A NOTE ON KERNELS, IMAGES, AND COKERNELS IN THE PERVERSE CATEGORY

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ABSTRACT. We discuss the relationship between kernels, images and cokernels of morphisms between perverse sheaves and induced maps on stalk cohomology.

1. INTRODUCTION

General references for the derived category and perverse sheaves are [5], [2], and [3]. As we are always considering the derived category, we follow the usual practice of omitting the "R"s in front of right derived functors.

We fix a complex analytic space X and a base ring, R, which is a commutative, regular, Noetherian ring, with finite Krull dimension (e.g., $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{C}$). We let $\mathrm{Perv}(X)$ denote the Abelian category of perverse sheaves of R-modules on X.

Suppose that $\mathbf{P}^\bullet$ and $\mathbf{Q}^\bullet$ are objects in $\mathrm{Perv}(X)$, and T is a morphism between them. We wish to look at the kernels, images, and cokernels of T in $\mathrm{Perv}(X)$ versus those of the induced maps on the stalk cohomology. The relationship between these – or seeming lack thereof – is well known to experts.

Consider the following simple, but illustrative, example.

Example 1.1. Let X be the union of the coordinate axes in $\mathbb{C}^2$; then, the shifted constant sheaf $\mathbf{R}_X^\bullet[1]$ is perverse. Let f denote the resolution of singularities map from two disjoint complex lines to X. Then, $\mathbf{I}_X^\bullet := f_* f^* \mathbf{R}_X^\bullet[1]$ is also perverse (it is, in fact, the intersection cohomology sheaf with constant coefficients on X).

The perverse sheaf $\mathbf{I}_X^\bullet$ is easy to describe; it is the direct sum of the extensions by zero of the shifted constant sheaves on each of the axes. In particular, the stalk cohomology at $x \in X$ of $\mathbf{I}_X^\bullet$ is zero outside of degree -1 and, in degree -1, is R at $x \neq \mathbf{0}$ and is $R \oplus R$ at $x = \mathbf{0}$.

There is a natural morphism $T : \mathbf{R}_X^\bullet[1] \to f_* f^* \mathbf{R}_X^\bullet[1] = \mathbf{I}_X^\bullet$ which is easy to describe on the level of stalks. Note that, in what follows, the superscript -1 is for the degree, **not** for the inverse map. For each $x \in X$, the map

$$T_x^{-1} : H^{-1}(\mathbf{R}_X^\bullet[1])_x \to H^{-1}(\mathbf{I}_X^\bullet)_x$$

is the diagonal map, i.e., the identity for $x \neq \mathbf{0}$ and the diagonal map $R \to R \oplus R$ when $x = \mathbf{0}$.

Thus, in all degrees, for all $x \in X$, $\ker T_x^i = 0$. However, we claim that $\ker T \neq 0$ in $\mathrm{Perv}(X)$.

How do you see this?

Consider the mapping cone $\mathbf{M}^\bullet$ of $T : \mathbf{R}_X^\bullet[1] \to \mathbf{I}_X^\bullet$ in the derived category, so that we have a distinguished triangle

$$\mathbf{R}_X^\bullet[1] \xrightarrow{T} \mathbf{I}_X^\bullet \to \mathbf{M}^\bullet \xrightarrow{[1]} \mathbf{R}_X^\bullet[1].$$

The complex $\mathbf{M}^\bullet$ is easy to describe; it is supported only at the origin, and has non-zero stalk cohomology only in degree -1, where it is the cokernel of the diagonal map $R \to R \oplus R$, i.e., is isomorphic to R.

The complex $\mathbf{M}^\bullet$ is not perverse; at an isolated point in the support of a perverse sheaf, the stalk cohomology can be non-zero only in degree 0. However, we may "turn the triangle" to obtain the distinguished triangle

$$\mathbf{M}^\bullet[-1] \xrightarrow{S} \mathbf{R}_X^\bullet[1] \xrightarrow{T} \mathbf{I}_X^\bullet \xrightarrow{[1]} \mathbf{M}^\bullet[-1],$$

where $\mathbf{M}^\bullet[-1]$ **is** a perverse sheaf. Now distinguished triangles of perverse sheaves are the short exact sequences in $\mathrm{Perv}(X)$. Therefore, in $\mathrm{Perv}(X)$, we have a short exact sequence

$$0 \to \mathbf{M}^\bullet[-1] \xrightarrow{S} \mathbf{R}_X^\bullet[1] \xrightarrow{T} \mathbf{I}_X^\bullet \to 0,$$

and so $\ker T \cong \mathbf{M}^\bullet[-1] \neq 0$ even though the kernels of all of the induced maps on stalk cohomology are zero.

Furthermore, T is a surjection, even though the induced maps on the stalk cohomology at $\mathbf{0}$ are not surjections in all degrees, and the map S is an injection, even though S induces the zero map on the stalk cohomology in all degrees.

In this note, we wish to clarify what is going on here by looking at induced maps on stalk cohomology after applying the shifted vanishing cycle functor.

We also show that, if the base ring is a field and T is an endomorphism on a perverse sheaf, then there is a nice relationship between the kernels and cokernels of the induced maps on stalk cohomology and the perverse kernel and cokernel of T.

We should remark that there are other, deeper, works which describe the category of perverse sheaves in more detail; see, for instance, [6] and [1] (and, also, [7] for an elaboration on [1]). However, those works do not give the stalk-wise characterization that we are after and, while our results may be derived from these other works, those derivations would be as long as the more elementary proofs here.

Our results do not require that the base ring is a field, nor do they refer to a Whitney stratification with respect to which the perverse sheaves are constructible; in particular, our results do not refer to a choice of a non-degenerate covector. This should be contrasted with, for example, the results in Section 5 of [6].

However, the price we pay for not picking specific strata and non-degenerate covectors is that, at each $x \in X$, we must vary our vanishing cycles ϕ_f over an infinite number of function germs $f : X \to \mathbb{C}$, where we use functions which have "isolated critical points" at x with respect to the given perverse sheaves, i.e., such that the vanishing cycles along f have x as an isolated point in their supports.

2. ENTER THE VANISHING CYCLES

We want to analyze kernels, images, and cokernels in $\mathrm{Perv}(X)$ by looking at stalks and homomorphisms of modules. As we shall see, we can do this if we first take vanishing cycles supported at isolated points.

Example 2.1. Let us look again at the map from Example 1.1, where we had $\mathbf{R}_X^\bullet[1] \xrightarrow{T} \mathbf{I}_X^\bullet$, where X was the union of the x- and y-axes in $\mathbb{C}^2$. We wish to see that taking vanishing cycles can explain why $\ker T$ must not be zero.

Let L be the restriction to X of the linear function $\hat{L}(x,y) = x + y$. Then the shifted vanishing cycles $\phi_L[-1]$ along L is an exact functor from $\mathrm{Perv}(X)$ to $\mathrm{Perv}(V(L)) = \mathrm{Perv}(\{\mathbf{0}\})$, and a perverse sheaf on an isolated point consists, up to isomorphism, of a finitely-generated R-module in degree 0 – its stalk cohomology at the point –and 0's in all other degrees.

Let $\mathbf{K}^\bullet := \ker T$ and $\mathbf{C}^\bullet := \mathrm{coker}\, T$, so that we have an exact sequence in $\mathrm{Perv}(X)$:

$$0 \to \mathbf{K}^\bullet \to \mathbf{R}_X^\bullet[1] \xrightarrow{T} \mathbf{I}_X^\bullet \to \mathbf{C}^\bullet \to 0.$$

Let $T_L := \phi_L[-1](T)$ and let $T_{L,\mathbf{0}}^0$ denote the homomorphism induced by T_L on the stalk cohomology at $\mathbf{0}$ in degree 0.

As $\phi_L[-1]$ is an exact functor, we have an exact sequence of perverse sheaves

$$0 \to \phi_L[-1]\mathbf{K}^\bullet \to \phi_L[-1]\mathbf{R}_X^\bullet[1] \xrightarrow{T_L} \phi_L[-1]\mathbf{I}_X^\bullet \to \phi_L[-1]\mathbf{C}^\bullet \to 0,$$

which are all supported at just the origin; so this exact sequence corresponds to the sequence of R-modules:

$$0 \to H^0\left(\phi_L[-1]\mathbf{K}^\bullet\right)_{\mathbf{0}} \to H^0\left(\phi_L[-1]\mathbf{R}_X^\bullet[1]\right)_{\mathbf{0}}$$

$$\xrightarrow{T_{L,\mathbf{0}}^0} H^0\left(\phi_L[-1]\mathbf{I}_X^\bullet\right)_{\mathbf{0}} \to H^0\left(\phi_L[-1]\mathbf{C}^\bullet\right)_{\mathbf{0}} \to 0.$$

Now, $H^0\left(\phi_L[-1]\mathbf{R}_X^\bullet[1]\right)_{\mathbf{0}} \cong \mathbb{Z}$, while $H^0\left(\phi_L[-1]\mathbf{I}_X^\bullet\right)_{\mathbf{0}} = 0$. Consequently,

$$H^0\left(\phi_L[-1]\mathbf{K}^\bullet\right)_{\mathbf{0}} \cong \mathbb{Z},$$

and so $\mathbf{K}^\bullet$ cannot be the zero complex.

We return to the case of a general analytic space X and a morphism $\mathbf{P}^\bullet \xrightarrow{T} \mathbf{Q}^\bullet$. We will use the vanishing cycles along various functions to analyze the situation. First we need two definitions.

Definition 2.2. We define an ordering on (isomorphism classes of) Noetherian R-modules by: $M \leq N$ if and only if there exists an R-module P such that $M \oplus P \cong N$.

Note that reflexivity and transitivity are immediate. Anti-symmetry reduces to proving the weak cancellation property that $M \oplus P \cong M$ implies that $P = 0$, provided that M is Noetherian; this is an easy exercise.

Definition 2.3. Let $x \in X$, let $\mathcal{U}$ be an open neighborhood of x in X, and let $f: (\mathcal{U}, x) \to (\mathbb{C}, 0)$ such that $\dim_x \mathrm{supp}\, \phi_f[-1]\left(\mathbf{P}_{|\mathcal{U}}^\bullet\right) \leq 0$ and $\dim_x \mathrm{supp}\, \phi_f[-1]\left(\mathbf{Q}_{|\mathcal{U}}^\bullet\right) \leq 0$.

Then we will say that f is $(\mathbf{P}^\bullet, \mathbf{Q}^\bullet)$-**isolating** at x, and we let $T_{f,x}^0$ denote the map induced by $\phi_f[-1]\left(T_{|\mathcal{U}}\right)$ from $H^0\left(\phi_f[-1](\mathbf{P}_{|\mathcal{U}}^\bullet)\right)_x$ to $H^0\left(\phi_f[-1](\mathbf{Q}_{|\mathcal{U}}^\bullet)\right)_x$. As this depends only on the germ of f at x, we will suppress the explicit restrictions to $\mathcal{U}$ below.

The cases above where $\dim_x < 0$ are meant to allow for the possibility that the supports are "empty at x", i.e., the cases where x is not in the supports. Note that, for all $x \in X$, there exists an f which is $(\mathbf{P}^\bullet, \mathbf{Q}^\bullet)$-isolating at x since we may select a common Whitney stratification $\mathfrak{S}$ of X with respect to which both $\mathbf{P}^\bullet$ and

$\mathbf{Q}^\bullet$ are constructible, and then, if S is the stratum containing x, take f to have a non-degenerate critical point at x with respect to $\mathfrak{S}$ in the sense of [4], 6.A.2.

Now we can state:

Theorem 2.4. *Given a morphism $T : \mathbf{P}^\bullet \to \mathbf{Q}^\bullet$, we have:*

  (1)

$\operatorname{supp}(\ker T)$

$$= \overline{\{x \in X \mid \text{ there exists a } (\mathbf{P}^\bullet, \mathbf{Q}^\bullet)\text{-isolating } f \text{ at } x \text{ such that } \ker T^0_{f,x} \neq 0\}}.$$

  (2) *Suppose that Y is an irreducible component of $\operatorname{supp}(\ker T)$, and let $d :=$ $\dim Y$. Then, for a generic point $x \in Y$, $H^*(\ker T)_x$ is zero, except in degree $-d$, where we have*

$$H^{-d}(\ker T)_x \cong M_{\min,x},$$

where $M_{\min,x}$ is the minimum non-zero module which occurs as $\ker\left(T^0_{f,x}\right)$ for some $(\mathbf{P}^\bullet, \mathbf{Q}^\bullet)$-isolating f at x.

Furthermore, the above statements remain true if each instance of ker *is replaced by* coker, *or if each instance of* ker *is replaced by* im.

Proof. The proofs for kernels, images, and cokernels are very similar; we prove the kernel and image statements, and leave the cokernel statements as exercises.

We define

$\operatorname{stalkker}_\phi$

$$:= \overline{\{x \in X \mid \text{ there exists a } (\mathbf{P}^\bullet, \mathbf{Q}^\bullet)\text{-isolating } f \text{ at } x \text{ such that } \ker T^0_{f,x} \neq 0\}},$$

and

$\operatorname{stalkim}_\phi$

$$:= \overline{\{x \in X \mid \text{ there exists a } (\mathbf{P}^\bullet, \mathbf{Q}^\bullet)\text{-isolating } f \text{ at } x \text{ such that } \operatorname{im} T^0_{f,x} \neq 0\}}.$$

Let $\mathbf{K}^\bullet := \ker T$, $\mathbf{I}^\bullet := \operatorname{im} T$, and $\mathbf{C}^\bullet := \operatorname{coker} T$, so that we have two short exact sequences of perverse sheaves

$$(\dagger) \qquad 0 \to \mathbf{K}^\bullet \to \mathbf{P}^\bullet \xrightarrow{\alpha} \mathbf{I}^\bullet \to 0 \quad \text{and} \quad 0 \to \mathbf{I}^\bullet \xrightarrow{\beta} \mathbf{Q}^\bullet \to \mathbf{C}^\bullet \to 0,$$

where $T = \beta \circ \alpha$; of course, we also have the combined exact sequence

$$(\ddagger) \qquad\qquad 0 \to \mathbf{K}^\bullet \to \mathbf{P}^\bullet \xrightarrow{T} \mathbf{Q}^\bullet \to \mathbf{C}^\bullet \to 0.$$

Kernel statements: We shall first show that $\operatorname{stalkker}_\phi \subseteq \operatorname{supp} \mathbf{K}^\bullet$ by showing that $X - \operatorname{supp}(\mathbf{K}^\bullet) = X - \operatorname{stalkker}_\phi$.

Suppose that $p \in X - \operatorname{supp} \mathbf{K}^\bullet$, i.e., suppose that $p \in X$ possesses an open neighborhood $\mathcal{U}$ such that, for all $x \in \mathcal{U}$, $H^*(\mathbf{K}^\bullet)_x = 0$, i.e., $\mathbf{K}^\bullet = 0$ on $\mathcal{U}$. Restricting to U, we are reduced to the case where ($\ddagger$) becomes the short exact sequence

$$0 \to \mathbf{P}^\bullet \xrightarrow{T} \mathbf{Q}^\bullet \to \mathbf{C}^\bullet \to 0.$$

Suppose that f is $(\mathbf{P}^\bullet, \mathbf{Q}^\bullet)$-isolating at a point $x \in \mathcal{U}$. Then, as $\phi_f[-1]$ is an exact functor, we have a short exact sequence

$$0 \to H^0(\phi_f[-1]\mathbf{P}^\bullet)_x \xrightarrow{T^0_{f,x}} H^0(\phi_f[-1]\mathbf{Q}^\bullet)_x \to H^0(\phi_f[-1]\mathbf{C}^\bullet)_x \to 0.$$

Thus, $\ker\left(T^0_{f,x}\right) = 0$; this shows that $p \in X - \text{stalkker}_\phi$. Hence, we have proved that
$$X - \text{supp}(\mathbf{K}^\bullet) \subseteq X - \text{stalkker}_\phi,$$
i.e., that $\text{stalkker}_\phi \subseteq \text{supp}(\mathbf{K}^\bullet)$.

We must show the reverse containment; in fact, we shall prove Item 2 of the theorem at the same time.

Suppose that $p \in \text{supp}(\mathbf{K}^\bullet)$. Let $\mathfrak{S}$ be a Whitney stratification, with connected strata, of X, with respect to which $\mathbf{P}^\bullet$, $\mathbf{Q}^\bullet$, $\mathbf{K}^\bullet$, and $\mathbf{C}^\bullet$ are all constructible. Then p must be in the closure of a maximal stratum S (ordered in the standard way by inclusion in the closure) of $\text{supp}(\mathbf{K}^\bullet)$.

Let $x \in S$ and let $d := \dim S$. Let f be a function from an open neighborhood of x in X to $\mathbb{C}$ with a complex non-degenerate critical point at x with respect to $\mathfrak{S}$ (in the sense of [4], 6.A.2). Then f is $(\mathbf{P}^\bullet, \mathbf{Q}^\bullet)$-isolating at x, and $H^i(\phi_f[-1]\mathbf{K}^\bullet)_x$ is zero if $i \neq 0$, while $H^0(\phi_f[-1]\mathbf{K}^\bullet)_x \cong H^{-d}(\mathbf{K}^\bullet)_x \neq 0$; note that $H^i(\phi_f[-1]\mathbf{K}^\bullet)_x$ is a minimum among such non-zero modules as f has a non-degenerate critical point at x.

Now, we have the exact sequence
$$0 \to H^0\left(\phi_f[-1]\mathbf{K}^\bullet\right)_x \to H^0\left(\phi_f[-1]\mathbf{P}^\bullet\right)_x$$
$$\xrightarrow{T^0_{f,x}} H^0\left(\phi_f[-1]\mathbf{Q}^\bullet\right)_x \to H^0\left(\phi_f[-1]\mathbf{C}^\bullet\right)_x \to 0.$$

Therefore, $\ker T^0_{f,x} \cong H^0\left(\phi_f[-1]\mathbf{K}^\bullet\right)_x \neq 0$, and we have shown that $p \in \text{stalkker}_\phi$, i.e., that $\text{supp}(\mathbf{K}^\bullet) \subseteq \text{stalkker}_\phi$.

Image statements: We shall first show that $\text{stalkim}_\phi \subseteq \text{supp}\,\mathbf{I}^\bullet$ by showing that $X - \text{supp}(\mathbf{I}^\bullet) = X - \text{stalkim}_\phi$.

Suppose that $p \in X - \text{supp}\,\mathbf{I}^\bullet$, i.e., suppose that $p \in X$ possesses an open neighborhood $\mathcal{U}$ such that, for all $x \in \mathcal{U}$, $H^*(\mathbf{I}^\bullet)_x = 0$, i.e., $\mathbf{I}^\bullet = 0$ on $\mathcal{U}$. Restricting to U, we are reduced to the case where ($\ddagger$) becomes the exact sequence
$$0 \to \mathbf{K}^\bullet \to \mathbf{P}^\bullet \xrightarrow{0} \mathbf{Q}^\bullet \to \mathbf{C}^\bullet \to 0.$$

Suppose that f is $(\mathbf{P}^\bullet, \mathbf{Q}^\bullet)$-isolating at a point $x \in \mathcal{U}$. Then, it follows immediately that $T^0_{f,x} = 0$, i.e., $\text{im}\left(T^0_{f,x}\right) = 0$. This shows that $p \in X - \text{stalkim}_\phi$. Hence, we have proved that
$$X - \text{supp}(\mathbf{I}^\bullet) \subseteq X - \text{stalkim}_\phi,$$
i.e., that $\text{stalkim}_\phi \subseteq \text{supp}(\mathbf{I}^\bullet)$.

We must show the reverse containment; in fact, we shall again prove Item 2 of the theorem at the same time.

Suppose that $p \in \text{supp}(\mathbf{I}^\bullet)$. Let $\mathfrak{S}$ be a Whitney stratification, with connected strata, of X, with respect to which $\mathbf{P}^\bullet$, $\mathbf{Q}^\bullet$, $\mathbf{K}^\bullet$, $\mathbf{I}^\bullet$, and $\mathbf{C}^\bullet$ are all constructible. Then p must be in the closure of a maximal stratum S of $\text{supp}(\mathbf{I}^\bullet)$.

Let $x \in S$ and let $d := \dim S$. Let f be a function from an open neighborhood of x in X to $\mathbb{C}$ with a complex non-degenerate critical point at x with respect to $\mathfrak{S}$ (in the sense of [4], 6.A.2). Then f is $(\mathbf{P}^\bullet, \mathbf{Q}^\bullet)$-isolating at x, and $H^i(\phi_f[-1]\mathbf{I}^\bullet)_x$ is zero if $i \neq 0$, while
$$H^0(\phi_f[-1]\mathbf{I}^\bullet)_x \cong H^{-d}(\mathbf{I}^\bullet)_x \neq 0;$$
note, as before, that $H^i(\phi_f[-1]\mathbf{I}^\bullet)_x$ is a minimum among such non-zero modules as f has a non-degenerate critical point at x.

Now, letting $\alpha_{f,x}^0$ and $\beta_{f,x}^0$ denote the maps induced on the stalk cohomology in degree 0 at x by the maps $\phi_f[-1](\alpha)$ and $\phi_f[-1](\beta)$, respectively, we have the short exact sequences

$$0 \to H^0\big(\phi_f[-1]\mathbf{K}^\bullet\big)_x \to H^0\big(\phi_f[-1]\mathbf{P}^\bullet\big)_x \xrightarrow{\alpha_{f,x}^0} H^0\big(\phi_f[-1]\mathbf{I}^\bullet\big)_x \to 0$$

and

$$0 \to H^0\big(\phi_f[-1]\mathbf{I}^\bullet\big)_x \xrightarrow{\beta_{f,x}^0} H^0\big(\phi_f[-1]\mathbf{Q}^\bullet\big)_x \to H^0\big(\phi_f[-1]\mathbf{C}^\bullet\big)_x \to 0,$$

where $T_{f,x}^0 = \beta_{f,x}^0 \circ \alpha_{f,x}^0$.

As $\alpha_{f,x}^0$ is a surjection and $\beta_{f,x}^0$ is an injection, we see that

$$\operatorname{im} T_{f,x}^0 = \operatorname{im}(\beta_{f,x}^0 \circ \alpha_{f,x}^0) \cong \operatorname{im}(\beta_{f,x}^0) \cong H^0\big(\phi_f[-1]\mathbf{I}^\bullet\big)_x \cong H^{-d}(\mathbf{I}^\bullet)_x \neq 0.$$

Therefore, we have shown that $p \in \operatorname{stalkim}_\phi$, i.e., that $\operatorname{supp}(\mathbf{I}^\bullet) \subseteq \operatorname{stalkim}_\phi$. $\quad\square$

Corollary 2.5. *The morphism $T : \mathbf{P}^\bullet \to \mathbf{Q}^\bullet$ is an injection (resp., zero morphism, surjection) if and only if, for all $x \in X$, for all $(\mathbf{P}^\bullet, \mathbf{Q}^\bullet)$-isolating f at x, $T_{f,x}^0$ is an injection (resp., zero morphism, surjection).*

Proof. This is immediate from Item 1 of the theorem in the kernel, image, and cokernel cases. $\quad\square$

3. The special case of an endomorphism over a field

It may seem strange, but – in the case of an endomorphism where the base ring is a field – we do not need to apply vanishing cycles in order to obtain a result along the lines of Theorem 2.4, though we must drop the conclusion about images.

We will use properties of characteristic cycles (see [5], [2], [8]):

- Characteristic cycles $\mathrm{CC}(\mathbf{P}^\bullet)$ of complexes of sheaves are additive over distinguished triangles; in particular, characteristic cycles are additive over exact sequences in $\mathrm{Perv}(X)$.
- For perverse sheaves $\mathbf{P}^\bullet$ with a field for a base ring, the subset X which underlies $\mathrm{CC}(\mathbf{P}^\bullet)$ equals $\operatorname{supp}\mathbf{P}^\bullet$; in particular, if $\mathrm{CC}(\mathbf{P}^\bullet) = 0$, then $\mathbf{P}^\bullet = 0$.

Theorem 3.1. *If the base ring is a field, and we have an endomorphism $T : \mathbf{P}^\bullet \to \mathbf{P}^\bullet$, then:*

(1)
$$\operatorname{supp}(\ker T) = \overline{\{x \in X \,|\, \ker T_x^* \neq 0\}}.$$

(2) *Suppose that Y is an irreducible component of $\operatorname{supp}(\ker T)$, and let $d := \dim Y$. Then, for a generic point $x \in Y$, $H^*(\ker T)_x$ is zero, except in degree $-d$, where we have*

$$H^{-d}(\ker T)_x \cong \ker\big(T_x^{-d}\big).$$

Furthermore, the above statements remain true if each instance of ker *is replaced by* coker.

Proof. We shall prove the statements about kernels; the cokernel proof is completely analogous. Let $\mathbf{K}^\bullet := \ker T$ and $\mathbf{C}^\bullet := \operatorname{coker} T$, and $\mathbf{I}^\bullet = \operatorname{im} T$.

Suppose that $p \in X - \mathrm{supp}(\mathbf{K}^\bullet)$. Then there exists an open neighborhood $\mathcal{U}$ of p such that the restriction of $\mathbf{K}^\bullet$ to $\mathcal{U}$ is zero. Thus, restricting to $\mathcal{U}$, we have a short exact sequence in $\mathrm{Perv}(X)$:

$$0 \to \mathbf{P}^\bullet \xrightarrow{T} \mathbf{P}^\bullet \to \mathbf{C}^\bullet \to 0.$$

But now the properties of characteristic cycles – the additivity and support properties – imply instantly that $\mathbf{C}^\bullet = 0$. Therefore, T is an isomorphism, i.e., induces isomorphisms on the stalk cohomology at every point. And so, for all $x \in \mathcal{U}$, for all degrees i, $\ker T_x^i = 0$. Consequently, $p \in X - \overline{\{x \in X \mid \ker T_x^* \neq 0\}}$. Thus, we have shown that $\overline{\{x \in X \mid \ker T_x^* \neq 0\}} \subseteq \mathrm{supp}(\mathbf{K}^\bullet)$.

We must show the reverse containment; in fact, we shall again prove Item 2 of the theorem at the same time.

We have the canonical exact sequence

$$0 \to \mathbf{K}^\bullet \to \mathbf{P}^\bullet \xrightarrow{T} \mathbf{P}^\bullet \to \mathbf{C}^\bullet \to 0.$$

Once again, using properties of characteristic cycles, we see that $\mathrm{CC}(\mathbf{K}^\bullet) = \mathrm{CC}(\mathbf{C}^\bullet)$, and so $\mathrm{supp}\,\mathbf{K}^\bullet = \mathrm{supp}\,\mathbf{C}^\bullet$.

Suppose that $p \in \mathrm{supp}(\mathbf{K}^\bullet)$. Let $\mathfrak{S}$ be a Whitney stratification, with connected strata, of X, with respect to which $\mathbf{P}^\bullet$, $\mathbf{K}^\bullet$, and $\mathbf{C}^\bullet$ are all constructible. Then p must be in the closure of a maximal stratum S of $\mathrm{supp}(\mathbf{K}^\bullet) = \mathrm{supp}(\mathbf{C}^\bullet)$.

Let $x \in S$ and let $d := \dim S$. Then, $H^*(\mathbf{K}^\bullet)_x$ and $H^*(\mathbf{C}^\bullet)_x$ are both non-zero precisely in degree $-d$.

From the short exact sequence

$$0 \to \mathbf{I}^\bullet \xrightarrow{\beta} \mathbf{P}^\bullet \to \mathbf{C}^\bullet \to 0,$$

and using that $H^{-d-2}(\mathbf{C}^\bullet)_x = H^{-d-1}(\mathbf{C}^\bullet)_x = 0$, we conclude that:

- $H^{-d-1}(\mathbf{I}^\bullet)_x \cong H^{-d-1}(\mathbf{P}^\bullet)_x$, and
- β_x^{-d} from $H^{-d}(\mathbf{I}^\bullet)_x$ to $H^{-d}(\mathbf{P}^\bullet)_x$ is an injection.

From the short exact sequence

$$0 \to \mathbf{K}^\bullet \to \mathbf{P}^\bullet \xrightarrow{\alpha} \mathbf{I}^\bullet \to 0,$$

and using that $H^{-d-1}(\mathbf{K}^\bullet)_x = H^{-d+1}(\mathbf{K}^\bullet)_x = 0$, we conclude that we have an exact sequence

$$0 \to H^{-d-1}(\mathbf{P}^\bullet)_x \to H^{-d-1}(\mathbf{I}^\bullet)_x \to H^{-d}(\mathbf{K}^\bullet)_x \to H^{-d}(\mathbf{P}^\bullet)_x \to H^{-d}(\mathbf{I}^\bullet)_x \to 0.$$

However, as we saw above, $H^{-d-1}(\mathbf{I}^\bullet)_x$ and $H^{-d-1}(\mathbf{P}^\bullet)_x$ are isomorphic finite-dimensional vector spaces and, consequently, the injection on the left of the above exact sequence is an isomorphism.

Therefore, we have a short exact sequence

$$0 \to H^{-d}(\mathbf{K}^\bullet)_x \to H^{-d}(\mathbf{P}^\bullet)_x \xrightarrow{\alpha_x^{-d}} H^{-d}(\mathbf{I}^\bullet)_x \to 0.$$

As β_x^{-d} is an injection, we conclude that

$$\ker T_x^{-d} = \ker(\beta_x^{-d} \circ \alpha_x^{-d}) = \ker \alpha_x^{-d} \cong H^{-d}(\mathbf{K}^\bullet)_x \neq 0.$$

This concludes the proof. $\qquad\square$

Remark 3.2. Our primary interest in Theorem 3.1 centers around eigenvalues of the monodromy for the nearby and vanishing cycles.

Suppose that our base ring is $\mathbb{C}$. Let $\mathbf{Q}^\bullet$ be a perverse sheaf on a complex analytic space Y and suppose that we have a complex analytic $f : Y \to \mathbb{C}$. Then, we have the perverse sheaves $\psi_f[-1]\mathbf{Q}^\bullet$ and $\phi_f[-1]\mathbf{Q}^\bullet$ on $X := V(f)$, together with their respective monodromy automorphisms, T_f and $\widetilde{T}_f$.

For each $\lambda \in \mathbb{C}$, there is the question: what does it mean for λ to be an eigenvalue of T_f (resp., $\widetilde{T}_f$)? Does it mean that $\ker(\lambda \cdot \mathrm{id} - T_f) \neq 0$ (resp., $\ker(\lambda \cdot \mathrm{id} - \widetilde{T}_f) \neq 0$) or does it mean that there is a point $x \in X$ and a degree i such that, in the stalk cohomology, we have $\ker(\lambda \cdot \mathrm{id} - T_{f,x}^i) \neq 0$ (resp., $\ker(\lambda \cdot \mathrm{id} - \widetilde{T}_{f,x}^i) \neq 0$)?

Item 1 of Theorem 3.1 tells us that these conditions are equivalent.

Example 3.3. The statement about images that one might expect to find in Theorem 3.1 is simply false.

Let R be a field and consider the injection $\mathbf{M}^\bullet[-1] \xrightarrow{S} \mathbf{R}_X^\bullet[1]$ from Example 1.1. Define an endomorphism

$$\mathbf{M}^\bullet[-1] \oplus \mathbf{R}_X^\bullet[1] \xrightarrow{T} \mathbf{M}^\bullet[-1] \oplus \mathbf{R}_X^\bullet[1]$$

by $T(a,b) = (0, S(a))$. Then the image of T is isomorphic to $\mathbf{M}^\bullet[-1]$, while all of the induced maps on stalk cohomology are zero.

It is interesting to note that $\ker T \ncong \mathrm{coker}\, T$, for $\ker T \cong \mathbf{R}_X^\bullet[1]$ and $\mathrm{coker}\, T \cong \mathbf{M}^\bullet[-1] \oplus \mathbf{I}_X^\bullet$, where $\mathbf{I}_X^\bullet$ is as in Example 1.1. And yet,

$$\mathrm{CC}\left(\mathbf{R}_X^\bullet[1]\right) = \left[T_{\mathbf{0}}^* \mathbb{C}^2\right] + \left[T_{V(y)}^* \mathbb{C}^2\right] + \left[T_{V(x)}^* \mathbb{C}^2\right] = \mathrm{CC}\left(\mathbf{M}^\bullet[-1] \oplus \mathbf{I}_X^\bullet\right)$$

(or, depending on one's shifting convention on the characteristic cycle, the characteristic cycle may be the negation of what we give).

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