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# COMPLETELY DECOMPOSABLE DIRECT SUMMANDS OF TORSION-FREE ABELIAN GROUPS OF FINITE RANK

ADOLF MADER AND PHILL SCHULTZ

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Abstract. Let $A$ be a finite rank torsion-free abelian group. Then there exist direct decompositions $A = B \oplus C$ where $B$ is completely decomposable and $C$ has no rank 1 direct summand. In such a decomposition $B$ is unique up to isomorphism and $C$ is unique up to near-isomorphism.

## 1. Introduction

All 'groups' in this article are torsion-free abelian groups of finite rank, and undefined notation comes from the standard texts [4] and [8].

Groups are best thought of as additive subgroups of finite dimensional $\mathbb{Q}$–vector spaces. The rank of a group $A$ is the dimension of the vector space $\mathbb{Q}A$ that $A$ generates. By reason of rank, such groups always have 'indecomposable decompositions', meaning direct decompositions with indecomposable summands. Indecomposable decompositions can be highly non-unique (see for example [4, Section 90]). A particularly striking result in this direction is due to A.L.S. Corner [2], [3].

Let $P = (r_1, \dots, r_t)$ be a partition of $n$, i.e., $r_i \geq 1$ and $r_1 + \cdots + r_t = n$. Then $G$ **realizes** $P$ if there is an indecomposable decomposition $G = G_1 \oplus \cdots \oplus G_t$ such that for all $i$, $r_i = \operatorname{rank}(G_i)$.

**Corner's Theorem.** *Given integers $n \geq k \geq 1$, there exists a group $G$ of rank $n$ such that $G$ realizes every partition of $n$ into $k$ parts $n = r_1 + \cdots + r_k$.*

Corner's Theorem constitutes a positive partial answer to the general

**Question.** Characterize the families of partitions of $n$ that can be realized by a group.

On the other hand, Corner comments that

> ... it can be shown quite readily that an equation such as $1+1+2 = 1+3$ ... cannot be realized.

Related questions were studied by Lee Lady [6] for finite rank **almost completely decomposable groups**, that is, finite extensions of completely decomposable groups. For a comprehensive exposition see [8].

**Near isomorphism** is a weakening of an isomorphism also due to Lady [7]. There are several equivalent definitions (see for example [8, Chapter 9]), the most

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useful one for us being that a group $A$ is nearly isomorphic to $B$, denoted $A \cong_{\rm nr} B$, if there exists a group $K$ such that $A \oplus K \cong B \oplus K$.

An important result due to Arnold [1, 12.9], [8, Theorem 12.2.5] is that if $A \cong_{\rm nr} A'$ and $A = X \oplus Y$, then $A' = X' \oplus Y'$ with $X \cong_{\rm nr} X'$ and $Y \cong_{\rm nr} Y'$. Conversely, if $X \cong_{\rm nr} X'$ and $Y \cong_{\rm nr} Y'$, then $X \oplus Y \cong_{\rm nr} X' \oplus Y'$.

By Arnold's Theorem, nearly isomorphic groups of rank $n$ realize the same partitions of $n$.

A group $G$ is **clipped** if it has no direct summands of rank 1, a property which is also preserved by near isomorphism. Lady's 'Main Decomposition Theorem' says that every almost completely decomposable group $G$ has a decomposition $G = G_{cd} \oplus G_{cl}$ where $G_{cd}$ is completely decomposable, $G_{cl}$ is clipped, $G_{cd}$ is unique up to isomorphism, and $G_{cl}$ is unique up to near-isomorphism.

Our main result, Theorem 2.5, is the generalization of the Main Decomposition Theorem to arbitrary groups, which then settles Corner's remark above about non-realizability.

## 2. Main decomposition

A **rank–1 group** is a group isomorphic to an additive subgroup of $\mathbb{Q}$, and a **completely decomposable group** is a direct sum of rank–1 groups. A **type** is the isomorphism class of a rank–1 group. It is easy to see that every rank–1 group is isomorphic to a **rational group**, by which we mean an additive subgroup of $\mathbb{Q}$ that contains 1. Types are commonly denoted by $\sigma, \tau, \dots$, and we use the same notation to mean a rational group of type $\sigma, \tau, \dots$. It will always be clear from the context whether $\tau$ is a rational group or a type. The advantage of this notation is that any completely decomposable group $A$ of finite rank $r$ can be expressed as $A = \sigma_1 v_1 \oplus \cdots \oplus \sigma_r v_r$ with $v_i \in A$. In this case $\{v_1, \dots, v_r\}$ is called a **decomposition basis** of $A$.

A completely decomposable group is called $\tau$–**homogeneous** if it is the direct sum of rank–1 groups of type $\tau$, and **homogeneous** if it is $\tau$–homogeneous for some type $\tau$. It is known [4, 86.6] that pure subgroups of homogeneous completely decomposable groups are direct summands.

**Definition 2.1.** A group $G$ is $\tau$–**clipped** if $G$ does not possess a rank–1 summand of type $\tau$.

**Lemma 2.2.** *Suppose that $G = D \oplus B = A \oplus C$ where $D$ and $A$ are completely decomposable $\tau$–homogeneous and $B$ and $C$ are $\tau$–clipped. Then $D \cong A$.*

*Proof.* Let $\delta, \beta, \alpha, \gamma \in \mathrm{End}(G)$ be the projections belonging to the given decompositions and let $0 \neq x \in D$, so that $x = x\alpha + x\gamma$. Denote by $\langle x \rangle_*$ the pure subgroup of $D$ generated by $x$, so $\langle x \rangle_*$ is a summand of $D$ of type $\tau$.

If $x\alpha = 0$, then $\langle x \rangle_* \subseteq \mathrm{Ker}\,\alpha = C$, so $\langle x \rangle_*$ is a subgroup and hence a summand of $C$ of type $\tau$, contradicting the fact that $C$ is $\tau$–clipped.

Hence $\alpha : D \to A$ is a monomorphism and therefore $\mathrm{rank}\, D \leq \mathrm{rank}\, A$. By symmetry $\mathrm{rank}\, A \leq \mathrm{rank}\, D$ and $D \cong A$ as desired. $\square$

The direct sum of $\tau$–clipped groups need not be $\tau$–clipped as Example 2.3 shows.

**Example 2.3.** Let $p, q$ be different primes and let $\sigma, \tau$ be rational groups that are incomparable as types and such that neither $\frac{1}{p}$ nor $\frac{1}{q}$ is contained in either $\sigma$ or $\tau$.

Recall that $\mathbb{Z}$ is a type. Let

$$X_1 = (\sigma v_1 \oplus \tau v_2) + \mathbb{Z}\frac{1}{p}(v_1 + v_2) \quad \text{and} \quad X_2 = (\sigma w_1 \oplus \tau w_2) + \mathbb{Z}\frac{1}{q}(w_1 + w_2).$$

It is easy to see that $X_1$ and $X_2$ are indecomposable and therefore clipped.

Let $X = X_1 \oplus X_2$. There exist integers $u_1, u_2$ such that $u_1 p + u_2 q = 1$. Now $\frac{1}{p}(v_1 + v_2) + \frac{1}{q}(w_1 + w_2) = \frac{1}{pq}((qv_1 + pw_1) + (qv_2 + pw_2))$. Set $v_1' = qv_1 + pw_1$, $v_2' = qv_2 + pw_2$, $w_1' = -u_1 v_1 + u_2 w_1$, and $w_2' = -u_1 v_2 + u_2 w_2$. Then (change of decomposition basis) $\sigma v_1 \oplus \sigma w_1 = \sigma v_1' \oplus \sigma w_1'$ and $\tau v_2 \oplus \tau w_2 = \tau v_2' \oplus \tau w_2'$. Hence $X = \sigma w_1' \oplus \tau w_2' \oplus \Big((\sigma v_1' \oplus \tau v_2') + \mathbb{Z}\frac{1}{pq}(v_1' + v_2')\Big)$, so $X$ has rank–1 summands of types $\sigma$ and $\tau$.

However, Lemma 2.4 settles positively a special case.

**Lemma 2.4.** *Let $G = A \oplus B$ where $A$ is completely decomposable $\tau$–clipped and $B$ is $\tau$–clipped. Then $G$ is $\tau$–clipped.*

*Proof.* We may assume that $\operatorname{rank} A = 1$. In fact, if $A = A_1 \oplus \cdots \oplus A_k$ where $\operatorname{rank} A_i = 1$, then $A_k \oplus B$ is $\tau$–clipped by the rank 1 case, $A_2 \oplus \cdots \oplus A_k \oplus B$ is $\tau$–clipped by induction, and $A \oplus B$ is $\tau$–clipped by the rank 1 case.

By way of contradiction assume that $G = \tau v \oplus C = \sigma a \oplus B$ with $\tau \ncong \sigma$ (as rational groups or $\tau \neq \sigma$ as types). Let $\alpha : G \to \sigma a \subseteq G$, $\beta : G \to B \subseteq G$, $\delta : G \to \tau v \subseteq G$, and $\gamma : G \to C \subseteq G$ be the projections (considered endomorphisms of $G$) that come with the stated decompositions.

(1) We have $v = v\alpha + v\beta$ uniquely. Suppose $v\alpha = 0$. Then $(\tau v)\alpha = 0$ and the summand $\tau v$ is contained in $\operatorname{Ker} \alpha = B$. Then $\tau v$ is a summand of $B$ contradicting the fact that $B$ is $\tau$–clipped. So $\alpha : \tau v \to \sigma a$ is a monomorphism and $\tau \leq \sigma$.

(2) We have $a = a\delta + a\gamma$. Suppose that $a\delta = 0$. Then $(\sigma a)\delta = 0$ and the summand $\sigma a$ is contained in $\operatorname{Ker} \delta = C$. Hence $C = \sigma a \oplus C'$ for some $C'$ and $G = \tau v \oplus \sigma a \oplus C' = \sigma a \oplus B$. Hence $\frac{G}{\sigma a} \cong \tau v \oplus C' \cong B$. This contradicts the fact that $B$ is $\tau$–clipped. So $\delta : \sigma a \to \tau v$ is a monomorphism and hence $\sigma \leq \tau$.

(3) By (1) and (2) we get the contradiction $\sigma = \tau$, saying that $G = \sigma a \oplus B$ does not have a rank–1 summand of type $\tau$, and the special case is proved. □

**Theorem 2.5** (Main Decomposition)**.** *Let $G$ be a group. Then $G$ has a decomposition $G = A_0 \oplus A_1$ in which $A_0$ is completely decomposable and $A_1$ is clipped. If also $G = B_0 \oplus B_1$ where $B_0$ is completely decomposable and $B_1$ is clipped, then $A_0 \cong B_0$ and consequently $A_1 \cong_{\mathrm{nr}} B_1$.*

*Proof.* Let $A_0$ be a completely decomposable summand of $G$ of maximal rank. Then $G = A_0 \oplus A_1$ and $A_1$ is clipped.

Let $A_0 = \bigoplus_\rho A_\rho$ and $B_0 = \bigoplus_\rho B_\rho$ be the homogeneous decompositions of the completely decomposable groups $A_0$ and $B_0$. By allowing $A_\rho$ and $B_\rho$ to be the zero group, we may assume that the summation index ranges over all types $\rho$.

We consider $G = A_\tau \oplus \Big(\bigoplus_{\rho \neq \tau} A_\rho \oplus A_1\Big) = B_\tau \oplus \Big(\bigoplus_{\rho \neq \tau} B_\rho \oplus B_1\Big)$. By Lemma 2.4 $\bigoplus_{\rho \neq \tau} A_\rho \oplus A_1$ and $\bigoplus_{\rho \neq \tau} B_\rho \oplus B_1$ are both $\tau$–clipped. Hence by Lemma 2.2 we conclude that $A_\tau \cong B_\tau$. Here $\tau$ was an arbitrary type, and the claim is clear.

The fact that $A_1 \cong_{\mathrm{nr}} B_1$ follows from the isomorphism $A_0 \oplus A_1 \cong A_0 \oplus B_1$. □

**Corollary 2.6.** *Suppose $G$ has rank $n$ and $G$ realizes the partitions $(1, \dots, 1, m)$ and $(1, \dots, 1, m')$. Then $m = m'$.*

*Proof.* The indecomposable summands of ranks $m$ and $m'$ are necessarily clipped. By Theorem 2.5, the completely decomposable parts of the decompositions are isomorphic and hence have the same rank, i.e., $n - m = n - m'$, and hence $m = m'$. $\square$

In particular there is no group that realizes both $(1, 1, 2)$ and $(1, 3)$.

We call a decomposition $G = G_{cd} \oplus G_{cl}$ with $G_{cd}$ completely decomposable and $G_{cl}$ clipped a **Main Decomposition of** $G$.

Main Decompositions are unique up to near isomorphism but not unique. For example, let $X = \tau v \oplus \Big( (\tau v_1 \oplus \sigma v_2) + \mathbb{Z}\frac{1}{p}(v_1 \oplus v_2) \Big)$, where $p$ is a prime not dividing $\sigma$ or $\tau$. The group $(\tau v_1 \oplus \sigma v_2) + \mathbb{Z}\frac{1}{p}(v_1 \oplus v_2)$ is indecomposable, hence clipped. We also have $X = \tau(v + v_1) \oplus \Big( (\tau v_1 \oplus \sigma v_2) + \mathbb{Z}\frac{1}{p}(v_1 \oplus v_2) \Big)$ and $\tau v \neq \tau(v + v_1)$.

On the other hand, if $G = G_{cd} \oplus G_{cl}$ and $\mathrm{Hom}(G_{cd}, G_{cl}) = 0$, then $G_{cd}$ is unique and direct complements of $G_{cd}$ are isomorphic ([8, Lemma 1.1.3]).

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Department of Mathematics, University of Hawaii at Manoa, 2565 McCarthy Mall, Honolulu, Hawaii 96922

*E-mail address*: adolf@math.hawaii.edu

School of Mathematics and Statistics, The University of Western Australia, Nedlands, WA, Australia, 6009

*E-mail address*: phill.schultz@uwa.edu.au