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A REMARK ON THE PRODUCT PROPERTY FOR THE GENERALIZED MÖBIUS FUNCTION

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ABSTRACT. We discuss an example related to the product property for the generalized Möbius function.

1. INTRODUCTION

For a domain $G \subset \mathbb{C}^n$ and a set $\varnothing \neq A \subset G$, the *generalized Möbius function* $\boldsymbol{m}_G(A, \cdot)$ *for* G *with poles at* A is defined by the formula:

$$\boldsymbol{m}_G(A, z) := \sup\{|f(z)| : f \in \mathcal{O}(G, \mathbb{D}),\ f|_A \equiv 0\}, \quad z \in G,$$

where $\mathbb{D} \subset \mathbb{C}$ stands for the unit disc (cf. [Jar-Pfl 2013, Definition 8.2.2]). For an arbitrary set $A \subset \mathbb{C}^n$ with $A \cap G \neq \varnothing$ we put $\boldsymbol{m}_G(A, \cdot) := \boldsymbol{m}_G(A \cap G, \cdot)$. It is an open problem whether the generalized Möbius function has the following *product property*:

(PP) for any $n_j \in \mathbb{N}$, $G_j \subset \mathbb{C}^{n_j}$, and $\varnothing \neq A_j \subset G_j$, $j = 1, 2$, we have

$$\boldsymbol{m}_{G_1 \times G_2}(A_1 \times A_2, (z_1, z_2)) = \max\{\boldsymbol{m}_{G_1}(A_1, z_1), \boldsymbol{m}_{G_2}(A_2, z_2)\},\ (z_1, z_2) \in G_1 \times G_2;$$

cf. [Jar-Pfl 2013, § 18.3]. So far the product property (PP) has been proved only in the case where $\min\{\#A_1, \#A_2\} = 1$ (cf. [Jar-Pfl 2013, Theorem 18.3.2]). On the other hand, it is known that the *generalized Green function* $\boldsymbol{g}_G(A, \cdot)$ *for* G *with poles at* A has the product property (cf. [Edi1997], [Edi2001]). Recall that

$$\boldsymbol{g}_G(A, z) := \sup\{u(z) : \ u : G \longrightarrow [0, 1),\ \log u \in \mathcal{PSH}(G),$$
$$\forall_{a \in A}\ \exists_{C > 0}\ \forall_{w \in G} :\ u(w) \leq C\|w - a\|\}, \quad z \in G.$$

Clearly, $\boldsymbol{m}_G(A, \cdot) \leq \boldsymbol{g}_G(A, \cdot)$. Thus, if $\boldsymbol{m}_{G_j}(A_j, \cdot) \equiv \boldsymbol{g}_{G_j}(A_j, \cdot)$, $j = 1, 2$, then (PP) is satisfied. Define $\boldsymbol{\Psi} : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$,

$$\boldsymbol{\Psi}(z, w) := \sum_{s=1}^{n} z_s w_s = \langle z, \overline{w} \rangle, \quad z = (z_1, \ldots, z_n),\ w = (w_1, \ldots, w_n) \in \mathbb{C}^n,$$

where $\langle\ ,\ \rangle$ denotes the standard complex Euclidean scalar product in $\mathbb{C}^n$.

Let $\mathcal{B}_n$ denote the class of all open unit balls $\boldsymbol{B} = \{z \in \mathbb{C}^n : \|z\| < 1\}$, where $\|\ \|$ is an arbitrary $\mathbb{C}$-norm.

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It is known (cf. [Jar-Pfl 2013, Proposition 18.3.1]) that the product property (PP) is equivalent to the following seemingly simpler condition:

(PP′) for any $n \in \mathbb{N}$, $\boldsymbol{B}_j \in \mathcal{B}_n$, and a finite set $\varnothing \neq A_j \subset \boldsymbol{B}_j$, $j = 1, 2$, such that $A_1 \times A_2 \subset \boldsymbol{\Psi}^{-1}(0)$ we have

$$|\boldsymbol{\Psi}(z_1, z_2)| \leq (\sup_{\boldsymbol{B}_1 \times \boldsymbol{B}_2} |\boldsymbol{\Psi}|) \max\{\boldsymbol{m}_{\boldsymbol{B}_1}(A_1, z_1), \boldsymbol{m}_{\boldsymbol{B}_2}(A_2, z_2)\},\ (z_1, z_2) \in \boldsymbol{B}_1 \times \boldsymbol{B}_2.$$

Notice the following example due to W. Zwonek (cf. [Jar-Pfl 2013, Example 8.2.28]):

If $\boldsymbol{B} := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2| < 1\}$ and $A := \{(t, \sqrt{t}), (t, -\sqrt{t})\}$ with $0 < t \ll 1$, then $\boldsymbol{m}_{\boldsymbol{B}}(A, (0, 0)) < \boldsymbol{g}_{\boldsymbol{B}}(A, (0, 0))$.

Thus even the simpler condition (PP′) cannot be a direct consequence of the product property for the generalized Green function.

For $\varnothing \neq S \subset \mathbb{C}^n$ define $S^\circ := \{w \in \mathbb{C}^n : \sup_{z \in S} |\boldsymbol{\Psi}(z, w)| < 1\}$.

Our aim is to prove the following two propositions.

Proposition 1. *The condition* (PP′) *is equivalent to the following one:*

(PP″) *for any $n \geq 2$ and $1 \leq d \leq n - 1$ we have:*

$$|\boldsymbol{\Psi}(z_1, z_2)| \leq \max\{\boldsymbol{m}_{\boldsymbol{B}}(M, z_1), \boldsymbol{m}_{\boldsymbol{B}^\circ}(M^\circ, z_2)\}, \quad (z_1, z_2) \in \boldsymbol{B} \times \boldsymbol{B}^\circ,$$

where $\boldsymbol{B} \in \mathcal{B}_n$ and $M = \mathbb{C}^d \times \{0\}^{n-d}$.

Note that in this case $M^\circ = \{0\}^d \times \mathbb{C}^{n-d}$. We *conjecture* that in the above situation we have

$$(*) \qquad \boldsymbol{m}_{\boldsymbol{B}}(M, \cdot) \equiv \boldsymbol{g}_{\boldsymbol{B}}(M, \cdot), \quad \boldsymbol{m}_{\boldsymbol{B}^\circ}(M^\circ, \cdot) = \boldsymbol{g}_{\boldsymbol{B}^\circ}(M^\circ, \cdot).$$

If (*) were true, we could get (PP″) (and hence the product property for the generalized Möbius function in the full generality) as a consequence of the product property for the generalized Green function.

So far we have verified (*) only in the following special case.

Proposition 2. *Assume that*

$$(**) \qquad \|(z, \lambda w)\| \leq \|(z, w)\|, \quad (z, w) \in \mathbb{C}^d \times \mathbb{C}^{n-d},\ \lambda \in \mathbb{T} := \partial\mathbb{D}.$$

Then (*) *is satisfied.*

2. Proofs

Remark 3. (a) $\|\cdot\|^\circ := \sup_{z \in \boldsymbol{B}} |\boldsymbol{\Psi}(z, \cdot)|$ is a $\mathbb{C}$-norm.

(b) If $M \subset \mathbb{C}^n$ is a $\mathbb{C}$-vector subspace, then

$$M^\circ = \{w \in \mathbb{C}^n : \forall_{z \in M} : \boldsymbol{\Psi}(z, w) = 0\} = \{\overline{w} : w \in M^\perp\}$$

($M^\perp$ is taken in the sense of the scalar product $\langle\ ,\ \rangle$). Consequently, M° is a $\mathbb{C}$-vector space and $\dim M^\circ = n - \dim M$.

(c) Let $U : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a unitary isomorphism. Put $U'(w) := \overline{U(\overline{w})}$. Then $\boldsymbol{\Psi}(U(z), U'(w)) = \boldsymbol{\Psi}(z, w)$, $z, w \in \mathbb{C}^n$. Consequently, $(U(S))^\circ = U'(S^\circ)$.

Proof of Proposition 1. We have to prove that (PP″) $\Longrightarrow$ (PP′). First we prove that (PP″) implies that

(a) for any n we have

$$|\boldsymbol{\Psi}(z, w)| \leq \max\{\boldsymbol{m}_{\boldsymbol{B}}(M, z), \boldsymbol{m}_{\boldsymbol{B}^\circ}(M^\circ, w)\}, \quad (z, w) \in \boldsymbol{B} \times \boldsymbol{B}^\circ,$$

where $\boldsymbol{B} \in \mathcal{B}_n$ and M is a $\mathbb{C}$-vector subspace of $\mathbb{C}^n$.

Put $M_0 := \mathbb{C}^d \times \{0\}^{n-d}$. Let $U : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a unitary mapping such that $U(M) = M_0$. Put $U'(w) := \overline{U(\overline{w})}$. Let us apply (PP$''$) to $(U(\boldsymbol{B}), U(M))$. Using Remark 3(c) and the fact that the generalized Möbius function is holomorphically invariant (cf. [Jar-Pfl 2013, Remark 8.2.4(l)]), we get for $(z,w) \in \boldsymbol{B} \times \boldsymbol{B}^\circ$:

$$\begin{aligned}|\boldsymbol{\Psi}(z,w)| &= |\boldsymbol{\Psi}(U(z), U'(w))| \\ &\le \max\{\boldsymbol{m}_{U(\boldsymbol{B})}(U(M), U(z)), \boldsymbol{m}_{(U(\boldsymbol{B}))^\circ}((U(M))^\circ, U'(w))\} \\ &= \max\{\boldsymbol{m}_{\boldsymbol{B}}(M,z), \boldsymbol{m}_{\boldsymbol{B}^\circ}(M^\circ, w)\},\end{aligned}$$

which gives (a).

Now we prove that (a) implies

(b) for any n we have

$$|\boldsymbol{\Psi}(z,w)| \le \max\{\boldsymbol{m}_{\boldsymbol{B}_1}(A_1,z), \boldsymbol{m}_{\boldsymbol{B}_2}(A_2,w)\}, \quad (z,w) \in \boldsymbol{B}_1 \times \boldsymbol{B}_2,$$

where $\boldsymbol{B}_j \in \mathcal{B}_n$, $\varnothing \ne A_j \subset \boldsymbol{B}_j$, A_j is finite, $j = 1,2$, are such that $A_1 \times A_2 \subset \boldsymbol{\Psi}^{-1}(0)$, and $\boldsymbol{\Psi}(\boldsymbol{B}_1 \times \boldsymbol{B}_2) \subset \mathbb{D}$.

Define $\boldsymbol{B} := \boldsymbol{B}_1$, $M := \operatorname{span} A_1$. Observe that $\boldsymbol{B}_2 \subset \boldsymbol{B}^\circ$ and $A_2 \subset M^\circ$. Consequently,

$$|\boldsymbol{\Psi}(z,w)| \le \max\{\boldsymbol{m}_{\boldsymbol{B}}(M,z), \boldsymbol{m}_{\boldsymbol{B}^\circ}(M^\circ,w)\} \le \max\{\boldsymbol{m}_{\boldsymbol{B}_1}(A_1,z), \boldsymbol{m}_{\boldsymbol{B}_2}(A_2,w)\}, \qquad (z,w) \in \boldsymbol{B} \times \boldsymbol{B}^\circ.$$

Notice that the cases $\dim M = 0$ or $\dim M = n$ follow from the fact that (PP) is true if $\max\{\#A_1, \#A_2\} = 1$.

Finally, we prove (b) $\Longrightarrow$ (PP$'$).

Let $C := \sup_{\boldsymbol{B}_1 \times \boldsymbol{B}_2} |\boldsymbol{\Psi}|$, $r := 1/\sqrt{C}$. Then $|\boldsymbol{\Psi}| \le 1$ on $(r\boldsymbol{B}_1) \times (r\boldsymbol{B}_2)$. Thus,

$$\begin{aligned}\tfrac{1}{C}|\boldsymbol{\Psi}(z_1,z_2)| &= |\boldsymbol{\Psi}(rz_1, rz_2)| \le \max\{\boldsymbol{m}_{r\boldsymbol{B}_1}(rA_1, rz_1), \boldsymbol{m}_{r\boldsymbol{B}_2}(rA_2, rz_2)\} \\ &= \max\{\boldsymbol{m}_{\boldsymbol{B}_1}(A_1,z_1), \boldsymbol{m}_{\boldsymbol{B}_2}(A_2,z_2)\}, \quad (z_1,z_2) \in \boldsymbol{B}_1 \times \boldsymbol{B}_2. \qquad \square\end{aligned}$$

Remark 4. (a) (**) implies that $\|(z,\lambda w)\| = \|(z,w)\|$, $(z,w) \in \mathbb{C}^d \times \mathbb{C}^{n-d}$, $\lambda \in \mathbb{T}$.

(b) By the maximum principle for plurisubharmonic functions we have $\|(z,\lambda w)\| \le \|(z,w)\|$, $(z,w) \in \mathbb{C}^n$, $\lambda \in \overline{\mathbb{D}}$. In particular, for every $(z,w) \in \boldsymbol{B}$ the set $\boldsymbol{B}_z := \{w \in \mathbb{C}^{n-d} : \|(z,w)\| < 1\}$ is a convex balanced domain.

(c) $(\boldsymbol{B}^\circ, M^\circ)$ satisfies the condition analogous to (**), namely $\|(\lambda u, v)\|^\circ \le \|(u,v)\|^\circ$, $(u,v) \in \mathbb{C}^d \times \mathbb{C}^{n-d}$, $\lambda \in \mathbb{T}$. Indeed, fix (u,v) and λ. Then

$$\begin{aligned}\|(\lambda u, v)\|^\circ &= \sup_{(z,w)\in\boldsymbol{B}} |\boldsymbol{\Psi}((z,w),(\lambda u, v))| \\ &= \sup_{(z,w)\in\boldsymbol{B}} |\boldsymbol{\Psi}((z,\overline{\lambda}w),(u,v))| \le \sup_{(z,w)\in\boldsymbol{B}} |\boldsymbol{\Psi}((z,w),(u,v))| = \|(u,v)\|^\circ.\end{aligned}$$

Proof of Proposition 2. By Remark 4(c), to get (*) we only need to consider the case of $(\boldsymbol{B}, M)$.

Let $h_{\boldsymbol{B}_z}$ denote the Minkowski functional of $\boldsymbol{B}_z$,

$$h_{\boldsymbol{B}_z}(w) := \inf\{t > 0 : \|(z, w/t)\| < 1\}, \quad w \in \mathbb{C}^{n-d}.$$

Using the holomorphic contractibility with respect to the mapping $\boldsymbol{B}_z \ni w \longmapsto (z,w) \in \boldsymbol{B}$ (cf. [Jar-Pfl 2013, Remark 8.2.4(l)]) and the fact that $\boldsymbol{B}_z$ is a convex balanced domain, gives

$$\boldsymbol{m}_{\boldsymbol{B}}(M,(z,w)) \le \boldsymbol{g}_{\boldsymbol{B}}(M,(z,w)) \le \boldsymbol{g}_{\boldsymbol{B}_z}(0,w) = h_{\boldsymbol{B}_z}(w), \quad (z,w) \in \boldsymbol{B};$$

cf. [Jar-Pfl 2013, Proposition 2.3.1(c)]. Fix a family $(L_i)_{i\in I}$ of $\mathbb{C}$-linear mappings $L_i : \mathbb{C}^n \longrightarrow \mathbb{C}$ such that $\|\ \| = \sup_{i\in I} |L_i|$. Write $L_i(z,w) = P_i(z)+Q_i(w)$. Observe that $\|(z,w)\| \le 1 \iff \forall_{\lambda\in\mathbb{T}} : \|(z,\lambda w)\| \le 1 \iff \forall_{i\in I,\,\lambda\in\mathbb{T}} : |L_i(z,\lambda w)| \le 1 \iff \forall_{i\in I} : |P_i(z)|+|Q_i(w)| \le 1$. Moreover, $\|(z,w)\| < 1 \Longrightarrow \forall_{i\in I} : |P_i(z)|+|Q_i(w)| < 1$. Consequently, for all $i \in I$ and $\lambda \in \mathbb{T}$ the function $f_{i,\lambda}(z,w) := \frac{Q_i(w)}{1-\lambda P_i(z)}$, $(z,w) \in \boldsymbol{B}$, is well defined, $|f_{i,\lambda}| \le 1$, and $f_{i,\lambda} = 0$ on $M \cap \boldsymbol{B}$. In particular, for $(z,w) \in \boldsymbol{B}$ we get

$$\boldsymbol{m}_{\boldsymbol{B}}(M,(z,w)) \ge \sup\{|f_{i,\lambda}(z,w)| : i \in I,\ \lambda \in \mathbb{T}\} = \sup\Big\{\frac{|Q_i(w)|}{1-|P_i(z)|} : i \in I\Big\}.$$

Observe that for $(z,w) \in \boldsymbol{B}$ we have

$$h_{\boldsymbol{B}_z}(w) = \inf\{t > 0 : \|(z,w/t)\| < 1\} = \inf\{t > 0 : \|(z,w/t)\| \le 1\}.$$

Indeed, we may assume that $w \ne 0$. Let $\inf\{t > 0 : \|(z,w/t)\| \le 1\} = 1/r$, where $r := \sup\{s > 0 : \|(z,sw)\| \le 1\}$. Note that $\|(z,rw)\| = 1$ and $r > 1$. In view of (**) we have $\|(z,\lambda w)\| = 1$ for all $|\lambda| = r$. Hence, by the maximum principle for plurisubharmonic functions, we get $\|(z,\lambda w)\| \le 1$ for all $|\lambda| \le r$ and thus, either $\|(z,\lambda w)\| < 1$ for all $|\lambda| < r$, or $\|(z,\lambda w)\| = 1$ for all $|\lambda| \le 1$. The second case is impossible because $\|(z,w)\| < 1$. Thus, finally, $1/r = \inf\{t > 0 : \|(z,w/t)\| < 1\}$.

Consequently, if $(z,w) \in \boldsymbol{B}$, then

$$\begin{aligned} h_{\boldsymbol{B}_z}(w) &= \inf\{t > 0 : \forall_{i\in I}\ |L_i(z,w/t)| \le 1\} \\ &= \inf\{t > 0 : \forall_{i\in I}\ |P_i(z)| + \tfrac{1}{t}|Q_i(w)| \le 1\} = \sup\Big\{\frac{|Q_i(w)|}{1-|P_i(z)|} : i \in I\Big\}, \end{aligned}$$

which finishes the proof. □

Remark 5. Let $A : \mathbb{C}^{n-d} \longrightarrow \mathbb{C}^d$ be $\mathbb{C}$-linear. Using the linear isomorphism $(z,w) \longmapsto (z + A(w), w)$ one may extend the equality $\boldsymbol{m}_{\boldsymbol{B}}(M,\cdot) \equiv \boldsymbol{g}_{\boldsymbol{B}}(M,\cdot)$ to all $\mathbb{C}$-norms such that

$$\|(z + (1-\lambda)A(w), \lambda w)\| \le \|(z,w)\|, \quad (z,w) \in \mathbb{C}^n,\ \lambda \in \mathbb{T}.$$

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