Φ, if $(x, t) \in \mathbb{T}^3$ with $\Phi(x, t) > 0$, then there exists $t_1 < t < t_2$, such that

$$(x, t_1) \in \Lambda_1, \ (x, t_2) \in \Lambda_2 \qquad \text{and} \qquad \Phi(x, s) > 0 \text{ for every } s \in (t_1, t_2).$$

Since Λ_1 and Λ_2 are compact and disjoint, there exists $\delta > 0$, such that $t_2 - t_1 \geq \delta$. This implies for every $x \in \mathbb{T}^2$, every connected component of $S_x \cap U^+$ has length $\geq \delta$. Denote by $k(x)$ the number of connected components of $S_x \cap U^+$ and by $(a_i(x), b_i(x)), i = 1, 2, \cdots, k(x)$ the connected components of $S_x \cap U^+$, i.e.,

$$S_x \cap U^+ = \{x\} \times \bigcup_{i=1}^{k(x)} (a_i(x), b_i(x)),$$

where $a_i(x) \in \Lambda_1, b_i(x) \in \Lambda_2$ for $i = 1, 2, \cdots, k(x)$.

Claim 3.4. $k : \mathbb{T}^2 \to \mathbb{N}$ is a constant function, i.e., there exists $k_0 \in \mathbb{N}$ such that

$$k(x) \equiv k_0, \qquad \forall x \in \mathbb{T}^2.$$

Moreover, $k_0 \leq 2/\delta$.

Proof of the claim. For every $x \in \mathbb{T}^2$ and every $s_1 < s_2$, if $(x, s_1) \in \Lambda_1$ and $(x, s_2) \in \Lambda_2$, then there must exist some point $s \in (s_1, s_2)$, such that $\Phi(x, s) > 0$.

We will first show that the function k is upper semi-continuous, i.e., if $\lim_{n\to\infty} x_n = x$, then $k(x) \geq \limsup_{n\to\infty} k(x_n)$. Actually, by taking subsequence if necessary, we can assume that

$$S_{x_n} \cap U^+ = \{x_n\} \times \bigcup_{i=1}^{l} (a_i(x_n), b_i(x_n)), \qquad \text{for } l = \limsup_{n\to\infty} k(x_n),$$

and

$$\lim_{n\to\infty} (x_n, a_i(x_n)) = (x, a_i) \in S_x \cap \Lambda_1, \qquad \lim_{n\to\infty} (x_n, b_i(x_n)) = (x, b_i) \in S_x \cap \Lambda_2.$$

This implies there exists some $c_i \in (a_i, b_i)$, such that $\Phi(x, c_i) > 0$, for $i = 1, 2, \cdots, l$. Since $\Phi(x, a_i) = \Phi(x, b_i) = 0$, $S_x \cap U^+$ has at least l connected components, i.e., $k(x) \geq l$, which implies $k : \mathbb{T}^2 \to \mathbb{N}$ is upper semi-continuous.

Assume that g is a skew-product diffeomorphism over a hyperbolic automorphism $A : \mathbb{T}^2 \to \mathbb{T}^2$. Then, for any $y \in W^u(x, A) \subset \mathbb{T}^2$, we have $k(y) = k(x)$ since both Λ_1 and Λ_2 are u-saturated. In fact, if $S_x \cap U^+ = \{x\} \times \bigcup_{i=1}^{k(x)} (a_i(x), b_i(x))$, then $h^u(a_i(x)) \in \Lambda_1 \cap S_y$ and $h^u(b_i(x)) \in \Lambda_2 \cap S_y$, where $h^u : S_x \to S_y$ is the holonomy map of the unstable foliation of g. We have $S_y \cap U^+ = \{y\} \times \bigcup_{i=1}^{k(x)} (h^u(a_i(x)), h^u(b_i(x)))$, and thus $k(y) = k(x)$.

Since every connected component of $S_x \cap U^+$ has length larger than δ, k is uniformly bounded by $2/\delta$. So we can choose the point $z \in \mathbb{T}^2$, where k takes the maximal value k_0 at z. Then, for very $w \in W^u(z, A) \subset \mathbb{T}^2$, $k(w) = k_0$. Since $W^u(z, A)$ is dense in $\mathbb{T}^2$ and k is upper semi-continuous, we have $k(x) \equiv k_0$ for every $x \in \mathbb{T}^2$. □

Now we will show the set U^+ is open in $\mathbb{T}^3$. Suppose on the contrary that there exists a point $(x, t) \in \mathbb{T}^3$ with $\Phi(x, t) > 0$, and a sequence of points $(x_n, t_n) \to (x, t)$ with $\Phi(x_n, t_n) \leq 0$. Denote that

$$S_{x_n} \cap U^+ = \{x_n\} \times \bigcup_{i=1}^{k_0} (a_i(x_n), b_i(x_n)).$$

Since $\Phi(x_n, t_n) \leq 0$, we may assume $t_n \in [b_j(x_n), a_{j+1}(x_n)]$ for some $1 \leq j \leq k_0$ by taking subsequence when necessary.

By taking subsequence when necessary, we may assume that $(x_n, a_i(x_n)) \to (x, a_i) \in \Lambda_1$ and $(x_n, b_i(x_n)) \to (x, b_i) \in \Lambda_2$. Moreover, we have $t \in [b_j, a_{j+1}]$. Since $\Phi(x, t) > 0$, $(x, a_i) \in \Lambda_1$, and $(x, b_i) \in \Lambda_2$, we must have $t \in (b_j, a_{j+1})$.

Now we have $(x, a_i) \in \Lambda_1$ and $(x, b_i) \in \Lambda_2$, which implies there exists some $c_i \in (a_i, b_i)$, such that $\Phi(x, c_i) > 0$, for $i = 1, 2, \cdots, k_0$. Moreover, we have $\Phi(x, t) > 0$ for $t \in (b_j, a_{j+1})$. However, $\Phi(x, a_i) = \Phi(x, b_i) = 0$, which implies $S_x \cap U^+$ has at least $k_0 + 1$ connected components. This is a contradiction to our claim. Thus U^+ is open in $\mathbb{T}^3$.

The same argument can show that U^- is open in $\mathbb{T}^3$. Since U^+ and U^- are both non-empty g-invariant open sets and disjoint, g is not transitive. □

Lemma 3.5. *The diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ is not transitive.*

Proof. By Proposition 3.3, we only need to show that f has two disjoint compact invariant u-saturated sets. Denote

$$\Lambda_0 = \overline{W^{uu}(P_0, f)} \qquad \text{and} \qquad \Lambda_1 = \overline{W^{uu}(P_1, f)}.$$

Since both P_0 and P_1 are fixed points, $W^{uu}(P_0, f)$ and $W^{uu}(P_1, f)$ are two invariant u-saturated sets. This implies Λ_0 and Λ_1 are two compact f-invariant u-saturated sets.

According to item 4 of Proposition 2.2,

$$\Lambda_i = \overline{W^{uu}(P_i, f)} \subset [i - \delta + \tau, i], \qquad \text{for } i = 0, 1.$$

Hence, we have $\Lambda_0 \cap \Lambda_1 = \emptyset$. This finishes the proof of this lemma. □

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