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DERIVATIVES OF BLASCHKE PRODUCTS WHOSE ZEROS LIE IN A STOLZ DOMAIN AND WEIGHTED BERGMAN SPACES

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Abstract. For a Blaschke product B whose zeros lie in a Stolz domain, we find a condition regarding ω which guarantees that B' belongs to the Bergman space A^p_ω. In addition, the sharpness of this condition is considered.

1. Introduction and main results

Let $\mathcal{H}(\mathbb{D})$ be the space of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$. For $0 < p < \infty$ and a weight ω, the weighted Bergman space A^p_ω consists of those $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA(z) < \infty,$$

where $dA(z)$ is the Lebesgue area measure on $\mathbb{D}$. A function $\omega : \mathbb{D} \to [0, \infty)$, which is integrable over $\mathbb{D}$, is called a weight. A weight ω is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. If $\omega(z) = (1 - |z|)^\alpha$ with $-1 < \alpha < \infty$, then we write $A^p_\omega = A^p_\alpha$. The notation A^p is used in the case $\alpha = 0$.

If $\{z_n\}$ is a sequence of points in $\mathbb{D}$ which satisfies the Blaschke condition $\sum_n (1 - |z_n|) < \infty$, then the Blaschke product associated with $\{z_n\}$ is defined by

$$B(z) = \prod_n \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z}_n z}, \quad z \in \mathbb{D}.$$

Information about Blaschke products can be found in [3] and [5]. In this note, we consider Blaschke products with zeros in Stolz domains,

$$\Omega_\eta(\xi) = \{z \in \mathbb{D} : |1 - \overline{\xi} z| \le \eta(1 - |z|)\},$$

where $\xi \in \mathbb{T}$ and $1 < \eta < \infty$ are given. Write $\Omega_\eta(1) = \Omega_\eta$ for short, and denote the family of all Blaschke products whose zeros lie in some Stolz domain by $\mathfrak{B}$.

Blaschke products B in $\mathfrak{B}$ have been studied earlier in [2, 4, 6–10, 12], to name a few. In particular, the behavior of B' has been studied there. For example, the main results of [6] show that $B' \in A^p$ for every $B \in \mathfrak{B}$ if and only if $0 < p < \frac{3}{2}$.

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Our purpose is to generalize this result for the weighted Bergman space A^p_ω. The first part of this generalization reads as follows.

Theorem 1. *Let $0<p<\infty$ and $B\in\mathfrak{B}$, and let ω be a radial weight on $\mathbb{D}$. Then:*
(a) *$B'\in A^p_\omega$ for all $0<p<\frac{1}{2}$.*
(b) *If $\int_0^1\omega(r)\log\left(\frac{1}{1-r}\right)dr<\infty$, then $B'\in A^{1/2}_\omega$.*
(c) *If $\frac{1}{2}<p<\infty$ and $\int_0^1\omega(r)(1-r)^{\frac{1}{2}-p}\,dr<\infty$, then $B'\in A^p_\omega$.*

For Theorem 2, which shows the sharpness of Theorem 1 in the case $\frac{1}{2}<p<\infty$, we need to recall some conditions on weights. We limit our investigations to the class $\widehat{\mathcal{D}}$ of doubling weights, which consists of radial weights ω satisfying $\widehat{\omega}(r)\lesssim\widehat{\omega}(\frac{1+r}{2})$, where $\widehat{\omega}(r)=\int_r^1\omega(s)\,ds$. The notation $a\lesssim b$ means that there exists a constant $C>0$ such that $a\le Cb$, while $a\gtrsim b$ is understood in an analogous manner. If $a\lesssim b$ and $a\gtrsim b$, then we write $a\asymp b$. A radial weight ω belongs to $\widehat{\mathcal{D}}$ if and only if

$$\widehat{\mathcal{D}}_p(\omega)=\sup_{0<r<1}\frac{(1-r)^p}{\widehat{\omega}(r)}\int_0^r\frac{\omega(s)}{(1-s)^p}\,ds<\infty \tag{1.1}$$

for some $0<p<\infty$ [16]. If (1.1) holds for some fixed p, then we write $\omega\in\widehat{\mathcal{D}}_p$. Regarding inner functions Θ, weights ω in $\widehat{\mathcal{D}}_p$ with some $0<p<\infty$ have the following property [17]:

$$\|\Theta'\|^p_{A^p_\omega}\asymp\int_{\mathbb{D}}\left(\frac{1-|\Theta(z)|}{1-|z|}\right)^p\omega(z)\,dA(z). \tag{1.2}$$

Inner functions are bounded analytic functions in $\mathbb{D}$ having unimodular radial limits at almost every point on the boundary $\mathbb{T}=\{z\in\mathbb{D}:|z|=1\}$ [5,13]. Hence (1.2) is valid, in particular, when Θ is a Blaschke product. Using (1.2), one can also show that a finite product of the derivatives of inner functions belongs to A^p_ω induced by $\omega\in\widehat{\mathcal{D}}_p$ if and only if all members of the product belong to A^p_ω.

We say that $\omega\in\mathcal{D}$ if there exist $C=C(\omega)\ge1$, $\alpha=\alpha(\omega)>0$ and $\beta=\beta(\omega)\ge\alpha$ such that

$$C^{-1}\left(\frac{1-r}{1-t}\right)^\alpha\widehat{\omega}(t)\le\widehat{\omega}(r)\le C\left(\frac{1-r}{1-t}\right)^\beta\widehat{\omega}(t),\quad 0\le r\le t<1. \tag{1.3}$$

In (1.3), only the left-hand side inequality is a restriction because the right-hand side inequality is valid for some $\beta>0$ if and only if $\omega\in\widehat{\mathcal{D}}$ [16]. In addition, we denote $\log_0 x=x$, $\log_1 x=\log x$ and $\log_{k+1}x=\log(\log_k x)$ for sufficiently large $0<x<\infty$ and $k\in\mathbb{N}$, and $y^{[1]}=y$, $y^{[2]}=y^y$ and $y^{[j+1]}=y^{y^{[j]}}$ for $0<y<\infty$ and $j\in\mathbb{N}$. Using this notation, we state the following theorem.

Theorem 2. *Let $N\in\mathbb{N}$, $\frac{1}{2}<p<\infty$ and $\omega\in\widehat{\mathcal{D}}_p$. For $1<p<\infty$, we assume in addition that the left-hand side inequality of (1.3) holds for some $\alpha=\alpha(\omega)>p-1$. There exists a Blaschke product B with zeros in the positive real axis such that if $B'\in A^p_\omega$, then*

$$\int_0^1\frac{\omega(r)}{(1-r)^{p-1/2}}\left(\log_N\frac{e^{[N]}}{1-r}\right)^{-\frac{1}{2}}dr<\infty.$$

Roughly speaking, for $\frac{1}{2}<p<\infty$, we are searching for a weighted Bergman space A^p_ω which contains $\{B':B\in\mathfrak{B}\}$ and is as close as possible to $A^p_{p-3/2}$.

Hence the hypotheses of ω in Theorem 2 are not very restricting, because if $\omega(z) = (1-|z|)^\alpha$ for all $z \in \mathbb{D}$, then these hypotheses are satisfied if and only if $p-2 < \alpha < p-1$.

We close this section with a concrete example which shows that Theorems 1 and 2 really improve the above-mentioned result from [6]. The next sections contain proofs of these theorems. Note that, in the following example, we use the interpretation $\prod_{k=1}^{0}(\cdot) = 1$.

Example. Let $\frac{1}{2} < p < \infty$, $M \in \mathbb{N}$ and

$$\upsilon_\alpha(z) = (1-|z|)^{p-3/2}\left(\log_M \frac{e^{[M]}}{1-|z|}\right)^{-\alpha}\prod_{k=1}^{M-1}\left(\log_k \frac{e^{[k]}}{1-|z|}\right)^{-1}, \quad z \in \mathbb{D},$$

for some $0 < \alpha < \infty$. Then $B' \in A^p_{\upsilon_\alpha}$ for every $B \in \mathfrak{B}$ if and only if $\alpha > 1$.

Theorem 1 implies that $B' \in A^p_{\upsilon_\alpha}$ for every $B \in \mathfrak{B}$ if $\alpha > 1$. Since υ_α satisfies the hypotheses of Theorem 2 for any α, the converse statement follows from this result by choosing $N = M+1$.

2. Proof and corollary of Theorem 1

Since the integral mean

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt\right)^{1/p}, \quad 0 < p < \infty,$$

of every $f \in \mathcal{H}(\mathbb{D})$ is increasing with r [5], Theorem 1 is a direct consequence of the following upper estimate of $M_p(r,B')$, where $B \in \mathfrak{B}$.

Theorem 3. *Let $0 < p < \infty$ and $B \in \mathfrak{B}$. Then there exists $R \in (0,1)$ such that*

$$M_p^p(r,B') \lesssim \begin{cases} 1, & p < \frac{1}{2}, \\ \log\left(\frac{1}{1-r}\right), & p = \frac{1}{2}, \\ (1-r)^{1/2-p}, & p > \frac{1}{2}, \end{cases} \tag{2.1}$$

for $R \le r < 1$.

It is worth noting that Theorem 3 for $p \ge 1$ is essentially proved in [9]. In fact, using [9, Corollary 1.10] together with the Schwarz-Pick lemma, we obtain

$$\frac{M_p^p(r,B')}{(1-r)^{1/2-p}} \longrightarrow 0^+, \quad r \to 1^-, \quad p \ge 1.$$

Furthermore, the case $0 < p \le \frac{1}{2}$ is a consequence of the proof of [7, Theorem 2.3], as we can see in the proof below. Hence our contribution to Theorem 3 limits to the case $\frac{1}{2} < p < 1$.

Proof of Theorem 3. Assume, without loss of generality, that the zero-sequence $\{z_n\}$ of B is contained in Ω_η for some $1 < \eta < \infty$. Then $|B'(z)| \lesssim |1-z|^{-2}$ for all $z \in \mathbb{D}$ by the proof of [7, Theorem 2.3]. Now the assertion in the case $p \le \frac{1}{2}$ follows directly from this fact by using [11, Theorem 1.7].

Let $\frac{1}{2} < p < \infty$, and set

$$f(\theta) = f_B(\theta) = \sum_n \frac{1-|z_n|}{(\theta + (1-|z_n|))^2}, \quad 0 < \theta < 2\pi.$$

By the proof of [6, Theorem 3], we find constants $0 < R < 1$, $K = K(\eta, R) > 0$ and $A = A(\eta, R) > 0$ such that

$$|B'(re^{it})| \le Af((1-r)+|t|)\exp\left(-K(1-r)f((1-r)+|t|)\right)$$

for $-\pi \le t \le \pi$ and $R \le r < 1$. Hence, by the change of variable $\theta = (1-r)+t$, we obtain

$$\begin{aligned} M_p^p(r,B') &\le 2A^p \int_0^\pi f((1-r)+t)^p \exp\left(-Kp(1-r)f((1-r)+t)\right)\,dt \\ &\lesssim \int_{1-r}^{2\pi} f(\theta)^p \exp\left(-Kp(1-r)f(\theta)\right)\,d\theta, \quad R \le r < 1. \end{aligned} \tag{2.2}$$

Since $\{z_n\}$ is a Blaschke sequence, there exists $C = C(B) > K^{\frac{2p-1}{2p+1}}$ such that $f(\theta) \le C\theta^{-2}$ for all θ. To estimate the last formula of (2.2), for $0 < r < 1$, we set $D_r = \{\theta \in (1-r, 2\pi) : f(\theta) \ge \theta^{-1-\frac{1}{2p}}\}$ and consider the auxiliary function

$$g(x) = x^p \exp(-Kp(1-r)x), \quad 0 < x < \infty.$$

A simple calculation shows that g is non-decreasing for $x \le K^{-1}(1-r)^{-1}$ and decreasing for $x > K^{-1}(1-r)$. If $\theta \ge \sqrt{CK}\sqrt{1-r}$, then

$$f(\theta) \le C\theta^{-2} \le K^{-1}(1-r)^{-1};$$

and if $\theta \in D_r$ and $\theta < K^{\frac{2p}{2p+1}}(1-r)^{\frac{2p}{2p+1}}$, then

$$f(\theta) \ge \theta^{-1-\frac{1}{2p}} > K^{-1}(1-r)^{-1}.$$

Since f is continuous and decreasing, by the inequalities above, there exists θ_r satisfying $K^{\frac{2p}{2p+1}}(1-r)^{\frac{2p}{2p+1}} < \theta_r \le \sqrt{CK}\sqrt{1-r}$ such that $g(f(\theta))$ is non-decreasing for $\theta \in D_r$ and $\theta \ge \theta_r$ and decreasing for $\theta \in D_r$ and $\theta < \theta_r$. Using (2.2) together with these monotonicity properties of $g(f(\theta))$, we obtain

$$\begin{aligned} M_p^p(r,B') &\lesssim \left(\int_{D_r} + \int_{(1-r,2\pi)\setminus D_r}\right) f(\theta)^p \exp\left(-Kp(1-r)f(\theta)\right)\,d\theta \\ &\le \left(\int_{D_r\cap\{\theta:\theta<\theta_r\}} + \int_{D_r\cap\{\theta:\theta\ge\theta_r\}}\right) f(\theta)^p \exp\left(-Kp(1-r)f(\theta)\right)\,d\theta \\ &\quad + \int_{(1-r,2\pi)\setminus D_r} \theta^{-p-\frac{1}{2}}\,d\theta \\ &\lesssim \int_{D_r\cap\{\theta:\theta<\theta_r\}} \theta^{-p-\frac{1}{2}} \exp\left(-Kp\frac{1-r}{\theta^{1+\frac{1}{2p}}}\right)\,d\theta \\ &\quad + \int_{D_r\cap\{\theta:\theta\ge\theta_r\}} \theta^{-2p} \exp\left(-CKp\frac{1-r}{\theta^2}\right)\,d\theta + (1-r)^{\frac{1}{2}-p} \\ &\lesssim \int_{1-r}^{2\pi} \theta^{-2p} \exp\left(-W\frac{1-r}{\theta^2}\right)\,d\theta + (1-r)^{\frac{1}{2}-p} = I_p(r), \quad R < r < 1, \end{aligned}$$

where $W = W(C,p,K) = CKp$. The change of variable $x = W(1-r)/\theta^2$ yields

$$\begin{aligned} I_p(r) &= \frac{W^{\frac{1}{2}-p}}{2}(1-r)^{\frac{1}{2}-p}\int_{\frac{W(1-r)}{4\pi^2}}^{\frac{W}{1-r}} x^{p-\frac{3}{2}}e^{-x}\,dx + (1-r)^{\frac{1}{2}-p} \\ &\lesssim (1-r)^{\frac{1}{2}-p}\left(\Gamma\left(p-\frac{1}{2}\right)+1\right) \asymp (1-r)^{\frac{1}{2}-p}, \quad 0 \le r < 1. \end{aligned}$$

This completes the proof. □

Recall that any inner function can be represented as the product of a Blaschke product (which may have zeros also at the origin) and a singular inner function

$$S(z) = S_\sigma(z) = \exp\left(\int_{\mathbb{T}} \frac{z+w}{z-w}\, d\sigma(w) + i\theta\right), \quad z \in \mathbb{D},$$

where $0 \le \theta < 2\pi$ and σ is a positive measure on $\mathbb{T}$ and singular with respect to the Lebesgue measure [5]. Regarding this fact, we state and prove the following consequence of Theorem 1.

Corollary 4. *Let $0 < p < \infty$ and $\omega \in \widehat{\mathcal{D}}_p$. Let $\Theta = BS$ be an inner function, where $B \in \mathfrak{B}$ and S is a non-constant singular inner function. Then $\Theta' \in A^p_\omega$ if and only if $S' \in A^p_\omega$.*

Proof. Using (1.2), one can show that $\Theta' \in A^p_\omega$ if and only if $S' \in A^p_\omega$ and $B' \in A^p_\omega$. Thus the case $0 < p < \frac{1}{2}$ is clear by Theorem 1. If $\int_0^1 \omega(r) \log\left(\frac{e}{1-r}\right) dr < \infty$, then $B' \in A^{1/2}_\omega$ by Theorem 1. Furthermore, (1.2) and [15, Theorem 4.4.5] yield

$$\|S'\|^{1/2}_{A^{1/2}_\omega} \asymp \int_{\mathbb{D}} \left(\frac{1-|S(z)|}{1-|z|}\right)^{\frac{1}{2}} \omega(z)\, dA(z) \gtrsim \int_0^1 \omega(r) \log\left(\frac{e}{1-r}\right) dr.$$

Hence the assertion for $p = \frac{1}{2}$ follows by combining these facts. The remaining case can be proved in a similar manner using the first corollary of [1, Theorem 5] or [15, Theorem 4.4.4]. □

It is obvious that the statement of Corollary 4 is not true in general if the hypothesis $B \in \mathfrak{B}$ is removed. For example, the derivative of the atomic singular inner function $S(z) = \exp\left(\frac{z+1}{z-1}\right)$ belongs to A^p for any $1 < p < \frac{3}{2}$, but $(BS)' \notin A^p$ if B is a Blaschke product associated with $\{z_n\}$, which is a finite union of separated sequences satisfying the condition $\sum_n (1-|z_n|)^{2-p} = \infty$. See, for instance, the main result of [14] and [18, Theorem 1].

3. Proof of Theorem 2

Let us begin by recalling [18, Corollary 6], referred to here as Lemma A. Write $\omega \in \mathcal{J}_p$ if

$$\mathcal{J}_p(\omega) = \sup_{0<r<1} \frac{(1-r)^p}{\widehat{\omega}(r)} \int_r^1 \frac{\omega(s)}{(1-s)^p}\, ds < \infty.$$

Furthermore, for $q \in \mathbb{R}$ and a weight ω, we write $\omega_q(z) = \omega(z)(1-|z|)^q$ for all $z \in \mathbb{D}$.

Lemma A. *Let $\frac{1}{2} < p < \infty$, $0 < q < \infty$ and $\omega \in \mathcal{D}$, and let Θ be an inner function. If*

(a) $1 < p < \infty$ *and* $\omega \in \widehat{\mathcal{D}}_p \cap \mathcal{J}_{p-1}$, *or*
(b) $p + q \le 1$ *and* $\omega \in \widehat{\mathcal{D}}_{2p-1}$, *or*
(c) $1 < p + q \le 1 + q$ *and* $\omega \in \widehat{\mathcal{D}}_{2p-1} \cap \mathcal{J}_{p-1}$,

then $\|\Theta'\|^p_{A^p_\omega} \asymp \|\Theta'\|^{p+q}_{A^{p+q}_{\omega_q}}$.

Proof of Theorem 2. Assume that B is the Blaschke product with zeros

$$z_n = 1 - \frac{1}{(\log_N n)^2 \prod_{k=0}^{N-1} \log_k n}, \quad n > e^{[N]}.$$

Let $\upsilon(r)$ be the number of zeros of B in $\{z \in \mathbb{D} : |z| < r\}$, and write $\lambda(r) = \upsilon(1-r)$. In addition, let δ be a function such that

$$\delta(r)^{-2} \int_0^{\delta(r)} \lambda(s)\, ds \asymp (1-r)^{-1}, \tag{3.1}$$

when $0 \le r < 1$ is close enough to one. Since the zeros of B lie in Ω_η for any $1 < \eta < \infty$, by simple modifications of [9, Theorems 1.9 and 1.14], we find $0 \le R < 1$ depending on the comparison constants of (3.1) such that

$$\int_0^{2\pi} (1 - |B(re^{i\theta})|)\, d\theta \asymp \delta(r), \quad R \le r < 1. \tag{3.2}$$

More precisely, by imitating the proofs of [9, Theorems 1.9 and 1.14], one can show that (3.2) is valid for δ_C satisfying $h(\delta_C(r)) = C(1-r)^{-1}$, where C is a positive constant and

$$h(t) = t^{-2} \int_0^t \lambda(s)\, ds, \quad 0 < t \le 1.$$

Since h is decreasing by [9, Lemma 1.4(c)], this fact implies (3.2) for any δ defined by (3.1). Now, if $p \le 1$ and $\delta(r) \asymp \sqrt{1-r}\left(\log_N \frac{e^{[N]}}{1-r}\right)^{-\frac{1}{2}}$ for $R \le r < 1$, then (1.2) yields

$$\begin{aligned} \|B'\|_{A^p_\omega}^p &\asymp \int_{\mathbb{D}} \left(\frac{1-|B(z)|}{1-|z|}\right)^p \omega(z)\, dA(z) \ge \int_{\mathbb{D}} (1-|B(z)|)\, \frac{\omega(z)}{(1-|z|)^p}\, dA(z) \\ &\asymp 1 + \int_R^1 \frac{\omega(r)}{(1-r)^{p-1/2}} \left(\log_N \frac{e^{[N]}}{1-r}\right)^{-\frac{1}{2}} dr. \end{aligned}$$

Hence, in the case $p \le 1$, it suffices to show that $\delta(r) \asymp \sqrt{1-r}\left(\log_N \frac{e^{[N]}}{1-r}\right)^{-\frac{1}{2}}$, when $0 \le r < 1$ is close enough to one.

We have

$$\lambda(t) \asymp \left[t \left(\log_N \frac{1}{t}\right)^2 \prod_{j=1}^{N-1} \log_j \frac{1}{t}\right]^{-1}, \quad 0 < t < e^{-[N]},$$

because, for $n > e^{[N]}$,

$$\begin{aligned} &\lambda(1-z_n) \\ &\asymp \frac{n(\log_N n)^2 \prod_{k=1}^{N-1} \log_k n}{\left[\log_N \left(n(\log_N n)^2 \prod_{k=1}^{N-1} \log_k n\right)\right]^2 \prod_{j=1}^{N-1} \log_j \left(n(\log_N n)^2 \prod_{k=1}^{N-1} \log_k n\right)} \asymp n, \end{aligned}$$

where the latter asymptotic equation is due to the estimates

$$\log_N\left(n(\log_N n)^2 \prod_{k=1}^{N-1} \log_k n\right) \asymp \log_N n \quad \text{and} \quad \log_j\left(n(\log_N n)^2 \prod_{k=1}^{N-1} \log_k n\right) \asymp \log_j n.$$

Moreover, since

$$\frac{d}{dt}\left(\log_N \frac{1}{t}\right)^{-1} = \left[t\left(\log_N \frac{1}{t}\right)^2 \prod_{j=1}^{N-1} \log_j \frac{1}{t}\right]^{-1}, \quad 0 < t < e^{-[N]},$$

we obtain

$$\int_0^t \lambda(s)\, ds \asymp \left(\log_N \frac{1}{t}\right)^{-1}, \quad 0 < t < e^{-[N]}.$$

Finally, we can verify that $\delta(r) \asymp \sqrt{1-r}\left(\log_N \frac{e^{[N]}}{1-r}\right)^{-\frac{1}{2}}$, when r is close enough to one, because δ is unique in the asymptotic sense by (3.2), and then

$$\delta(r)^{-2} \int_0^{\delta(r)} \lambda(s)\, ds \asymp \delta(r)^{-2}\left(\log_N \frac{1}{\delta(r)}\right)^{-1} \asymp (1-r)^{-1}.$$

Thus the assertion for $p \leq 1$ has been proved.

Let $1 < p < \infty$, and assume that there exists $\alpha = \alpha(\omega) > p-1$ such that

$$\frac{\widehat{\omega}(t)}{(1-t)^\alpha} \lesssim \frac{\widehat{\omega}(r)}{(1-r)^\alpha}, \quad 0 \leq r \leq t < 1.$$

Since

$$\frac{\widehat{\omega}(t)}{(1-t)^{p-1}} \lesssim 2^\alpha \widehat{\omega}\left(\frac{1}{2}\right)(1-t)^{\alpha+1-p} \longrightarrow 0, \quad t \to 1^-,$$

an integration by parts yields

$$\int_r^1 \frac{\omega(s)}{(1-s)^{p-1}}\, ds = \frac{\widehat{\omega}(r)}{(1-r)^{p-1}} + (p-1)\int_r^1 \frac{\widehat{\omega}(s)}{(1-s)^p}\, ds.$$

In particular, $\omega \in \mathcal{J}_{p-1}$ because

$$\begin{aligned}\int_r^1 \frac{\widehat{\omega}(s)}{(1-s)^p}\, ds &= \int_r^1 \frac{\widehat{\omega}(s)}{(1-s)^\alpha} \frac{ds}{(1-s)^{p-\alpha}} \\ &\lesssim \frac{\widehat{\omega}(r)}{(1-r)^\alpha}\int_r^1 \frac{ds}{(1-s)^{p-\alpha}} \asymp \frac{\widehat{\omega}(r)}{(1-r)^{p-1}}, \quad 0 \leq r < 1.\end{aligned}$$

Now, using these properties of ω, we obtain

$$\frac{\widehat{\omega_{1-p}}(t)}{(1-t)^{\alpha+1-p}} \lesssim \frac{\widehat{\omega}(t)}{(1-t)^\alpha} \lesssim \frac{\widehat{\omega}(r)}{(1-r)^\alpha} \leq \frac{\widehat{\omega_{1-p}}(r)}{(1-r)^{\alpha+1-p}}, \quad 0 \leq r \leq t < 1.$$

Since $\omega \in \widehat{\mathcal{D}}_p$ implies that $\omega_{1-p} \in \widehat{\mathcal{D}}_1$, we have $\omega_{1-p} \in \mathcal{D} \cap \widehat{\mathcal{D}}_1$. Thus, by Lemma A and the reasoning in the case $p = 1$, we find $0 \leq R < 1$ such that

$$\begin{aligned}\|B'\|_{A^p_\omega}^p \asymp \|B'\|_{A^1_{\omega_{1-p}}} &\asymp 1 + \int_R^1 \frac{\omega_{1-p}(r)}{(1-r)^{1/2}}\left(\log_N \frac{e^{[N]}}{1-r}\right)^{-\frac{1}{2}} dr \\ &= 1 + \int_R^1 \frac{\omega(r)}{(1-r)^{p-1/2}}\left(\log_N \frac{e^{[N]}}{1-r}\right)^{-\frac{1}{2}} dr.\end{aligned}$$

This completes the proof. □

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