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GLOBAL-IN-TIME SMOOTHING EFFECTS FOR SCHRÖDINGER EQUATIONS WITH INVERSE-SQUARE POTENTIALS

HARUYA MIZUTANI

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Abstract. The purpose of this note is to prove global-in-time smoothing effects for the Schrödinger equation with potentials exhibiting critical singularity. A typical example of admissible potentials is the inverse-square potential $a|x|^{-2}$ with $a > -(n-2)^2/4$. This particularly gives an affirmative answer to a question raised by T. A. Bui et al. (J. Differential Equations **262** (2017), 2771–2807). The proof employs a uniform resolvent estimate proved by Barceló, Vega, and Zubeldia (Adv. Math. **240** (2013), 636–671) an abstract perturbation method by Bouclet and Mizutani (preprint).

1. Introduction

This note is concerned with smoothing properties of the time-dependent Schrödinger equation

$$i\partial_t u(t,x) = Hu(t,x) + F(t,x); \quad u(0,x) = \psi(x), \tag{1.1}$$

with given data $\psi \in L^2(\mathbb{R}^n)$ and $F \in L^1_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^n))$, where $H = -\Delta + V(x)$ is a Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$, with a real-valued function V which decays at spatial infinity in a suitable sense and has a critical singularity at the origin. A typical example of potentials we have in mind is the inverse-square potential $V(x) = a|x|^{-2}$ satisfying $a > -(n-2)^2/4$.

Let us first recall several known results for the free case, describing the motivation of this paper. It is well-known that the solution $u = e^{it\Delta}\psi$ to the free Schrödinger equation

$$i\partial_t u(t,x) = -\Delta u(t,x); \quad u|_{t=0} = \psi \in L^2(\mathbb{R}^n),$$

satisfies the global-in-time smoothing effect

$$||\langle x\rangle^{-\rho}|D|^{1/2}e^{it\Delta}\psi||_{L^2(\mathbb{R}^{1+n})} \leq C||\psi||_{L^2(\mathbb{R}^n)}, \tag{1.2}$$

where $\langle x\rangle = (1+|x|^2)^{1/2}$, $\rho > 1/2$ and $|D| = (-\Delta)^{1/2}$ (see Ben-Artzi and Klainerman [2] for $n \geq 3$ and Chihara [8] for $n = 2$). When $n \geq 3$, the estimate of the form

$$||w(x)e^{it\Delta}\psi||_{L^2(\mathbb{R}^{1+n})} \leq C||w||_{L^n(\mathbb{R}^n)}||\psi||_{L^2(\mathbb{R}^n)} \tag{1.3}$$

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was proved by Kato and Yajima [18]. The estimate (1.3) also follows from Hölder's inequality and the endpoint Strichartz estimate proved by Keel and Tao [19]:

$$||e^{it\Delta}\psi||_{L^2(\mathbb{R};L^{\frac{2n}{n-2}}(\mathbb{R}^n))} \le C||\psi||_{L^2(\mathbb{R}^n)}, \tag{1.4}$$

which can also be regarded as a smoothing property in L^p-spaces. All of these three estimates are fundamental tools in the study of the Cauchy problem and scattering theory for both linear and nonlinear Schrödinger equations (see [7,17,18,27,28] and references therein). It is also worth noting that (1.2) and (1.3) are closely connected with uniform estimates for the resolvent $(-\Delta - z)^{-1}$ with respect to $z \in \mathbb{C} \setminus [0,\infty)$ (see the next section for more details).

There is a vast literature on extending estimates (1.2)–(1.4) to the Schrödinger operator $H = -\Delta + V$ with potential $V(x)$. For the case when V has enough regularity and decays sufficiently fast at spatial infinity, we refer to [12,13,22,24,26] and references therein. There are also several results in the case when V has critical singularity. In particular, the Schrödinger operator with the inverse-square potential of the form

$$H_a = -\Delta + a|x|^{-2}, \quad a > -\frac{(n-2)^2}{4},$$

has attracted increasing attention since it represents a borderline case for the validity of (1.2)–(1.4) ([11,14]), where we note that $(n-2)^2/4$ is the best constant in Hardy's inequality

$$\frac{(n-2)^2}{4}\int |x|^{-2}|u|^2 dx \le \int |\nabla u|^2 dx, \quad u \in C_0^\infty(\mathbb{R}^n). \tag{1.5}$$

We refer to [1,3,5,6] for Kato–Yajima type estimates (1.3) and to [3,5,6,23] for Strichartz estimates (1.4). Concerning the estimate (1.2), in a recent paper [4], the authors showed, among others, the following theorem.

Theorem 1.1 ([4, Theorem 1.2])**.** *Let $n \ge 3$, $a \ge -(n-2)^2/4 + 1/4$ and $\varepsilon > 0$. Then there exists $C_\varepsilon > 0$ such that for all $\psi \in L^2(\mathbb{R}^n)$, $e^{-itH_a}\psi$ satisfies*

$$||w(|x|)|D|^{1/2}e^{-itH_a}\psi||_{L^2(\mathbb{R}^{1+n})} \le C_\varepsilon||\psi||_{L^2(\mathbb{R}^n)} \tag{1.6}$$

where $w(r) = r^{(\varepsilon-1)/2}(1+r^\varepsilon)^{-1}$.

The condition $a \ge -(n-2)^2/4 + 1/4$ was used to ensure that w satisfies some conditions for two-sided weighted norm estimates

$$C_1||w(x)(-\Delta)^{s/2}f||_{L^2} \le ||w(x)H_a^{s/2}f||_{L^2} \le C_2||w(x)(-\Delta)^{s/2}f||_{L^2}$$

established by the same paper (see [4, Theorem 1.1]). Then the authors raised a question whether (1.6) holds under the condition $a > -(n-2)^2/4$. The main purpose of the present note is to give an affirmative answer to this question. More precisely, we prove global-in-time smoothing effects of the form (1.6) for Schrödinger operators $H = -\Delta + V(x)$ with a large class of real-valued potentials which particularly includes the inverse-square potential with $a > -(n-2)^2/4$. Furthermore, global-in-time smoothing effects for the solution to (1.1) with the inhomogeneous term F are also studied for the same class of potentials. The proofs are based on an abstract perturbation method from our previous work [3] and a uniform estimate proved by [1] for the weighted resolvent $|x|^{-1}(H-z)^{-1}|x|^{-1}$ with respect to $z \in \mathbb{C} \setminus [0,\infty)$.

In order to state the main results, we introduce some notation. From now on we let $n \geq 3$ and impose the following condition:

Assumption A. *Let $V(x)$ be a real-valued function on $\mathbb{R}^n$ such that $|x|V \in L^{n,\infty}(\mathbb{R}^n)$ and $x \cdot \nabla V \in L^{n/2,\infty}(\mathbb{R}^n)$. Moreover, there exists $\delta > 0$ such that $-\Delta + V \geq -\delta\Delta$ and $-\Delta - V - x \cdot \nabla V \geq -\delta\Delta$ in the sense of forms; that is, for all $u \in C_0^\infty(\mathbb{R}^n)$,*

$$\langle(-\Delta + V)u, u\rangle \geq \delta||\nabla u||_{L^2}, \quad \langle(-\Delta - V - x \cdot (\nabla V))u, u\rangle \geq \delta||\nabla u||_{L^2}. \tag{1.7}$$

Here $\langle f, g\rangle = \int f\overline{g}dx$ is the inner product in $L^2(\mathbb{R}^n)$ and $L^{p,q}$ is the Lorentz space (see the end of this section). A typical example satisfying Assumption A is the inverse-square potential $V(x) = a|x|^{-2}$ with $a > -(n-2)^2/4$. In this case, it follows from Hardy's inequality (1.5) that (1.7) is satisfied with $\delta = 1 - 4|a|/(n-2)^2 > 0$ if $a < 0$ or $\delta = 1$ if $a \geq 0$. Moreover, Assumption A is general enough to include some potentials such that $|x|^2 V \notin L^\infty$. For instance, we let $c_1, c_2 > 0$, $\alpha \in \mathbb{R}^n$ and $\chi \in C^1(\mathbb{R})$ such that $0 \leq \chi \leq 1$ and $|\chi^{(k)}(t)| \leq |t|^{-k-1}$ for $|t| \geq 1$. Then

$$V(x) = \frac{-(n-2)^2/4 + c_1}{|x|^2} - \frac{c_2\chi(|x-\alpha|)}{|x-\alpha|}$$

satisfies Assumption A with $\delta = c_1 - c_2(2 + \sup|\chi'|)(|\alpha| + 1)$ if

$$0 < c_2 < \frac{c_1}{(2 + \sup|\chi'|)(|\alpha| + 1)}.$$

Under Assumption A, Hardy's inequality (1.5) implies that the sesquilinear form

$$Q_H(u, v) = \langle(-\Delta + V)u, v\rangle, \quad u, v \in C_0^\infty(\mathbb{R}^n),$$

is symmetric, non-negative and closable such that the domain of its closure $\overline{Q}_H$ satisfies $D(\overline{Q}_H) = \mathcal{H}^1(\mathbb{R}^n)$. Let H be the Friedrichs extension of $\overline{Q}_H$, e^{-itH} the unitary group on $L^2(\mathbb{R}^n)$ generated by H via Stone's theorem and Γ_H the inhomogeneous propagator defined by

$$\Gamma_H F(t) = \int_0^t e^{-i(t-s)H} F(s)ds, \quad F \in L^1_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^n)). \tag{1.8}$$

Then a unique (mild) solution to the Schrödinger equation (1.1) is given by

$$u(t) = e^{-itH}\psi - i\Gamma_H F(t). \tag{1.9}$$

We say that $v(x)$ belongs to the Muckenhoupt A_2 class if $v, 1/v \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $v \geq 0$ and

$$\Big(\frac{1}{|B|}\int_B v(x)dx\Big)\Big(\frac{1}{|B|}\int_B \frac{1}{v(x)}dx\Big) \leq C$$

for all balls $B \subset \mathbb{R}^n$ with some constant $C > 0$ independent of B.

The main result in this paper then is as follows.

Theorem 1.2. *Let $n \geq 3$, V satisfy Assumption* A *and $w \in L^2(\mathbb{R})$. Suppose $w(|x|)^2 \in A_2$ and, for any $j = 1, 2, \ldots, n$, there exists $C_j > 0$ such that*

$$w(|x|) \leq C_j w(x_j), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Then there exists $C > 0$, independent of w, such that e^{-itH} satisfies

$$||w(|x|)|D|^{1/2}e^{-itH}\psi||_{L^2(\mathbb{R}^{1+n})} \leq C||w||_{L^2(\mathbb{R})}||\psi||_{L^2(\mathbb{R}^n)}, \quad \psi \in L^2(\mathbb{R}^n).$$

Assuming $0 < \varepsilon < 1$ without loss of generality, it is easy to see that $w(r) = r^{(\varepsilon-1)/2}(1+r^\varepsilon)^{-1}$ fulfills the above conditions. Another typical example of w is $\langle x\rangle^{-\rho}$, in which case we have

Theorem 1.3. *Let $n \ge 3$, $\rho > 1/2$ and $\mathcal{A}, \mathcal{B} \in \{\dot{\mathcal{H}}^{-1/2,\rho}(\mathbb{R}^n),\ L^{\frac{2n}{n+2},2}(\mathbb{R}^n)\}$. Suppose V satisfies Assumption* A. *Then the solution u to* (1.1) *given by* (1.9) *satisfies*

$$||u||_{L^2(\mathbb{R};\mathcal{B}^*)} \le C||\psi||_{L^2(\mathbb{R}^n)} + C||F||_{L^2(\mathbb{R};\mathcal{A})} \tag{1.10}$$

for all $\psi \in L^2(\mathbb{R}^n)$ and $F \in L^1_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R};\mathcal{A})$.

Here $\dot{\mathcal{H}}^{s,\mu}(\mathbb{R}^n)$ denotes the weighted homogeneous Sobolev space equipped with the norm $||f||_{\dot{\mathcal{H}}^{s,\mu}} = ||\langle x\rangle^\mu |D|^s f||_{L^2}$. Note that $(L^{\frac{2n}{n+2},2}(\mathbb{R}^n))^* = L^{\frac{2n}{n-2},2}(\mathbb{R}^n)$ and $(\dot{\mathcal{H}}^{-1/2,\rho}(\mathbb{R}^n))^* = \dot{\mathcal{H}}^{1/2,-\rho}(\mathbb{R}^n)$. If $\mathcal{A} = \mathcal{B} = L^{\frac{2n}{n+2},2}(\mathbb{R}^n)$, (1.10) becomes the endpoint Strichartz estimate and was proved by our previous work [3]. If $\mathcal{A} = \mathcal{B} = \dot{\mathcal{H}}^{-1/2,\rho}(\mathbb{R}^n)$, (1.10) is a generalization of (1.2) and seems to be new under Assumption A. Here we stress that $\mathcal{A}$ and $\mathcal{B}$ do not have to coincide.

Notation. Throughout the paper we use the following notation. For $T > 0$ and a Banach space $\mathcal{X}$, we denote $||F||_{L^p_T\mathcal{X}} = ||F||_{L^p([-T,T];\mathcal{X})}$. $L^{p,q}(\mathbb{R}^n)$ denotes the Lorentz space equipped with the norm $||\cdot||_{L^{p,q}(\mathbb{R}^n)}$ satisfying

$$||f||_{L^{p,q}(\mathbb{R}^n)} \sim ||td_f(t)^{1/p}||_{L^q(\mathbb{R}_+,t^{-1}dt)},$$

where $d_f(t) := \mu(\{x \in \mathbb{R}^n \mid |f(x)| > t\})$ is the distribution function of f. We use the convention $L^{\infty,\infty} = L^\infty$. For $1 \le p, p_1, p_2 < \infty$ and $1 \le q, q_1, q_2 \le \infty$ satisfying $1/p = 1/p_1 + 1/p_2$, $1/q = 1/q_1 + 1/q_2$, we have Hölder's inequality for Lorentz spaces:

$$||fg||_{L^{p,q}} \le C||f||_{L^{p_1,q_1}}||g||_{L^{p_2,q_2}}, \quad ||fg||_{L^{p,q}} \le C||f||_{L^\infty}||g||_{L^{p,q}}. \tag{1.11}$$

When $n \ge 3$, we also have Sobolev's inequality in Lorentz spaces:

$$||f||_{L^{\frac{2n}{n-2},2}} \le C||\nabla f||_{L^2}, \quad f \in \mathcal{H}^1. \tag{1.12}$$

We refer to [15] for more details on Lorentz spaces. In what follows we often omit $\mathbb{R}^n$ from $L^p(\mathbb{R}^n)$ and so on, if there is no confusion.

The rest of the paper is organized as follows. We first recall in the next section the abstract perturbation method developed in [3], which plays an important role in the proof of the main theorems. The proof of Theorems 1.2 and 1.3 is given in section 3.

2. An abstract perturbation method

Here we recall the abstract method developed in [3]. We begin by recalling the notion of the (super)smoothness in the sense of Kato [17] and Kato-Yajima [18]. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $||\cdot||$, H a self-adjoint operator on $\mathcal{H}$ and A a densely defined closed operator on $\mathcal{H}$. Note that A^* is also a densely defined closed operator (see [25, Theorem VIII.1]). Let $R_H(z) := (H-z)^{-1}$, $z \notin \sigma(H)$. Then we say that A is H*-smooth* with bound a if

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}} |\langle (R_H(z) - R_H(\overline{z}))A^*\psi, A^*\psi\rangle| \le \frac{a^2}{2}||\psi||^2, \quad \psi \in D(A^*).$$

We say that A is H-*supersmooth* with bound a if

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}} |\langle R_H(z)A^*\psi, A^*\psi\rangle| \le \frac{a}{2}||\psi||^2, \quad \psi \in D(A^*).$$

Note that if A is H-supersmooth with bound a, then A is H-smooth with bound $(2a)^{1/2}$. The H-(super)smoothness is closely connected with smoothing effects.

Proposition 2.1. (1) *A is H-smooth with bound a if and only if, for any $\psi \in \mathcal{H}$, $e^{-itH}\psi$ belongs to $D(A)$ for a.e. $t \in \mathbb{R}$ and*

$$||Ae^{-itH}\psi||_{L^2(\mathbb{R};\mathcal{H})} \le a||\psi||. \tag{2.1}$$

(2) *Suppose A is H-supersmooth with bound a. Then, for any simple function $F : \mathbb{R} \to D(A^*)$ and $t \in \mathbb{R}$, $Ae^{-i(t-s)H}A^*F(s)$ is Bochner integrable in s over $[0, t]$ (or $[t, 0]$) and satisfies*

$$\Big|\Big|e^{-|\varepsilon t|}\int_0^t Ae^{-i(t-s)H}A^*F(s)ds\Big|\Big|_{L^2(\mathbb{R};\mathcal{H})} \le a||e^{-|\varepsilon t|}F||_{L^2(\mathbb{R};\mathcal{H})} \tag{2.2}$$

for all $\varepsilon \in \mathbb{R}$. Conversely, if the estimate (2.2) *holds for all simple functions $F : \mathbb{R} \to D(A^*)$ and $|\varepsilon| < \varepsilon_0$ with some $\varepsilon_0 > 0$, then A is H-supersmooth with bound a.*

Proof. The first statement is due to [17, Lemma 3.6 and Theorem 5.1] (see also [25, Theorem XIII.25]). The second assertion was proved by [10, Theorem 2.3]. ☐

Note that if A is H-smooth, then A is infinitesimally H-bounded ([25, Theorem XIII.22]). Also note that the estimate (2.2) can be replaced by

$$\Big|\Big|e^{-|\varepsilon t|}\int_0^t Ae^{-i(t-s)H}A^*F(s)ds\Big|\Big|_{L^2_T\mathcal{H}} \le a||e^{-|\varepsilon t|}F||_{L^2_T\mathcal{H}}. \tag{2.3}$$

Indeed, (2.2) implies (2.3) since $s \in [-T, T]$ if $t \in [-T, T]$ and $s \in [0, t]$ (or $s \in [-t, 0]$). Conversely, since (2.3) implies that

$$\Big|\Big|e^{-|\varepsilon t|}\int_0^t Ae^{-i(t-s)H}A^*F(s)ds\Big|\Big|_{L^2_T\mathcal{H}} \le a||e^{-|\varepsilon t|}F||_{L^2(\mathbb{R};\mathcal{H})}$$

and a is independent of T, one has (2.2) by letting $T \to \infty$. Let Γ_H be the inhomogeneous propagator defined by the formula (1.8) and set

$$\Gamma_H^*F(t) = \mathbb{1}_{[0,\infty)}(t)\int_t^T e^{-i(t-s)H}F(s)ds - \mathbb{1}_{(-\infty,0]}(t)\int_{-T}^t e^{-i(t-s)H}F(s)ds. \tag{2.4}$$

By a direct calculation, $\langle\langle \Gamma_H F, G\rangle\rangle_T = \langle\langle F, \Gamma_H^* G\rangle\rangle_T$ for $F, G \in L^1_{\mathrm{loc}}\mathcal{H}$, where

$$\langle\langle F, G\rangle\rangle_T := \int_{-T}^T \langle F(t), G(t)\rangle dt.$$

Hence Γ_H^* is the adjoint of Γ_H in $L^2_T\mathcal{H}$. In the abstract theorem below, the operators $A\Gamma_H$ and $A\Gamma_H^*$ for some H-smooth operator A play important roles. These operators are a priori well-defined on $L^1_{\mathrm{loc}}(\mathbb{R}; D(H))$ since, for some $z \notin \sigma(H)$ and each $T > 0$,

$$||A\Gamma_H F||_{L^2_T\mathcal{H}} + ||A\Gamma_H^* F||_{L^2_T\mathcal{H}} \le C_T||AR_0(z)||_{\mathbb{B}(\mathcal{H})}||R_0(z)F||_{L^1_T\mathcal{H}} < \infty.$$

The next lemma provides a rigorous definition of $A\Gamma_H F(t)$ for $F \in L^1_{\mathrm{loc}}(\mathbb{R}; \mathcal{H})$.

Lemma 2.2 ([3, Lemma 4.3 and Lemma 4.5]). *Suppose that A is H-smooth with bound a. Then $A\Gamma_H$ and $A\Gamma_H^*$ extend to bounded operators from $L^1_T\mathcal{H}$ to $L^2_T\mathcal{H}$ such that*

$$||A\Gamma_H F||_{L^2_T\mathcal{H}} \le Ca||F||_{L^1_T\mathcal{H}}, \quad ||A\Gamma_H^* F||_{L^2_T\mathcal{H}} \le Ca||F||_{L^1_T\mathcal{H}} \tag{2.5}$$

for any $T > 0$ and $F \in L^1_T\mathcal{H}$ with some $C > 0$ independent of a, T and F. Moreover, we have

$$A\Gamma_H F(t) = \int_0^t Ae^{-i(t-s)H}F(s)ds \tag{2.6}$$

for all simple functions $F : [-T,T] \to \mathcal{H}$ and a.e. $t \in [-T,T]$. In particular $A\Gamma_H F(t)$ and $\int_0^t Ae^{-i(t-s)H}F(s)ds$ coincide in $L^2_T\mathcal{H}$.

Proof. For the sake of self-containedness, we give the proof for Γ_H in detail. The proof for Γ_H^* is analogous. Let $\chi \in C_0^\infty(\mathbb{R})$ be such that $\chi \equiv 1$ near 0 and $0 \le \chi \le 1$ and set $\chi_\varepsilon(t) = \chi(\varepsilon t)$. Note that $A\Gamma_H\chi_\varepsilon(H) = A\chi_\varepsilon(H)\Gamma_H$ is well-defined on $L^1_T\mathcal{H}$ since A is H-bounded. Moreover, the H-smoothness of A shows that

$$\Big|\Big|Ae^{-itH}\int_{[0,T]} e^{isH}\chi_\varepsilon(H)F(s)ds\Big|\Big|_{L^2_T\mathcal{H}} \le a||F||_{L^1_T\mathcal{H}}.$$

Using the Christ-Kiselev lemma [9], one can replace $[0,T]$ by $[0,t]$ in the left hand side to obtain

$$||A\chi_\varepsilon(H)\Gamma_H F||_{L^2_T\mathcal{H}} \le Ca||F||_{L^1_T\mathcal{H}}$$

with some universal constant $C > 0$ independent of F. This estimate implies that, for all $F \in L^1_T\mathcal{H}$, $A\chi_\varepsilon(H)\Gamma_H F$ converges in $L^2_T\mathcal{H}$ as $\varepsilon \to 0$ and the limit denoted by the same symbol $A\Gamma_H F$ satisfies the first estimate in (2.5). This proves the first half of this lemma.

Next, let $F : [-T,T] \to \mathcal{H}$ be a simple function. As we have shown, $A\chi_{\varepsilon_n}(H)\Gamma_H$ converges to $A\Gamma_H F$ in $L^2_T\mathcal{H}$ as $n \to \infty$ for any sequence $\varepsilon_n > 0$ with $\varepsilon_n \to 0$. Then one can find a subsequence ε_{n_k} and a null set $\mathcal{N} \subset [-T,T]$ such that $A\chi_{\varepsilon_{n_k}}(H)\Gamma_H F \to A\Gamma_H F(t)$ in $\mathcal{H}$ as $k \to \infty$ for all $t \in [-T,T] \setminus \mathcal{N}$. Hence, in order to show (2.6), it suffices to check that

$$\Big|\Big|\int_0^t Ae^{-i(t-s)H}F(s)ds - A\chi_{\varepsilon_{n_k}}(H)\Gamma_H F(t)\Big|\Big|_{\mathcal{H}} \to 0$$

as $k \to \infty$ for all $t \in [-T,T] \setminus \mathcal{N}$. Let us write $F = \sum_{j=1}^N \mathbb{1}_{E_j}(t)f_j$ with $f_j \in \mathcal{H}$ and measurable sets $E_j \subset [-T,T]$. Then we have for all $t \in [-T,T] \setminus \mathcal{N}$,

$$\begin{aligned}
&\Big|\Big|\int_0^t Ae^{-i(t-s)H}F(s)ds - A\chi_{\varepsilon_{n_k}}(H)\Gamma_H F(t)\Big|\Big|_{\mathcal{H}} \\
&\le \sum_{j=1}^N \Big|\Big|\int_{[0,t]\cap M_j} Ae^{isH}(1-\chi_{\varepsilon_{n_k}}(H))e^{-itH}f_j ds\Big|\Big|_{\mathcal{H}} \\
&\le \sum_{j=1}^N T^{1/2}\Big(\int_{\mathbb{R}} ||Ae^{isH}e^{-itH}(1-\chi_{\varepsilon_{n_k}}(H))f_j||^2_{\mathcal{H}}ds\Big)^{1/2} \\
&\le CT^{1/2}\sum_{j=1}^N ||(1-\chi_{\varepsilon_{n_k}}(H))f_j||_{\mathcal{H}} \to 0
\end{aligned}$$

as $k \to \infty$, where we have used Hölder's inequality in the third line and the H-smoothness of A and the unitarity of e^{-itH} in the last line. This completes the proof. $\square$

In what follows, $A\Gamma_H$ and $A\Gamma_H^*$ denote such extensions. By Proposition 2.1(1), (2.3) and this lemma, if A is H-supersmooth with bound a, then $A\Gamma_H A^*$ is well-defined on $L^1_{\mathrm{loc}}(\mathbb{R};(D(A^*)))$ and satisfies $||A\Gamma_H A^* F||_{L^2_T\mathcal{H}} \le a||F||_{L^2_T\mathcal{H}}$ for all simple functions $F:[-T,T]\to D(A^*)$.

Now we are ready to recall an abstract theorem from [3]. Let (H_0,H) be a pair of self-adjoint operators on $\mathcal{H}$ such that $H = H_0 + V_1^* V_2$ in the following sense:

- V_1, V_2 are densely defined closed operators on $\mathcal{H}$ such that we have continuous embeddings $D(H_0) \subset D(V_1)$, $D(H) \subset D(V_1)$ and $D(H) \subset D(V_2)$.
- $\langle Hf, g\rangle = \langle f, H_0 g\rangle + \langle V_2 f, V_1 g\rangle$ for $f \in D(H)$ and $g \in D(H_0)$.

Note that, under the above conditions, V_1^*, V_2^* are also densely defined. Recall that two Banach spaces $(\mathcal{A},\mathcal{B})$ are said to be a Banach couple if both $\mathcal{A}$ and $\mathcal{B}$ are algebraically and topologically embedded in a Hausdorff topological vector space $\mathcal{C}$.

Proposition 2.3 ([3, Theorem 4.7]). *Let $\mathcal{A},\mathcal{B}$ be two Banach spaces such that $(\mathcal{A},\mathcal{H})$ and $(\mathcal{B},\mathcal{H})$ are Banach couples. Suppose that V_1 is H_0-smooth and V_2 is both H_0-smooth and H-smooth. Consider the following series of estimates:*

$$|\langle\langle e^{-itH_0}\psi, G\rangle\rangle_T| \le s_1||\psi||_{\mathcal{H}}||G||_{L^2_T\mathcal{B}}, \tag{2.7}$$

$$|\langle\langle \Gamma_{H_0}F, G\rangle\rangle_T| \le s_2||F||_{L^2_T\mathcal{A}}||G||_{L^2_T\mathcal{B}}, \tag{2.8}$$

$$||V_1\Gamma_{H_0}^* G||_{L^2_T\mathcal{H}} \le s_3||G||_{L^2_T\mathcal{B}}, \tag{2.9}$$

$$||V_1\Gamma_{H_0}F||_{L^2_T\mathcal{H}} \le s_4||F||_{L^2_T\mathcal{A}}, \tag{2.10}$$

$$||V_2\Gamma_{H_0}F||_{L^2_T\mathcal{H}} \le s_5||F||_{L^2_T\mathcal{A}}, \tag{2.11}$$

$$||V_2 e^{-itH}\psi||_{L^2_T\mathcal{H}} \le s_6||\psi||_{\mathcal{H}}, \tag{2.12}$$

$$||V_2\Gamma_H V_2^*\widetilde{G}||_{L^2_T\mathcal{H}} \le s_7||\widetilde{G}||_{L^2_T\mathcal{H}}. \tag{2.13}$$

(1) *Suppose there exist constants $s_1, s_3, s_6 > 0$ such that* (2.7), (2.9) *and* (2.12) *are satisfied for all $\psi \in \mathcal{H}$ and simple function $G:[-T,T]\to\mathcal{H}\cap\mathcal{B}$. Then one has*

$$|\langle\langle e^{-itH}\psi, G\rangle\rangle_T| \le (s_1 + s_3 s_6)||\psi||_{\mathcal{H}}||G||_{L^2_T\mathcal{B}}$$

for all $\psi \in \mathcal{H}$ and simple function $G:[-T,T]\to\mathcal{H}\cap\mathcal{B}$.

(2) *Suppose that there exist constants $s_2, s_3, s_4, s_5, s_7 > 0$ such that* (2.8), (2.9), (2.10), (2.11) *and* (2.13) *hold for all simple functions $F:[-T,T]\to\mathcal{H}\cap\mathcal{A}$, $G:[-T,T]\to\mathcal{H}\cap\mathcal{B}$ and $\widetilde{G}:[-T,T]\to D(H)$. Then one has*

$$|\langle\langle \Gamma_H F, G\rangle\rangle_T| \le (s_2 + s_3 s_5 + s_3 s_4 s_7)||F||_{L^2_T\mathcal{A}}||G||_{L^2_T\mathcal{B}}$$

for all simple functions $F:[-T,T]\to\mathcal{H}\cap\mathcal{A}$ and $G:[-T,T]\to\mathcal{H}\cap\mathcal{B}$.

Proof. We give the proof in detail for the sake of self-containedness. Let us first show the first statement. The Duhamel formula implies that

(2.14)

$$\langle e^{-itH}\psi,\varphi\rangle = \langle e^{-itH_0}\psi,\varphi\rangle - i\int_0^t \langle V_2 e^{-irH}\psi, V_1 e^{-i(r-t)H_0}\varphi\rangle dr, \quad \psi,\varphi \in L^2.$$

Plugging in $\varphi = G(t)$, integrating over $t \in [-T, T]$ and using Fubini's theorem, we learn from the formula (2.4) of the adjoint $\Gamma^*_{H_0}$ that

$$\langle\langle e^{-itH}\psi, G\rangle\rangle_T = \langle\langle e^{-itH_0}\psi, G\rangle\rangle_T - i\langle\langle V_2 e^{-itH}\psi, V_1\Gamma^*_{H_0}G\rangle\rangle_T.$$

Applying (2.7), (2.9) and (2.12), we then obtain the first assertion (1).

In order to prove the second assertion (2), we replace t by $t-s$ and plug in $\psi = F(s), \varphi = G(t)$ and integrate over $s \in [0,t]$ in (2.14) to obtain

$$\begin{aligned}
&\langle \Gamma_H F(t), G(t)\rangle \\
&\quad = \langle \Gamma_{H_0}F(t), G(t)\rangle - i\int_0^t\int_s^t \langle V_2 e^{-i(\tau-s)H}F(s), V_1 e^{-i(\tau-t)H_0}G(t)\rangle d\tau ds \\
&\quad = \langle \Gamma_{H_0}F(t), G(t)\rangle - i\int_0^t \langle V_2\Gamma_H F(\tau), V_1 e^{-i(\tau-t)H_0}G(t)\rangle d\tau.
\end{aligned}$$

As above, integrating in $t \in [-T, T]$ and using (2.4) implies that

$$\langle\langle \Gamma_H F, G\rangle\rangle_T = \langle\langle \Gamma_{H_0}F, G\rangle\rangle_T - i\langle\langle V_2\Gamma_H F, V_1\Gamma^*_{H_0}G\rangle\rangle_T. \tag{2.15}$$

Exchanging the roles of H and H_0, we also obtain

$$\langle\langle \Gamma_H F, G\rangle\rangle_T = \langle\langle \Gamma_{H_0}F, G\rangle\rangle_T - i\langle\langle V_1\Gamma_{H_0} F, V_2\Gamma^*_{H}G\rangle\rangle_T. \tag{2.16}$$

Now applying (2.8), (2.9) to (2.15) implies that

$$|\langle\langle \Gamma_H F, G\rangle\rangle_T| \le s_2||F||_{L^2_T\mathcal{A}}||G||_{L^2_T\mathcal{B}} + s_3||V_2\Gamma_H F||_{L^2_T\mathcal{H}}||G||_{L^2_T\mathcal{B}}. \tag{2.17}$$

It remains to deal with $||V_2\Gamma_H F||_{L^2_T\mathcal{H}} = \sup_{||\widetilde{G}||_{L^2_T\mathcal{H}}=1}|\langle\langle V_2\Gamma_H F, \widetilde{G}\rangle\rangle_T|$. Since $D(H)$ is dense in $\mathcal{H}$, we may assume $\widetilde{G}(t) \in D(H)$. Then, taking $D(H) \subset D(V_2^*)$ into account, we use (2.16) with $G = V_2^*\widetilde{G}$, (2.10), (2.11) and (2.13) to obtain $|\langle\langle V_2\Gamma_H F, \widetilde{G}\rangle\rangle_T| \le (s_5 + s_4 s_7)||F||_{L^2_T\mathcal{A}}$, which, together with (2.16), gives us the second assertion. This completes the proof. □

3. Proof of Theorems 1.2 and 1.3

Let $H = -\Delta + V(x)$ be as in Theorem 1.2. This section is devoted to the proof of the main theorems. In what follows we use a standard notation $2^* = 2n/(n-2)$, $2_* = 2n/(n+2)$. We write $\Gamma_0 = \Gamma_{-\Delta}$. Let us first recall various estimates for the free Schrödinger equation.

Lemma 3.1. *There exists $C > 0$ such that, for any $v \in L^{n,\infty}(\mathbb{R}^n)$ and $T > 0$,*

$$||e^{it\Delta}\psi||_{L^2_T L^{2^*,2}} \le C||\psi||_{L^2}, \tag{3.1}$$

$$||\Gamma_0 F||_{L^2_T L^{2^*,2}} \le C||F||_{L^2_T L^{2_*,2}}, \tag{3.2}$$

$$||v(x)\Gamma_0 F||_{L^2_T L^2} \le C||v||_{L^{n,\infty}}||F||_{L^2_T L^{2_*,2}}, \tag{3.3}$$

$$||v(x)\Gamma_0^* F||_{L^2_T L^2} \le C||v||_{L^{n,\infty}}||F||_{L^2_T L^{2_*,2}}. \tag{3.4}$$

Proof. (3.1) and (3.2) are endpoint Strichartz estimates proved by [19, Theorem 10.1]. The latter two estimates follow from (3.2), Hölder's inequality (1.11) and the duality. □

Lemma 3.2. *Let $w \in L^2(\mathbb{R})$ be as in Theorem 1.2, $\rho > 1/2$ and $v \in L^{n,\infty}(\mathbb{R}^n)$. Then there exists $C > 0$, independent of w, v, and $T > 0$, such that, for all $\psi \in L^2$ and simple function $F : \mathbb{R} \to \mathcal{S}(\mathbb{R}^n)$, one has*

$$||w(|x|)|D|^{1/2}e^{it\Delta}\psi||_{L^2_T L^2} \le C||w||_{L^2(\mathbb{R})}||\psi||_{L^2}, \tag{3.5}$$

$$||w(|x|)|D|^{1/2}\Gamma_0 F||_{L^2_T L^2} \le C||w||_{L^2(\mathbb{R})}||F||_{L^2_T L^{2_*,2}}, \tag{3.6}$$

$$||v(x)\Gamma_0^* F||_{L^2_T L^2} \le C||v||_{L^{n,\infty}}||w||_{L^2(\mathbb{R})}||w(|x|)^{-1}|D|^{-1/2}F||_{L^2_T L^2}, \tag{3.7}$$

$$||\langle x\rangle^{-\rho}|D|^{1/2}\Gamma_0 F||_{L^2_T L^2} \le C||\langle x\rangle^{\rho}|D|^{-1/2}F||_{L^2_T L^2}, \tag{3.8}$$

$$||\langle x\rangle^{-\rho}|D|^{1/2}\Gamma_0 F||_{L^2_T L^2} \le C||F||_{L^2_T L^{2_*,2}}, \tag{3.9}$$

$$||v(x)\Gamma_0^* F||_{L^2_T L^2} \le C||v||_{L^{n,\infty}}||\langle x\rangle^{\rho}|D|^{-1/2}F||_{L^2_T L^2}, \tag{3.10}$$

$$||v(x)\Gamma_0 F||_{L^2_T L^2} \le C||v||_{L^{n,\infty}}||\langle x\rangle^{\rho}|D|^{-1/2}F||_{L^2_T L^2}. \tag{3.11}$$

Proof. Let us first consider (3.5). When $n = 1$, it was proved by [20] that

$$\sup_{x\in\mathbb{R}} |||D_x|^{1/2}e^{it\partial_x^2}f||_{L^2(\mathbb{R}_t)} \le C||f||_{L^2(\mathbb{R}_x)},$$

which, together with the unitarity of $e^{it\Delta_{\widehat{x}_j}}$ in $L^2(\mathbb{R}^{n-1})$, implies that

$$|||D_j|^{1/2}e^{it\Delta}\psi||_{L^\infty_{x_j}L^2_T L^2_{\widehat{x}_j}} \le C||\psi||_{L^2}, \quad j = 1, 2, \dots, n,$$

uniformly in $T > 0$, where $\widehat{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in \mathbb{R}^{n-1}$ and $D_j = -i\partial_{x_j}$. (3.5) then is derived from this estimate as follows. Let $\{C_j(\xi)\}$ be a conical partition of unity on $\mathbb{R}^n$ so that $I = \sum_{j=1}^n C_j(\xi)$, where $C_j \in C^\infty(\mathbb{R}^n \setminus \{0\})$ such that $\operatorname{supp} C_j \subset \{2|\xi_j| > |\xi|\}$ and $\partial_\xi^\alpha C_j(\xi) = O(|\xi|^{-|\alpha|})$. If we set $\widetilde{C}_j(\xi) = C_j(\xi)|\xi|^{1/2}|\xi_j|^{-1/2}$, then $\widetilde{C}_j$ also satisfies $\partial_\xi^\alpha C_j(\xi) = O(|\xi|^{-|\alpha|})$ and $|\xi|^{1/2} = \sum_{j=1}^n \widetilde{C}_j(\xi)|\xi_j|^{1/2}$. Since $w(|x|)^2$ belongs to the Muckenhoupt A_2-class, $\widetilde{C}_j(D)$ is bounded on a weighted space $L^2(\mathbb{R}^n, w(|x|)^2dx)$ by weighted Mikhlin's multiplier theorem (see [21]). Thus we conclude that

$$\begin{aligned}
||w(|x|)|D|^{1/2}e^{it\Delta}\psi||^2_{L^2_T L^2} &\le \sum_{j=1}^n ||w(|x|)\widetilde{C}_j(D)|D_j|^{1/2}e^{it\Delta}\psi||^2_{L^2_T L^2} \\
&\le C\sum_{j=1}^n ||w(|x|)|D_j|^{1/2}e^{it\Delta}\psi||^2_{L^2_T L^2} \\
&\le C\sum_{j=1}^n ||w||^2_{L^2(\mathbb{R})}|||D_j|^{1/2}e^{it\Delta}\psi||^2_{L^\infty_{x_j}L^2_T L^2_{\widehat{x}_j}} \\
&\le C||w||^2_{L^2(\mathbb{R})}||\psi||^2_{L^2}
\end{aligned}$$

uniformly in $T > 0$, where we used the properties $w(|x|) \le C_j w(x_j)$ and $w \in L^2(\mathbb{R})$ in the third line. Next, by the same argument as above, (3.6) follows from the estimate

$$|||D_j|^{1/2}\Gamma_0 F||_{L^\infty_{x_j}L^2_T L^2_{\widehat{x}_j}} \le C||F||_{L^2_T L^{2_*,2}},$$

which is a slight generalization of [16, Lemma 4], in which the same estimate with $L^{2_*,2}$ replaced by L^{2_*} was proved. Although the proof is essentially the same as that of [16, Lemma 4], we briefly recall its strategy for the reader's convenience. Without loss of generality, we may assume $j=1$. Then it suffices to show that

$$\sup_{x_1} |||D_j|^{1/2}\widetilde{\Gamma}_0 F||_{L^2_T L^2_{\widehat{x}_1}} \le C||F||_{L^2_T L^{2_*,2}}, \tag{3.12}$$

where $\widetilde{\Gamma}_0$ is defined by

$$\widetilde{\Gamma}_0 F(t) := \int_{-\infty}^t e^{i(t-s)\Delta} F(s) ds.$$

Indeed, the corresponding estimate for $\Gamma_0 - \widetilde{\Gamma}_0$ follows from (3.5) and the dual estimate of (3.1). The only difference from the proof of [16, Lemma 4] is an interpolation step. While they used the complex interpolation, we will use a real interpolation technique as in [19, section 6]. Let $I_\pm = \mathbf{1}_\pm(D_1)|D_1|^{1/2}\widetilde{\Gamma}_0$, where $\mathbf{1}_\pm(t) = 1$ for $\pm t \ge 0$ and $\mathbf{1}_\pm(t) = 0$ for $\mp t \ge 0$. It suffices to show that I_+ is bounded from $L^2_T L^{2_*,2}$ to $L^2_T L^2_{\widehat{x}_1}$ uniformly in x_1 since the proof for I_- is analogous. By the TT^* argument, $I_+ \in \mathbb{B}(L^2_T L^{2_*,2}, L^2_T L^2_{\widehat{x}_1})$ if $I_+^* I_+$ is bounded from $L^2_T L^{2_*,2}$ to $L^2_T L^{2^*,2}$. Hence, if we define a bilinear form I by

$$I(F,G) := \iint \langle I_+ F(s,\cdot), I_+ G(t,\cdot)\rangle ds dt,$$

then it suffices to show that

$$|I(F,G)| \le C||F||_{L^2_T L^{2_*,2}} ||G||_{L^2_T L^{2_*,2}} \tag{3.13}$$

uniformly in x_1 and $T > 0$. It was shown by [16] that $I_+^* I_+$ is bounded on $L^2(\mathbb{R}^n)$ and the kernel of $I_+^* I_+$, denoted by $K_+(t,s,x,y)$, satisfies the dispersive estimate:

$$|K_+(t,s,x,y)| \le C|t-s|^{-n/2}, \quad t \ne s.$$

We then decompose $I(F,G)$ as

$$I(F,G) = \sum_{k\in\mathbb{Z}} I_k(F,G),\ I_k(F,G) := \iint_{t-2^{k+1}}^{t-2^k} \langle I_+ F(s,\cdot), I_+ G(t,\cdot)\rangle ds dt.$$

By using the same argument as in [19, Lemma 4.1], we see that

$$|I_k(F,G)| \le C2^{-k\beta(a,b)}||F||_{L^2_T L^{a'}}||G||_{L^2_T L^{b'}}, \quad \beta(a,b) = \frac{n}{2} - 1 - \frac{n}{2}(\frac{1}{a} - \frac{1}{b})$$

uniformly in $k \in \mathbb{Z}$, where (a,b) satisfies one of the following conditions:

$$\text{(i)}\ \ \frac{1}{a} = \frac{1}{b} = 0; \quad \text{(ii)}\ \ \frac{n-1}{2n} \le \frac{1}{a} \le \frac{1}{2} \text{ and } \frac{1}{b} = \frac{1}{2};$$
$$\text{(iii)}\ \ \frac{n-1}{2n} \le \frac{1}{b} \le \frac{1}{2} \text{ and } \frac{1}{a} = \frac{1}{2}.$$

In other words, a vector valued sequence $(I_k)_{k\in\mathbb{Z}}$ is bounded from $L^2_T L^{a'}_x \times L^2_T L^{b'}_x$ to $\ell^\infty_{\beta(a,b)}$, where $\ell^p_s = L^p(\mathbb{Z}, 2^{js}dj)$ is a weighted ℓ^p space with the counting measure dj. Then (3.13) follows from the technique by [19, Section 6] based on a bilinear real interpolation.

The estimate (3.7) follows from the dual estimate of (3.6) and Hölder's inequality (1.11). For (3.8), we refer to [8]. (3.9) and (3.10) follow from (3.6) and (3.7) since

$\langle x\rangle^{-\rho}$ satisfies the condition on w in Theorem 1.2. In order to derive (3.11), we observe from the formula (2.4) that Γ_0 can be brought to the form

$$\Gamma_0 F(t) = -\Gamma_0^* F(t) \pm \Gamma_0^0 F(t) + \Gamma_0^{\mp} F(t)$$

for $\pm t \geq 0$, where

$$\Gamma_0^0 F(t) = \int_{-T}^{T} e^{-i(t-s)H} F(s)ds, \quad \Gamma_0^{\pm} F(t) = \int_0^{\pm T} e^{-i(t-s)H} F(s)ds.$$

Then the desired estimate for Γ_0^* is nothing but (3.9); the desired estimates for Γ_0^0 and $\Gamma_0^{\mp}$ follow from (3.1), the dual estimate of (3.5) with $w = \langle x\rangle^{-\rho}$ and Hölder's inequality (1.11). $\square$

The following fact, proved by [1, Theorem 1.6 and (1.23)] (see also [3, Theorem 6.1 and Appendix B]), also plays an important role.

Proposition 3.3. *Let* $n \geq 3$ *and* V *satisfy Assumption* A. *Then* $|x|^{-1}$ *is* H*-supersmooth.*

We are in a position to show the main theorems.

Proof of Theorem 1.2. Let us set $V_1 = |x|V$ and $V_2 = |x|^{-1}$. By Sobolev's inequality (1.12),

$$||V_j f||_{L^2} \leq C||V_j||_{L^{n,\infty}} ||f||_{L^{2^*,2}} \leq C||V_j||_{L^{n,\infty}} ||\nabla f||_{L^2}$$

and hence $D(V_j) \supset \mathcal{H}^1 \supset D(\Delta) \cup D(H)$. Moreover, (3.1) and Proposition 2.1(1) show that both V_1 and V_2 are Δ-smooth. On the other hand, Propositions 2.1 and 3.3 show that

$$||V_2 e^{-itH}\psi||_{L^2_T L^2} \leq C||\psi||_{L^2} \tag{3.14}$$

uniformly in $T > 0$. Let $\mathcal{B}$ be the completion of C_0^∞ with respect to the norm $||w(|x|)^{-1}|D|^{-1/2} f||_{L^2}$. By virtue of (3.5), (3.7) and (3.14), one can use Proposition 2.3 with $H_0 = -\Delta$, $H = -\Delta + V$ and this $\mathcal{B}$ to obtain

$$|\langle\langle e^{-itH}\psi, G\rangle\rangle_T| \leq C||w||_{L^2(\mathbb{R})} ||\psi||_{L^2} ||G||_{\mathcal{B}}$$

for all $\psi \in L^2$ and simple function $G : [-T, T] \to \mathcal{S}$ uniformly in $T > 0$. Then the desired estimate follows from density and duality arguments. $\square$

Proof of Theorem 1.3. We use the same decomposition $V = V_1 V_2$ as above. Since V_2 is H-supersmooth, we learn by Proposition 2.1 and a remark after Lemma 2.2 that

$$||V_2 \Gamma_H V_2 \widetilde{G}||_{L^2_T L^2} \leq C||\widetilde{G}||_{L^2_T L^2} \tag{3.15}$$

for all simple functions $\widetilde{G} : \mathbb{R} \to D(V_2)$ with the constant C independent of T and $\widetilde{G}$. By virtue of (3.1)–(3.4), (3.5) with $w = \langle x\rangle^{-\rho}$, (3.8)–(3.11) with $v \in \{V_1, V_2\}$, (3.14) and (3.15), we can use Proposition 2.3 with $\mathcal{A}, \mathcal{B} \in \{\dot{\mathcal{H}}^{-1/2,\rho}, L^{2_*,2}\}$ to obtain

$$|\langle\langle e^{-itH}\psi, G\rangle\rangle_T| \leq C||\psi||_{L^2} ||G||_{L^2_T \mathcal{B}}, \quad |\langle\langle \Gamma_H F, G\rangle\rangle_T| \leq C||F||_{L^2_T \mathcal{A}} ||G||_{L^2_T \mathcal{B}}$$

uniformly in $T > 0$, $\psi \in L^2$ and simple functions $F, G : \mathbb{R} \to \mathcal{S}$. Then the assertion follows from density of simple functions $F : \mathbb{R} \to \mathcal{S}$ in $L^2_T \mathcal{A}$ and $L^2_T \mathcal{B}$ and the formula (1.9). $\square$

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Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail address: `haruya@math.sci.osaka-u.ac.jp`