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PROBABILISTIC WELL-POSEDNESS OF GENERALIZED KDV

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Abstract. We consider the Cauchy problem of the generalized Korteweg-de Vries (gKdV) equation. We prove the local well-posedness of the mass supercritical gKdV equations for the scaling supercritical regularity $s < s_c = \frac{1}{2} - \frac{2}{\kappa}$ in the sense of the probabilistic manner. The main ingredient is to establish the probabilistic local smoothing estimate.

1. Introduction

In this paper, we consider the following generalized Korteweg-de Vries (gKdV) equation:

$$\begin{cases} \partial_t u + \partial_x^3 u + F(u) = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = \phi(x) \in H^s, \end{cases} \tag{1.1}$$

where $F(u) = \mu u^\kappa \partial_x u$, $\kappa \geq 1$ is an integer and $\mu = \pm 1$.

By Duhamel's formula, (1.1) is equivalent to the following integral equation:

$$u = U(t)\phi - \int_0^t U(t-t')(\mu u^\kappa(t')\partial_x u(t'))\, dt'. \tag{1.2}$$

Here we define the linear solution $U(t)f$ to the linear problem $\partial_t z + \partial_x^3 z = 0$ with an initial datum f. Then it is formally given by

$$U(t)f = \mathcal{F}^{-1} e^{-it\xi^3} \mathcal{F} f = (2\pi)^{-1} \int_{\mathbb{R}} e^{i(x\cdot\xi - t\xi^3)} \widehat{f}(\xi)\, d\xi, \tag{1.3}$$

where $\widehat{f} = \mathcal{F} f$ denotes the Fourier transform of f such that $\widehat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\, dx$ and we denote its inverse Fourier transform by $\mathcal{F}^{-1} g(x) = (2\pi)^{-1} \int e^{ix\cdot\xi} g(\xi)\, d\xi$.

The equation (1.1) admits the scaling symmetry under the transform

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda^{\frac{2}{\kappa}} u(\lambda^3 t, \lambda x)$$

for $\lambda > 0$ and this implies the scaling invariance for the initial data in $\dot{H}^{\frac{1}{2}-\frac{2}{\kappa}}(\mathbb{R})$. We denote this scaling critical exponent $\frac{1}{2} - \frac{2}{\kappa}$ by s_c.

When $\kappa = 1$ or 2, we particularly call (1.1) KdV and modified KdV equations, respectively. Those equations are very famous in various points of view and have been widely studied by several researchers. See [10] and the references therein. When $\kappa = 4$, we call (1.1) the mass-critical gKdV equation, and it was solved by Kenig-Ponce-Vega [21] for local and (conditionally) global results, and by Dodson [13] for the global result without any condition of the initial data.

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In the following, we only consider the mass-supercritical equations which are of the cases when $\kappa \geq 5$. The scaling sub-critical ($s > s_c$) and critical ($s = s_c$) Cauchy problems of the gKdV equation (1.1) have already been studied by Kenig-Ponce-Vega [21] (see also [15, 16, 31]). They showed local / (small data) global well-posedness of (1.1) in H^s for $s \geq s_c$ by taking appropriate norms to (1.3).

Theorem A (Theorem 2.15 – Corollary 2.18 in [21])**.** *Let $\kappa \geq 5$ and $s \geq s_c$. Then for any $\phi \in H^s(\mathbb{R})$, there exists a unique strong solution $u \in C([-T,T]; H^s(\mathbb{R}))$ of* (1.1)*. Moreover when the initial data is sufficiently small, then the solution can be extended to $T = \infty$.*

The main tool used in the proof of Theorem A is the *local smoothing estimate*. Since (1.1) has one derivative in the nonlinear term, the smoothing effect is essential to make the solution map as in (1.3) a contraction map. The local smoothing effect is naturally inherent to the dispersive phenomenon of linear dispersive equations. Kato [20] first showed that the solution of (1.1) possesses such a smoothing effect, and Kenig-Ponce-Vega [21] developed and used it as the following form:

Lemma 1.1 (Theorem 3.5, Lemma 3.18 in [21])**.** *Let (q, r) and α satisfy*

$$\frac{2}{q} + \frac{1}{r} = \frac{1}{2}, \;\; 4 \leq q < \infty, \;\; 2 \leq r < \infty, \;\; \text{and} \;\; \alpha = \frac{1}{q} + \frac{3}{r} - \frac{1}{2}. \tag{1.4}$$

Then, we have

$$\|D^\alpha S(t) f\|_{L_x^q L_t^r} \lesssim \|f\|_{L_x^2},$$

where D^α is a homogeneous fractional derivative with respect to the x variable.

Now we focus on the supercritical case. In the supercritical regime, many dispersive equations have been known to be ill-posed; see [9] and the references therein. Nonetheless, putting the problem in stochastic perspective, some positive results can be achieved. It was pioneered by Bourgain [4] and continued in Burq-Tzvetkov [7,8], Colliander-Oh [11], Lührmann-Mendelson [24] and Bényi-Oh-Pocovinicu [1,2]. See also [6, 12, 14, 17–19, 25, 27, 29, 34]. For the gKdV equation on the periodic domain, probabilistic local and global well-posedness results have been established below critical thresholds in [5, 26, 28, 30]. In this article, we focus on a probabilistic local well-posedness of the gKdV equation on the real line.

In the following, we briefly introduce the randomization ϕ^ω of the initial data ϕ of (1.1) as follows: Let $\psi \in \mathcal{S}(\mathbb{R})$ be a function satisfying

$$\text{supp } \psi \subset [-1, 1] \quad \text{and} \quad \sum_{n \in \mathbb{Z}} \psi(\xi - n) \equiv 1.$$

We define a pseudo-differential operator $\psi(D - n)$ as a Fourier multiplier

$$\psi(D - n) u(x) = \mathcal{F}^{-1} \psi(\xi - n) \mathcal{F} u.$$

Then, for a given function $f \in L^2(\mathbb{R})$, we have

$$f = \sum_{n \in \mathbb{Z}} \psi(D - n) f.$$

Let $\{g_n\}_{n \in \mathbb{Z}}$ be a sequence of independent mean zero complex-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where g_n is independent and endowed with a probability distribution μ_n. We assume there exists a constant $c > 0$ such that

$$\Big| \int_{\mathbb{R}} e^{\gamma x} d\mu_n \Big| \leq e^{c \gamma^2}, \tag{1.5}$$

for all $\gamma \in \mathbb{R}, n \in \mathbb{Z}$. Now we define the Wiener randomization of f by

$$f^\omega := \sum_{n\in\mathbb{Z}} g_n(\omega)\psi(D-n)f. \tag{1.6}$$

The main difficulty dealing with (1.1) comes from the one derivative in the nonlinearity. To overcome it, we establish the probabilistic local smoothing (see Proposition 2.4 below).

Remark 1.2. In the previous works (NLS, NLW, and so on) associated to the probabilistic well-posedness theory, the main ingredient is the probabilistic Strichartz estimate. It provides additional integrability to the randomization of initial data, and hence it prevents a derivative loss coming from Sobolev embedding. In contrast to previous works, the study on the KdV-type equations is highly dependent on the local smoothing effect. However, the local smoothing norm has the opposite order of Lebesgue spaces from the Strichartz norm, so additional integrability of the local smoothing norm could not be obtained in a similar way. Nevertheless, we can detect additional integrability from the probabilistic local smoothing estimate.

The following is the main result in this paper.

Theorem 1.3. *Let $\kappa \ge 5$, $\max(\frac{1}{\kappa+1}\cdot(\frac{1}{2}-\frac{2}{\kappa}), \frac{1}{4}-\frac{2}{\kappa}) < s < \frac{1}{2}-\frac{2}{\kappa}$ and $\phi \in H^s(\mathbb{R})$. Consider the randomization ϕ^ω defined as in* (1.6) *with a probability space $(\Omega, \mathcal{F}, P)$ satisfying the condition* (1.5)*. Then* (1.1) *is almost surely locally well-posed in the sense that there exist C, c, γ and $\sigma(> s_c)$ such that for each $T \ll 1$, there exists a set $\Omega_T \subset \Omega$ with the following properties:*

(1) $P(\Omega\backslash\Omega_T) \le C\exp\big(-\frac{c}{T^\gamma\|\phi\|_{H^s}^2}\big)$.
(2) For each $\omega \in \Omega_T$, there exists a unique solution $u \in C([0,T]; H^s(\mathbb{R}))$ to (1.1) *with initial data ϕ^ω.*
(3) The Duhamel part of the solution is smoother than initial data, i.e.,

$$u - U(t)\phi^\omega \in C([0,T]; H^\sigma(\mathbb{R})).$$

Remark 1.4. The lower bound of s in Theorem 1.3 satisfies $\frac{1}{\kappa+1}\cdot(\frac{1}{2}-\frac{2}{\kappa}) < s < \frac{1}{2}-\frac{2}{\kappa}$, if $5 \le \kappa \le 8$, and $\frac{1}{4}-\frac{2}{\kappa} < s < \frac{1}{2}-\frac{2}{\kappa}$, otherwise.

The rest of the paper is organized as follows: In Section 2, we will introduce several probabilistic estimates. In particular, we will establish the probabilistic local smoothing estimate which is the most important tool in our analysis. In Section 3, we will provide the definition of $X^{s,b}$ space and also give basic properties associated to this space. Finally, we will prove Theorem 1.3 in Section 4.

Notation. For $x, y \in \mathbb{R}_+$, $x \lesssim y$ means that there exists $C > 0$ such that $x \le Cy$, and $x \sim y$ means $x \lesssim y$ and $y \lesssim x$.

2. Probabilistic estimates

In this section, we mainly establish the local smoothing estimate in the sense of the probabilistic manner. First, we briefly review several well-known probabilistic estimates. The first lemma is of the Khintchine inequality kind, which will be essentially used to prove all probabilistic estimates.

Lemma 2.1 (Lemma 3.1 in [7]). *For given $\{c_n\} \in \ell^2(\mathbb{Z})$ and $p \geq 2$, there exists $C > 0$ such that*

$$\|\sum_{n\in\mathbb{Z}} g_n(\omega)c_n\|_{L^p(\Omega)} \leq C\sqrt{p}\|c_n\|_{l^2_n(\mathbb{Z})}.$$

The next lemma is used to describe that the randomization does not make a given function smoother, but it guarantees to almost surely keep the same regularity as the original function.

Lemma 2.2 (Lemma 2.2 in [1]). *Given $f \in H^s(\mathbb{R})$, we have for any $\lambda > 0$,*

$$P\Big(\|f^\omega\|_{H^s(\mathbb{R})} > \lambda\Big) \leq Ce^{-c\lambda^2\|f\|^{-2}_{H^s}}.$$

We remark that the initial data ϕ and its randomization ϕ^ω are still lain in the supercritical Sobolev space, but this randomization makes the initial value problem a sub-critical problem for a certain regularity σ. See Section 4.

Now we shall verify the probabilistic local smoothing estimate. For that purpose, we recall the smoothing estimate on modulation space.

Lemma 2.3 (Lemma 6.2 in [35]). *We have for $4 \leq p \leq \infty$ and $n \in \mathbb{Z}$ that*

$$\|\psi(D-n)U(t)f\|_{L^p_xL^\infty_t} \lesssim \|\psi(D-n)f\|_{\dot{H}^{\frac{1}{p}}}.$$

Based on Lemma 2.3, we have the following probabilistic local smoothing estimate.

Proposition 2.4. *Let (q,r) be a pair satisfying $\frac{2}{q}+\frac{1}{r} \leq \frac{1}{2}, 4 \leq q < \infty, 2 \leq r < \infty$. Then we have*

$$P(\|D^{\frac{2}{r}-\frac{1}{q}}U(t)f^\omega\|_{L^q_xL^r_t(\mathbb{R}\times\mathbb{R})} > \lambda) \leq C\exp(-c\lambda^2\|f\|^{-2}_{L^2})$$

for all $\lambda > 0$. In particular, for any given small $\varepsilon > 0$, the following holds:

$$\|D^{\frac{2}{r}-\frac{1}{q}}U(t)f^\omega\|_{L^q_xL^r_t(\mathbb{R}\times\mathbb{R})} \lesssim (\log\frac{1}{\varepsilon})^{\frac{1}{2}}\|f\|_{L^2}$$

outside a set of probability at most ε.

Proof. The proof is quite similar to the proof of the probabilistic Strichartz estimate in [1,2]. Let $p \geq \max(q,r)$. By using the Minkowski inequality and Lemma 2.1, we obtain

$$\begin{aligned}
&(\mathbb{E}\|D^{\frac{2}{r}-\frac{1}{q}}U(t)f^\omega\|^p_{L^q_xL^r_t})^{\frac{1}{p}} \\
&\leq \Big\|\|D^{\frac{2}{r}-\frac{1}{q}}U(t)f^\omega\|_{L^p(\Omega)}\Big\|_{L^q_xL^r_t} \leq \sqrt{p}\Big\|\|D^{\frac{2}{r}-\frac{1}{q}}\psi(D-n)U(t)f\|_{l^2_n}\Big\|_{L^q_xL^r_t} \\
&\leq \sqrt{p}\Big\|\|D^{\frac{2}{r}-\frac{1}{q}}\psi(D-n)U(t)f\|_{L^q_xL^r_t}\Big\|_{l^2_n}.
\end{aligned}$$

We interpolate $L^q_xL^r_t$ between an $L^\infty_xL^2_t$ estimate in Lemma 1.1 and an $L^p_xL^\infty_t$ estimate in Lemma 2.3 for an appropriate $p \geq 4$ to obtain that

$$\|D^{\frac{2}{r}-\frac{1}{q}}\psi(D-n)U(t)f\|_{L^q_xL^r_t} \lesssim \|\psi(D-n)f\|_{L^2}$$

for each $n \in \mathbb{Z}$. We note that the implicit constant does not depend on $n \in \mathbb{Z}$. So we have

$$(\mathbb{E}\|D^{\frac{2}{r}-\frac{1}{q}}U(t)f^\omega\|^p_{L^q_xL^r_t})^{\frac{1}{p}} \lesssim \sqrt{p}\big\|\|\psi(D-n)f\|_{L^2}\big\|_{l^2_n} \sim \sqrt{p}\|f\|_{L^2}.$$

Then Chebyshev's inequality gives

$$P(\|D^{\frac{2}{r}-\frac{1}{q}}U(t)f^\omega\|_{L^q_xL^r_t} > \lambda) < (\frac{C_0p^{\frac{1}{2}}\|f\|_{L^2}}{\lambda})^p.$$

Let $p_0 = (\frac{\lambda}{C_0 e\|f\|_{L^2}})^2$. If $p_0 \geq 2$, then we have from above that

$$P(\|D^{\frac{2}{r}-\frac{1}{q}}U(t)f^\omega\|_{L^q_xL^r_t} > \lambda) < (\frac{C_0 p_0^{\frac{1}{2}}\|f\|_{L^2}}{\lambda})^{p_0} = e^{-p_0} = \exp(-c\lambda^2\|f\|_{L^2}^{-2}).$$

Otherwise, we can choose C such that $Ce^{-2} \geq 1$. Then we have

$$P(\|D^{\frac{2}{r}-\frac{1}{q}}U(t)f^\omega\|_{L^q_xL^r_t} > \lambda) \leq 1 \leq Ce^{-2} \leq C\exp(-c\lambda^2\|f\|_{L^2}^{-2}).$$

Therefore, we complete the proof of Proposition 2.4. □

3. Bourgain space

In this section, we introduce $X^{s,b}$ space and its associated properties. We first define $X^{s,b}$ space as follows: For $s, b \in \mathbb{R}$, $X^{s,b}$ space is defined to be the closure of Schwartz functions $\varphi \in \mathcal{S}(\mathbb{R}\times\mathbb{R})$ under the $X^{s,b}$-norm

$$\|\varphi\|_{X^{s,b}} := \|\langle\xi\rangle^s\langle\tau-\xi^3\rangle^b\widetilde{\varphi}(\tau,\xi)\|_{L^2(\mathbb{R}\times\mathbb{R})},$$

where $\langle a\rangle = 1+|a|$ and $\widetilde{\varphi}$ denotes the time-space Fourier transform of φ. The $X^{s,b}$ space was first introduced in its form by Bourgain [3] in the context of periodic nonlinear Schrödinger and gKdV equations, and has been developed by several researchers [22, 32]. As is well known, $X^{s,b}$ analysis is the most useful tool to reflect the dispersive effect of linear and nonlinear dispersive phenomena. In what follows we mention a few well-known properties of $X^{s,b}$ space. Let η be a smooth cutoff function supported on $[-2,2], \eta = 1$ on $[-1,1]$, and let $\eta_T(t) = \eta(t/T)$. The following lemma is the typical property (linear estimate) in the $X^{s,b}$ analysis.

Lemma 3.1. *Let $T \in (0,1)$ and $b \in \left(\frac{1}{2}, \frac{3}{2}\right]$. Then for $s \in \mathbb{R}$ and $\theta \in \left[0, \frac{3}{2}-b\right)$ the following hold:*

$$\|\eta_T(t)U(t)f\|_{X^{s,b}(\mathbb{R}\times\mathbb{R})} \lesssim T^{\frac{1}{2}-b}\|f\|_{H^s(\mathbb{R})},$$

$$\|\eta_T(t)\int_0^t U(t-t')\eta_T(t')F(t')dt'\|_{X^{s,b}(\mathbb{R}\times\mathbb{R})} \lesssim T^\theta\|F\|_{X^{s,b-1+\theta}(\mathbb{R}\times\mathbb{R})}.$$

Proof. Readers can find several references for the proof. We refer particularly Section 2.6 in [33]. □

The next lemma shows that $X^{s,b}$ space is a suitable tool to detect the local smoothing effect of linear KdV flow, and the proof follows immediately from Lemma 1.1 and Lemma 2.9 in [33].

Lemma 3.2. *Let (q,r) and α satisfy* (1.4). *Then for $b > \frac{1}{2}$ we have*

$$\|D^\alpha u\|_{L^q_xL^r_t(\mathbb{R}\times\mathbb{R})} \lesssim \|u\|_{X^{0,b}(\mathbb{R}\times\mathbb{R})}. \tag{3.1}$$

The starting point of the proof of the main proposition in the next section is the duality argument. In order to complete our analysis, we need an $L^q_xL^2_t$ estimate for a test function in the duality argument. The following lemma gives such an $L^q_xL^2_t$ estimate.

Lemma 3.3. *Let $q \geq 2$. Then for $b > \frac{1}{2}$ we have*

$$\|D^{1-\frac{1}{q}}u\|_{L^q_xL^2_t} \lesssim \|u\|_{X^{0,b(1-\frac{2}{q})}(\mathbb{R}\times\mathbb{R})}.$$

Proof. It follows directly from the interpolation between (3.1) and the trivial estimate $\|u\|_{L^2_{x,t}} \lesssim \|u\|_{X^{0,0}}$. We omit the detailed proof. □

If one considers the interaction of two different frequency localized data, it is possible to get the improved version of the bilinear local smoothing estimate. Throughout the paper we denote by $A(N)$ the set $\{\xi : N/2 \le |\xi| \le 2N\}$.

Lemma 3.4 (Lemma 2.2 in [23]). *Let $d \ge 2$. Suppose that* supp $\widehat{f} \subset A(N_1)$ *and* supp $\widehat{g} \subset A(N_2)$ *with $N_1 \le N_2$. Then,*

$$\|D^\alpha U(t) f D^\alpha U(t) g\|_{L_x^{\frac{q}{2}} L_t^{\frac{r}{2}}} \lesssim \Big(\frac{N_1}{N_2}\Big)^{\frac{5\theta}{12}} \|f\|_{L_x^2} \|g\|_{L_x^2},$$

where

$$-\alpha + \frac{1}{q} + \frac{3}{r} = \frac{1}{2}, \quad \Big(\frac{1}{q}, \frac{1}{r}\Big) = \Big(\frac{\theta}{6} + \frac{1-\theta}{4}, \frac{\theta}{6}\Big), 0 \le \theta \le 1.$$

In particular, we have

$$\|U(t) f U(t) g\|_{L_x^{\frac{5}{2}} L_t^{5}} \lesssim \Big(\frac{N_1}{N_2}\Big)^{\frac{1}{4}} \|f\|_{L_x^2} \|g\|_{L_x^2}.$$

The last lemma shows the *Transference Principle* of $X^{s,b}$ space.

Lemma 3.5. *Let $d \ge 2$. Consider $u, v \in X^{0,b}$ for $b > \frac{1}{2}$. If* supp $\widehat{u} \subset A(N_1)$ *and* supp $\widehat{v} \subset A(N_2)$ *with $N_1 \le N_2$, then*

$$\|uv\|_{L_x^{\frac{5}{2}} L_t^{5}} \lesssim \Big(\frac{N_1}{N_2}\Big)^{\frac{1}{4}} \|u\|_{X^{0,b}} \|v\|_{X^{0,b}}.$$

Proof. It is a multilinear version of Lemma 2.9 in [33]. Hence it follows from Lemma 3.4 above and Lemma 2.9 in [33]. □

4. Proof of the main theorem

In this section, we will give the proof of Theorem 1.3. We use (1.1) with initial data ϕ^ω. Let $z(t) := U(t)\phi^\omega$ and $v(t) := u(t) - U(t)\phi^\omega$. Then (1.1) becomes

$$\begin{cases} \partial_t v + \partial_x^3 v + F(v+z) = 0, \\ v(0,x) = 0 \in H^s. \end{cases} \tag{4.1}$$

By Duhamel's principle, (4.1) is written as an integral equation

$$v(t) = \eta_T(t) \int_0^t U(t-t')\eta_T(t') F(\eta_T(t')v + \eta_T(t')z)(t')dt'.$$

So we define $\mathcal{D}$ by

$$\mathcal{D}v(t) = \eta_T(t) \int_0^t U(t-t')\eta_T(t') F(\eta_T(t')v + \eta_T(t')z)(t')dt'. \tag{4.2}$$

The main task is to show that the map $\mathcal{D}$ is a contraction on a certain ball in $X^{\sigma,b}$ for some $\sigma > s_c$. The purpose of following proposition is to reduce Theorem 1.3.

Proposition 4.1. *Let $\max(\frac{1}{\kappa+1}\cdot(\frac{1}{2}-\frac{2}{\kappa}), \frac{1}{4}-\frac{2}{\kappa}) < s < \frac{1}{2}-\frac{2}{\kappa}$. Given $\phi \in H^s$, let ϕ^ω be its randomization. Then, there exist $\sigma(> s_c)$, $b = \frac{1}{2}+$ and $\theta = 0+$ such that for each small $T \ll 1$ and $R > 0$, we have*

$$\|\mathcal{D}v\|_{X^{\sigma,b}} \le C_1 T^\theta \big(\|v\|_{X^{\sigma,b}}^{\kappa+1} + R^{\kappa+1}\big),$$
$$\|\mathcal{D}v - \mathcal{D}w\|_{X^{\sigma,b}} \le C_2 T^\theta \big(\|v\|_{X^{\sigma,b}}^{\kappa} + \|w\|_{X^{\sigma,b}}^{\kappa} + R^\kappa\big)\|v-w\|_{X^{\sigma,b}},$$

outside a set of probability at most $C\exp\Big(-c\frac{R^2}{\|\phi\|_{H^s}^2}\Big)$.

We briefly sketch that Proposition 4.1 implies Theorem 1.3. We first assume to hold Proposition 4.1. Let B_1* be a ball of $X^{\sigma,b}$ given by

$$B_1 = \{v \in X^{\sigma,b} : \|v\|_{X^{\sigma,b}} \leq 1\}.$$

Then, for given $0 < T \ll 1$, it suffices to show that the map $\mathcal{D}$ is a contraction on B_1. By choosing $R = R(T) \sim T^{-\frac{\beta}{\kappa}}$ for some $\beta \in (0, \frac{\kappa\theta}{\kappa+1})$ such that

$$C_1 T^\theta (1 + R^{\kappa+1}) \leq 1 \quad \text{and} \quad C_2 T^\theta (2 + R^\kappa) \leq \frac{1}{2},$$

then, for $v, w \in B_1$ and $\gamma := \frac{2\beta}{\kappa}$, we have from Proposition 4.1 that

$$\|\mathcal{D}v\|_{X^{\sigma,b}} \leq 1,$$
$$\|\mathcal{D}v - \mathcal{D}w\|_{X^{\sigma,b}} \leq \frac{1}{2}\|v - w\|_{X^{\sigma,b}}$$

outside a set of probability at most

$$C \exp\Big(- c\frac{R^2}{\|\phi\|_{H^s}^2}\Big) = C \exp\Big(- \frac{c}{T^\gamma \|\phi\|_{H^s}^2}\Big).$$

Hence, we can define a set Ω_T as a complement of this set, and we can also find a unique solution v for the initial data ϕ^ω when $\omega \in \Omega_T$. Therefore, we complete the proof of Theorem 1.3.

Proof of Proposition 4.1. For given $\max(\frac{1}{\kappa+1} \cdot (\frac{1}{2} - \frac{2}{\kappa}), \frac{1}{4} - \frac{2}{\kappa}) < s < \frac{1}{2} - \frac{2}{\kappa}$, we choose σ, ε and $\widetilde{b}$ as follows:

(4.3)

$$\frac{1}{2} - \frac{3}{2(\kappa-1)} < \sigma < \min\left((\kappa+1)s, s + \frac{1}{4}\right),$$
$$0 < \varepsilon < \min\left(\frac{1}{2}(s + \frac{1}{4} - \sigma), \frac{1}{2}((\kappa+1)s - \sigma), \frac{\kappa}{3}(\sigma - (\frac{1}{2} - \frac{2}{\kappa})), (\kappa-1)(\sigma - (\frac{1}{2} - \frac{3}{2(\kappa-1)})\right),$$
$$\frac{1}{2} < \widetilde{b} < \frac{1}{2(1-2\varepsilon)}.$$

Let $b := \frac{1}{\kappa+3}(\frac{\kappa+4}{2} - \widetilde{b}(1-2\varepsilon))$ and $\theta := b - \frac{1}{2}$. We remark that $b > \frac{1}{2}$ and $\widetilde{b}(1-2\varepsilon) = 1 - b - (\kappa+2)\theta$. We only prove the first estimate, since the second estimate follows similarly. By using Lemma 3.1 and duality, we get

$$\begin{aligned}\|\mathcal{D}v(t)\|_{X^{\sigma,b}} &\lesssim T^{(\kappa+2)\theta} \|F(\eta_T v + \eta_T z)\|_{X^{\sigma, b-1+(\kappa+2)\theta}} \\ &= T^{(\kappa+2)\theta} \sup_{\|v_0\|_{X^{0,1-b-(\kappa+2)\theta}} \leq 1} \Big| \int\int_{\mathbb{R}\times\mathbb{R}} \langle D\rangle^\sigma [F(\eta_T v + \eta_T z)] v_0 dx dt\Big|.\end{aligned}$$

In the following, we ignore μ in $F(u)$, since it does not affect our analysis. Then Hölder inequality and Lemma 3.3 (with $\widetilde{b}$ as in (4.3)) yield

$$\begin{aligned}&\Big| \int\int_{\mathbb{R}\times\mathbb{R}} \langle D\rangle^\sigma \partial_x [(\eta_T v + \eta_T z)^{\kappa+1}] v_0 dx dt\Big| \\ &\lesssim \|D^{2\varepsilon} \langle D\rangle^\sigma (\eta_T v + \eta_T z)^{\kappa+1}\|_{L_x^{\frac{1}{1-\varepsilon}} L_t^2} \|D^{1-2\varepsilon} v_0\|_{L_x^{\frac{1}{\varepsilon}} L_t^2} \\ &\lesssim \|D^{2\varepsilon} \langle D\rangle^\sigma (\eta_T v + \eta_T z)^{\kappa+1}\|_{L_x^{\frac{1}{1-\varepsilon}} L_t^2} \|v_0\|_{X^{0,1-b-(\kappa+2)\theta}}.\end{aligned}$$

*Since v is in the sub-critical regularity space, it is enough to consider B_1 thanks to the scaling-rescaling argument.

First, we deal with $\|D^{2\varepsilon}\langle D\rangle^\sigma(\eta_T v)^{\kappa+1}\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2}$. By using the fractional Leibniz rule, the Hölder inequality and Lemma 3.2, we obtain

$$\begin{aligned}\|D^{2\varepsilon}\langle D\rangle^\sigma(\eta_T v)^{\kappa+1}\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2} &\lesssim \prod_{i=1}^{\kappa}\|v\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}}\|D^{2\varepsilon}\langle D\rangle^\sigma v\|_{L_x^{\frac{5}{1-2\varepsilon}}L_t^{\frac{10}{1+8\varepsilon}}}\\ &\lesssim \prod_{i=1}^{\kappa}\|v\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}}\|v\|_{X^{\sigma,b}}.\end{aligned}$$

Sobolev embedding and Lemma 3.2 give

$$\|v\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}} \lesssim \|D^{\frac{\kappa-4+4\varepsilon}{4\kappa}}v\|_{L_x^{\frac{20\kappa}{5\kappa-4+8\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}} \lesssim \|v\|_{X^{\sigma,b}},$$

for $\sigma > \frac{1}{2} - \frac{3}{2(\kappa-1)}, 0 < \varepsilon < \frac{\kappa}{3}(\sigma - (\frac{1}{2} - \frac{2}{\kappa}))$.

In order to handle the remaining terms, we make a dyadic decomposition for v, z. Hereafter we assume that the Fourier transforms of z_i, v_i are supported in a set $\{\xi : |\xi| \sim N_i\}$. Also, we drop the smooth cutoff function η_T from $\eta_T v_i$ and $\eta_T z_i$, and simply denote them by v_i and z_i, respectively. There exist two kinds of terms, a $z_1\cdots z_{\kappa+1}$-term and a $z_1\cdots z_j v_{j+1}\cdots v_{\kappa+1}$-term.

Case 1. $z_1\cdots z_{\kappa+1}$-term.

We may assume $N_1 \le \cdots \le N_{\kappa+1}$.

We first consider the case of $N_{\kappa+1} \le 1$. By using the fractional Leibniz rule, the Hölder inequality and Lemma 3.2, we obtain

$$\begin{aligned}\|D^{2\varepsilon}\langle D\rangle^\sigma(z_1\cdots z_{\kappa+1})\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2} &\lesssim \prod_{i=1}^{\kappa}\|z_i\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}}\|D^{2\varepsilon}\langle D\rangle^\sigma z_{\kappa+1}\|_{L_x^{\frac{5}{1-2\varepsilon}}L_t^{\frac{10}{1+8\varepsilon}}}\\ &\lesssim \prod_{i=1}^{\kappa}\|z_i\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}}\|z_{\kappa+1}\|_{X^{\sigma,b}}\\ &\sim \prod_{i=1}^{\kappa}\|z_i\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}}\|z_{\kappa+1}\|_{X^{0,b}}.\end{aligned}$$

Moreover, Sobolev embedding and Lemma 3.2 give

$$\|z_i\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}} \lesssim \|D^{\frac{\kappa-4+4\varepsilon}{4\kappa}}z_i\|_{L_x^{\frac{20\kappa}{5\kappa-4+8\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}} \lesssim \|z_i\|_{X^{\sigma,b}} \sim \|z_i\|_{X^{0,b}},$$

for $\sigma > \frac{1}{2} - \frac{3}{2(\kappa-1)}, 0 < \varepsilon < \frac{\kappa}{3}(\sigma - (\frac{1}{2} - \frac{2}{\kappa}))$. Then sums over $N_1, \cdots, N_{\kappa+1}$ and Lemma 3.1 give

$$\sum_{N_{\kappa+1}\le 1}\cdots\sum_{0<N_1\le N_2}\|D^{2\varepsilon}\langle D\rangle^\sigma(z_1\cdots z_{\kappa+1})\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2} \lesssim T^{-(\kappa+1)\theta}\|\phi^\omega\|_{H^s}^{\kappa+1}.$$

Now, we focus on the case of $N_{\kappa+1} > 1$. By using the fractional Leibniz rule and the Hölder inequality, we have that

$$\begin{aligned}\|D^{2\varepsilon}\langle D\rangle^\sigma(z_1\cdots z_{\kappa+1})\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2} &\lesssim \|z_1(D^{2\varepsilon}\langle D\rangle^\sigma z_{\kappa+1})\|_{L_x^{\frac{5}{2}}L_t^5}\|z_2\cdots z_\kappa\|_{L_x^{\frac{5}{3-5\varepsilon}}L_t^{\frac{10}{3}}}\\ &\lesssim \|z_1(D^{2\varepsilon}\langle D\rangle^\sigma z_{\kappa+1})\|_{L_x^{\frac{5}{2}}L_t^5}\prod_{i=2}^{\kappa}\|z_i\|_{L_x^{\frac{5(\kappa-1)}{3-5\varepsilon}}L_t^{\frac{10(\kappa-1)}{3}}}.\end{aligned}$$

Lemma 3.5 and Lemma 3.1 give

$$\begin{aligned}\|z_1(D^{2\varepsilon}\langle D\rangle^\sigma z_{\kappa+1})\|_{L_x^{\frac52}L_t^5} &\lesssim N_{\kappa+1}^{\sigma-\frac14-s+2\varepsilon}N_1^{\frac14}\|z_{\kappa+1}\|_{X^{s,b}}\|z_1\|_{X^{0,b}}\\ &\lesssim T^{-2\theta}N_{\kappa+1}^{\sigma-\frac14-s+2\varepsilon}N_1^{\frac14}\|P_{N_{\kappa+1}}\phi^\omega\|_{H^s}\|P_{N_1}\phi^\omega\|_{L^2}.\end{aligned}$$

If $N_1>1$, we obtain

$$\begin{aligned}\|\langle D\rangle^{\sigma+2\varepsilon}(z_1\cdots z_{\kappa+1})\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2} &\lesssim T^{-2\theta}N_{\kappa+1}^{\sigma-\frac14-s+2\varepsilon}N_1^{\frac14-s}\|P_{N_1}\phi^\omega\|_{H^s}\|P_{N_{\kappa+1}}\phi^\omega\|_{H^s}\\ &\quad\times\prod_{i=2}^{\kappa}N_i^{-s-\frac{\varepsilon}{\kappa-1}}\|D^{\frac{\varepsilon}{\kappa-1}}\langle D\rangle^s z_i\|_{L_x^{\frac{5(\kappa-1)}{3-5\varepsilon}}L_t^{\frac{10(\kappa-1)}{3}}}.\end{aligned}$$

Since

$$0<\sigma<(\kappa+1)s$$

and

$$0<\varepsilon<\frac12((\kappa+1)s-\sigma),$$

we have that $N_{\kappa+1}^{\sigma-\frac14-s+2\varepsilon}N_1^{\frac14-s}\prod_{i=2}^{\kappa}N_i^{-s-\frac{\varepsilon}{\kappa-1}}$ is summable over $1<N_1\le\cdots\le N_{\kappa+1}$. Hence we conclude from Lemma 2.2 and Proposition 2.4 that

$$\|\langle D\rangle^{\sigma+2\varepsilon}(\eta_T z)^{\kappa+1}\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2}\lesssim T^{-2\theta}R^{\kappa+1}$$

outside a set of probability at most $C\exp\Big(-c\frac{R^2}{\|\phi\|_{H^s}^2}\Big)$.

Otherwise, let $\widetilde\kappa$ be a number such that $N_{\widetilde\kappa}\le1$ and $N_{\widetilde\kappa+1}>1$. Then we get

$$\begin{aligned}&\|D^{2\varepsilon}\langle D\rangle^\sigma(z_1\cdots z_{\kappa+1})\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2}\\ &\lesssim T^{-2\theta}N_{\kappa+1}^{\sigma-\frac14-s+2\varepsilon}N_1^{\frac14}\|P_{N_1}\phi^\omega\|_{L^2}\|P_{N_{\kappa+1}}\phi^\omega\|_{H^s}\\ &\times\prod_{i=2}^{\widetilde\kappa}N_i^{-\frac{\varepsilon}{\kappa-1}}\|D^{\frac{\varepsilon}{\kappa-1}}z_i\|_{L_x^{\frac{5(\kappa-1)}{3-5\varepsilon}}L_t^{\frac{10(\kappa-1)}{3}}}\prod_{i=\widetilde\kappa+1}^{\kappa}N_i^{-s-\frac{\varepsilon}{\kappa-1}}\|D^{\frac{\varepsilon}{\kappa-1}}\langle D\rangle^s z_i\|_{L_x^{\frac{5(\kappa-1)}{3-5\varepsilon}}L_t^{\frac{10(\kappa-1)}{3}}}.\end{aligned}$$

By choosing appropriate σ and ε as $0<\sigma<s+\frac14$ and $0<\varepsilon<\frac12(s+\frac14-\sigma)$, we know that

$$N_1^{\frac14}\prod_{i=2}^{\widetilde\kappa}N_i^{-\frac{\varepsilon}{\kappa-1}}\quad\text{and}\quad N_{\kappa+1}^{\sigma-\frac14-s+2\varepsilon}\prod_{i=\widetilde\kappa+1}^{\kappa}N_i^{-s-\frac{\varepsilon}{\kappa-1}}$$

are ℓ^p-summable for any $p\ge1$ over $0<N_1\le\cdots\le N_{\widetilde\kappa}\le1$ and $1<N_{\widetilde\kappa+1}\le\cdots\le N_{\kappa+1}$, respectively. Hence, we have

(4.4)

$$\begin{aligned}&\|D^{2\varepsilon}\langle D\rangle^\sigma(z_1\cdots z_{\kappa+1})\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2}\\ &\lesssim\sum_{0<N_1\le\cdots\le N_{\widetilde\kappa}\le1<N_{\widetilde\kappa+1}\le\cdots\le N_{\kappa+1}}T^{-2\theta}N_{\kappa+1}^{\sigma-\frac14-s+2\varepsilon}N_1^{\frac14}\|P_{N_1}\phi^\omega\|_{L^2}\|P_{N_{\kappa+1}}\phi^\omega\|_{H^s}\\ &\quad\times\prod_{i=2}^{\widetilde\kappa}N_i^{-\frac{\varepsilon}{\kappa-1}}\|D^{\frac{\varepsilon}{\kappa-1}}z_i\|_{L_x^{\frac{5(\kappa-1)}{3-5\varepsilon}}L_t^{\frac{10(\kappa-1)}{3}}}\prod_{i=\widetilde\kappa+1}^{\kappa}N_i^{-s-\frac{\varepsilon}{\kappa-1}}\|D^{\frac{\varepsilon}{\kappa-1}}\langle D\rangle^s z_i\|_{L_x^{\frac{5(\kappa-1)}{3-5\varepsilon}}L_t^{\frac{10(\kappa-1)}{3}}}\\ &\lesssim T^{-2\theta}\|\phi^\omega\|_{L^2}\|\phi^\omega\|_{H^s}\|D^{\frac{\varepsilon}{\kappa-1}}z\|^{\widetilde\kappa-1}_{L_x^{\frac{5(\kappa-1)}{3-5\varepsilon}}L_t^{\frac{10(\kappa-1)}{3}}}\|D^{\frac{\varepsilon}{\kappa-1}}\langle D\rangle^s z\|^{\kappa-\widetilde\kappa}_{L_x^{\frac{5(\kappa-1)}{3-5\varepsilon}}L_t^{\frac{10(\kappa-1)}{3}}}.\end{aligned}$$

In conclusion, from Lemma 2.2 and Proposition 2.4, we have

$$\|D^{2\varepsilon}\langle D\rangle^{\sigma}(\eta_T z)^{\kappa+1}\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2} \lesssim T^{-2\theta}R^{\kappa+1}$$

outside a set of probability at most $C\exp\Big(-c\frac{R^2}{\|\phi\|_{H^s}^2}\Big)$.

Case 2. $z_1\cdots z_j v_{j+1}\cdots v_{\kappa+1}$-term

We may assume $N_1 \le \cdots \le N_j$ and $N_{j+1} \le \cdots \le N_{\kappa+1}$.

⋄ *Subcase* 1: $N_j \le N_{\kappa+1}$

By using the Hölder inequality, we have

$$\begin{aligned}
&\|z_1\cdots z_j v_{j+1}\cdots v_\kappa(D^{2\varepsilon}\langle D\rangle^{\sigma}v_{\kappa+1})\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2}\\
&\lesssim \prod_{i_1=1}^{j}\|z_{i_1}\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}}\prod_{i_2=j+1}^{\kappa}\|v_{i_2}\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}}\|D^{2\varepsilon}\langle D\rangle^{\sigma}v_{\kappa+1}\|_{L_x^{\frac{5}{1-2\varepsilon}}L_t^{\frac{10}{1+8\varepsilon}}}\\
&\lesssim \prod_{i_1=1}^{j}\|z_{i_1}\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}}\prod_{i_2=j+1}^{\kappa}\|v_{i_2}\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}}\|v_{\kappa+1}\|_{X^{\sigma,b}}.
\end{aligned}$$

Let $\widetilde{j}$ be a number such that $N_{\widetilde{j}} \le 1$ and $N_{\widetilde{j}+1} > 1$. When $i_1 \le \widetilde{j}$, by using Sobolev embedding, Lemma 3.2 and Lemma 3.1, we have

$$\begin{aligned}
\|z_{i_1}\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}} &\lesssim \|D^{\frac{\kappa-4+4\varepsilon}{4\kappa}}z_{i_1}\|_{L_x^{\frac{20\kappa}{5\kappa-4+8\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}} \lesssim \|z_{i_1}\|_{X^{\sigma,b}}\\
&\lesssim T^{-\theta}\|P_{N_{i_1}}\phi^{\omega}\|_{H^s}.
\end{aligned}\tag{4.5}$$

Otherwise ($i_1 > \widetilde{j}$), we have from the Bernstein inequality that

$$\|z_{i_1}\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}} \lesssim N_{i_1}^{\frac{\varepsilon}{\kappa}-s}\|D^{-\frac{\varepsilon}{\kappa}}\langle D\rangle^{s}z_{i_1}\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}}.\tag{4.6}$$

We note that $N_{i_1}^{\frac{\varepsilon}{\kappa}-s}$ is summable over $N_{i_1} > 1$.

For v_{i_2}, since $\frac{1}{2}-\frac{2}{\kappa}+\frac{3\varepsilon}{\kappa} < \sigma$, we always have from Lemma 3.2 that

$$\|v_{i_2}\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}} \lesssim \|D^{\frac{\kappa-4+4\varepsilon}{4\kappa}}v_{i_2}\|_{L_x^{\frac{20\kappa}{5\kappa-4+8\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}} \lesssim \|v_{i_2}\|_{X^{\sigma,b}}.\tag{4.7}$$

Then, we carry out summations of (4.5), (4.6) and (4.7) and apply Lemma 2.2 and Proposition 2.4 to get

$$\sum\|z_1\cdots z_j v_{j+1}\cdots v_\kappa\langle D\rangle^{\sigma+2\varepsilon}v_{\kappa+1}\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2} \lesssim T^{-\widetilde{j}\theta}R^j\|v\|_{X^{\sigma,b}}^{\kappa+1-j}$$

outside a set of probability at most $C\exp\Big(-c\frac{R^2}{\|\phi\|_{H^s}^2}\Big)$.

⋄ *Subcase* 2: $N_j \ge N_{\kappa+1}$

We first consider the case of $N_j \le 1$. In this case, since there is no effect of derivatives $D^{2\epsilon}\langle D\rangle^{\sigma}$, by using the fractional Leibniz rule, the Hölder inequality, Lemma 3.2, (4.5) and (4.7), we obtain

$$
\begin{aligned}
&\|z_1 \cdots z_{j-1}(D^{2\varepsilon}\langle D\rangle^\sigma z_j)v_{j+1}\cdots v_{\kappa+1}\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2}\\
&\lesssim \prod_{i_1=1}^{j-1}\|z_{i_1}\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}}\|D^{2\varepsilon}\langle D\rangle^\sigma z_j\|_{L_x^{\frac{5}{1-2\varepsilon}}L_t^{\frac{10}{1+8\varepsilon}}}\prod_{i_2=j+1}^{\kappa+1}\|v_{i_2}\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}}\\
&\lesssim \prod_{i_1=1}^{j-1}\|z_{i_1}\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}}\|z_j\|_{X^{0,b}}\prod_{i_2=j+1}^{\kappa+1}\|v_{i_2}\|_{L_x^{\frac{5\kappa}{4-3\varepsilon}}L_t^{\frac{5\kappa}{2-4\varepsilon}}}.
\end{aligned}
$$

Then, we carry out summations and apply Lemma 2.2 and Proposition 2.4 to get

$$
\begin{aligned}
&\sum\|z_1 \cdots z_{j-1}(D^{2\varepsilon}\langle D\rangle^\sigma z_j)v_{j+1}\cdots v_{\kappa+1}\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2}\\
&\lesssim T^{-j\theta}R^j\|v\|_{X^{\sigma,b}}^{\kappa+1-j}
\end{aligned}
$$

outside a set of probability at most $C\exp\Big(-c\frac{R^2}{\|\phi\|_{H^s}^2}\Big)$.

Now in what follows, we assume $N_j > 1$. We consider two cases separately:

- $j \geq 2$.
- $j = 1$.

When $j \geq 2$, by using the Hölder inequality, we get

$$
\begin{aligned}
&\|z_1 \cdots z_{j-1}(D^{2\varepsilon}\langle D\rangle^\sigma z_j)v_{j+1}\cdots v_{\kappa+1}\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2}\\
&\lesssim \|z_1(D^{2\varepsilon}\langle D\rangle^\sigma z_j)\|_{L_x^{\frac{5}{2}}L_t^5}\prod_{i_1=2}^{j-1}\|z_{i_1}\|_{L_x^{\frac{5(\kappa-1)}{3-5\varepsilon}}L_t^{\frac{10(\kappa-1)}{3}}}\prod_{i_2=j+1}^{\kappa+1}\|v_{i_2}\|_{L_x^{\frac{5(\kappa-1)}{3-5\varepsilon}}L_t^{\frac{10(\kappa-1)}{3}}}.
\end{aligned}
$$

From Sobolev embedding and Lemma 3.2, we know that

$$
\|v_{i_2}\|_{L_x^{\frac{5(\kappa-1)}{3-5\varepsilon}}L_t^{\frac{10(\kappa-1)}{3}}} \lesssim \|D^{\frac{\kappa-4+4\varepsilon}{4(\kappa-1)}}v_{i_2}\|_{L_x^{\frac{20(\kappa-1)}{5\kappa-8}}L_t^{\frac{10(\kappa-1)}{3}}} \lesssim \|v_{i_2}\|_{X^{\sigma,b}} \tag{4.8}
$$

for $\sigma > \frac{1}{2} - \frac{3}{2(\kappa-1)}$ and $0 < \varepsilon < (\kappa-1)(\sigma - (\frac{1}{2} - \frac{3}{2(\kappa-1)})$. In addition to (4.8), we use similar argument as in (4.4) for $\|z_1(D^{2\varepsilon}\langle D\rangle^\sigma z_j)\|_{L_x^{\frac{5}{2}}L_t^5}\prod_{i_1=1}^{j-1}\|z_{i_2}\|_{L_x^{\frac{5(\kappa-1)}{3-5\varepsilon}}L_t^{\frac{10(\kappa-1)}{3}}}$ to obtain that

$$
\|z_1 \cdots z_{j-1}(D^{2\varepsilon}\langle D\rangle^\sigma z_j)v_{j+1}\cdots v_{\kappa+1}\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2} \lesssim T^{-2\theta}R^j\|v\|_{X^{\sigma,b}}^{\kappa+1-j}
$$

outside a set of probability at most $C\exp\Big(-c\frac{R^2}{\|\phi\|_{H^s}^2}\Big)$.

When $j = 1$, from the Hölder inequality, we get

$$
\begin{aligned}
&\|(D^{2\varepsilon}\langle D\rangle^\sigma z_1)v_2\cdots v_{\kappa+1}\|_{L_x^{\frac{1}{1-\varepsilon}}L_t^2}\\
&\lesssim \|v_{\kappa+1}(D^{2\varepsilon}\langle D\rangle^\sigma z_1)\|_{L_x^{\frac{5}{2}}L_t^5}\prod_{i_2=2}^{\kappa+1}\|v_{i_2}\|_{L_x^{\frac{5(\kappa-1)}{3-5\varepsilon}}L_t^{\frac{10(\kappa-1)}{3}}}.
\end{aligned}
$$

Lemma 3.5 and Lemma 3.1 give

$$
\begin{aligned}
\|v_{\kappa+1}(D^{2\varepsilon}\langle D\rangle^\sigma z_1)\|_{L_x^{\frac{5}{2}}L_t^5} &\lesssim N_1^{\sigma-\frac{1}{4}-s+2\varepsilon}N_{\kappa+1}^{\frac{1}{4}}\|z_1\|_{X^{s,b}}\|v_{\kappa+1}\|_{X^{0,b}}\\
&\lesssim T^{-\theta}N_1^{\sigma-\frac{1}{4}-s+2\varepsilon}N_{\kappa+1}^{\frac{1}{4}}\|P_{N_1}\phi^\omega\|_{H^s}\|v_{\kappa+1}\|_{X^{0,b}}.
\end{aligned}
$$

When $N_{\kappa+1} \leq 1$, thanks to the conditions of σ and ε as $0 < \sigma < \frac{1}{4} + s$ and $0 \leq \varepsilon \leq \frac{1}{2}(s + \frac{1}{4} - \sigma)$, we have

$$\sum_{N_{\kappa+1} \leq 1} \sum_{1 \leq N_1} \|v_{\kappa+1}(D^{2\varepsilon}\langle D\rangle^{\sigma} z_1)\|_{L_x^{\frac{5}{2}} L_t^5} \lesssim T^{-\theta} \|\phi^{\omega}\|_{H^s} \|v\|_{X^{\sigma,b}}.$$

Otherwise ($N_{\kappa+1} > 1$), since we have further derivative gain ($N_{\kappa+1}^{-\sigma}$), we also obtain

$$\begin{aligned} &\sum_{1 < N_{\kappa+1}} \sum_{N_{\kappa+1} \leq N_1} \|v_{\kappa+1}(D^{2\varepsilon}\langle D\rangle^{\sigma} z_1)\|_{L_x^{\frac{5}{2}} L_t^5} \\ &\lesssim \sum_{1 < N_{\kappa+1}} \sum_{N_{\kappa+1} \leq N_1} T^{-\theta} N_1^{\sigma - \frac{1}{4} - s + 2\varepsilon} N_{\kappa+1}^{\frac{1}{4} - \sigma} \|P_{N_1}\phi^{\omega}\|_{H^s} \|v_{\kappa+1}\|_{X^{\sigma,b}} \\ &\lesssim T^{-\theta} \|\phi^{\omega}\|_{H^s} \|v\|_{X^{\sigma,b}} \end{aligned}$$

for $\sigma < s + \frac{1}{4}$ and $0 < \varepsilon < \frac{1}{2}(s + \frac{1}{4} - \sigma)$. Then, similarly to the case when $j \geq 2$, we have from (4.8) and the same argument as in (4.4) that

$$\sum \|z_1 \cdots z_{j-1}(D^{2\varepsilon}\langle D\rangle^{\sigma} z_j) v_{j+1} \cdots v_{\kappa+1}\|_{L_x^{\frac{1}{1-\varepsilon}} L_t^2} \lesssim T^{-2\theta} R^j \|v\|_{X^{\sigma,b}}^{\kappa+1-j}$$

outside a set of probability at most $C \exp\Big(-c\frac{R^2}{\|\phi\|_{H^s}^2}\Big)$. This completes the proof of Proposition 4.1. $\square$

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