PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 3, March 2018, Pages 1181–1187
http://dx.doi.org/10.1090/proc/13792
Article electronically published on October 23, 2017

RANK OF A CO-DOUBLY COMMUTING SUBMODULE IS 2

ARUP CHATTOPADHYAY, B. KRISHNA DAS, AND JAYDEB SARKAR

(Communicated by Stephan Ramon Garcia)

Dedicated to the memory of our friend and colleague Sudipta Dutta

ABSTRACT. We prove that the rank of a non-trivial co-doubly commuting submodule is 2. More precisely, let $\varphi, \psi \in H^\infty(\mathbb{D})$ be two inner functions. If $\mathcal{Q}_\varphi = H^2(\mathbb{D})/\varphi H^2(\mathbb{D})$ and $\mathcal{Q}_\psi = H^2(\mathbb{D})/\psi H^2(\mathbb{D})$, then

$$\text{rank } (\mathcal{Q}_\varphi \otimes \mathcal{Q}_\psi)^\perp = 2.$$

An immediate consequence is the following: Let $\mathcal{S}$ be a co-doubly commuting submodule of $H^2(\mathbb{D}^2)$. Then rank $\mathcal{S} = 1$ if and only if $\mathcal{S} = \Phi H^2(\mathbb{D}^2)$ for some one variable inner function $\Phi \in H^\infty(\mathbb{D}^2)$. This answers a question posed by R. G. Douglas and R. Yang [Integral Equations Operator Theory 38(2000), pp207–221]

1. INTRODUCTION

Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting bounded linear operators on a Hilbert space $\mathcal{H}$. For a subset $E \subseteq \mathcal{H}$ we denote $[E]_T$ by the closed subspace $\overline{\text{span}}\{T_1^{k_1} \cdots T_n^{k_n} E : k_j \in \mathbb{N}, j = 1, \ldots, n\}$ of $\mathcal{H}$. Then the *rank* of T [3] is the unique number

$$\text{rank}(T) = \min\{\#E : [E]_T = \mathcal{H}, E \subseteq \mathcal{H}\}.$$

Throughout the paper, $\mathbb{N}$ will denote the set of positive integers with 0. A closed subspace $\mathcal{S}$ of $H^2(\mathbb{D}^n)$, the Hardy space over the unit polydisc $\mathbb{D}^n$, is said to be shift invariant if $M_{z_i}(\mathcal{S}) \subseteq \mathcal{S}$ where M_{z_i} is the co-ordinate multiplication operator on $H^2(\mathbb{D}^n)$ for all $i = 1, 2, \ldots, n$. The *rank* of a shift invariant subspace $\mathcal{S}$ of $H^2(\mathbb{D}^n)$ is the rank of the corresponding n-tuple of restricted co-ordinate shift operators, that is,

$$\text{rank } \mathcal{S} = \text{rank } (M_{z_1}|_{\mathcal{S}}, \ldots, M_{z_n}|_{\mathcal{S}}).$$

The rank of a bounded linear operator (or, of a commuting tuple of bounded linear operators) on a Hilbert space is an important numerical invariant. Very briefly, the rank of a bounded linear operator is the cardinality of a minimal generating set. One of the most intriguing and important problems in operator theory and function theory is the existence of a finite generating set for a commuting tuple of operators. Alternatively, one may ask when the rank of a commuting tuple of operators is finite.

Received by the editors February 7, 2017, and, in revised form, April 25, 2017.
2010 *Mathematics Subject Classification.* Primary 47A13, 47A15, 47A16, 46M05, 46C99, 32A70.
Key words and phrases. Hardy space over bidisc, rank, joint invariant subspaces, semi-invariant subspaces.

Prototype examples of rank one operators are the co-ordinate multiplication operator tuple $(M_{z_1}, \dots, M_{z_n})$ on the Hardy space, the (weighted) Bergman space over the unit ball and the polydisc in $\mathbb{C}^n$, $n \geq 1$, and the Drury-Arveson space over the unit ball in $\mathbb{C}^n$. Moreover, a particular version of the celebrated invariant subspace theorem of Beurling says: A shift invariant (or, shift co-invariant) subspace of the one variable Hardy space is of rank one.

Computation of ranks of shift invariant as well as shift co-invariant subspaces beyond the case of the one variable Hardy space is an excruciatingly difficult problem, even if one considers only shift invariant (as well as co-invariant) subspaces of the Hardy space over the unit polydisc in $\mathbb{C}^n$, $n > 1$ (see however [2, 6–8, 14]).

The purpose of this paper is to compute the rank of a tractable class of shift invariant subspaces of the two variables Hardy space, $H^2(\mathbb{D}^2)$, over the bidisc $\mathbb{D}^2$ in $\mathbb{C}^2$. In order to state the precise contribution of this paper, we need to introduce first some definitions and notation.

We denote the open unit disc of $\mathbb{C}$ by $\mathbb{D}$, and the unit circle by $\mathbb{T}$. The Hardy space over the unit disc $\mathbb{D}$ (bidisc $\mathbb{D}^2$), denoted by $H^2(\mathbb{D})$ ($H^2(\mathbb{D}^2)$), is the Hilbert space of all square summable holomorphic functions on $\mathbb{D}$ (on $\mathbb{D}^2$). Also we will denote by M_z and M_w the multiplication operators on $H^2(\mathbb{D}^2)$ by the coordinate functions z and w, respectively. It is easy to see that (M_z, M_w) is a pair of commuting isometries, that is,

$$M_z M_w = M_w M_z, \quad M_z^* M_z = M_w^* M_w = I_{H^2(\mathbb{D}^2)}.$$

Identifying $H^2(\mathbb{D}^2)$ with the 2-fold Hilbert space tensor product $H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$, one can represent (M_z, M_w) as $(M_z \otimes I_{H^2(\mathbb{D})}, I_{H^2(\mathbb{D})} \otimes M_w)$.

Let $\mathcal{S}$ and $\mathcal{Q}$ be closed subspaces of $H^2(\mathbb{D}^2)$. Then $\mathcal{S}$ is said to be a *submodule* if $M_z(\mathcal{S}) \subseteq \mathcal{S}$ and $M_w(\mathcal{S}) \subseteq \mathcal{S}$. We say that $\mathcal{Q}$ is a *quotient module* if $\mathcal{Q}^\perp$ is a submodule.

A well-known result due to Beurling states that if $\mathcal{S}$ is a submodule of $H^2(\mathbb{D})$ (that is, $\mathcal{S}$ is a closed subspace of $H^2(\mathbb{D})$ and $M_z\mathcal{S} \subseteq \mathcal{S}$), then $\mathcal{S}$ can be represented as

$$\mathcal{S} = \mathcal{S}_\varphi := \varphi H^2(\mathbb{D}),$$

where $\varphi \in H^\infty(\mathbb{D})$ is an inner function (that is, φ is a bounded holomorphic function on $\mathbb{D}$ and $|\varphi| = 1$ a.e. on $\mathbb{T}$). Consequently, a quotient module $\mathcal{Q}$ (that is, $\mathcal{Q}$ is a closed subspace of $H^2(\mathbb{D})$ and $M_z^*\mathcal{Q} \subseteq \mathcal{Q}$) of $H^2(\mathbb{D})$ can be represented as

$$\mathcal{Q} = \mathcal{Q}_\varphi := (\mathcal{S}_\varphi)^\perp = H^2(\mathbb{D})/\varphi H^2(\mathbb{D}).$$

It readily follows that

$$\text{rank } (M_z|_{\mathcal{S}_\varphi}) = \text{rank } (P_{\mathcal{Q}_\varphi} M_z|_{\mathcal{Q}_\varphi}) = 1.$$

Rudin [10], however, pointed out that there exists a submodule $\mathcal{S}$ of $H^2(\mathbb{D}^2)$ such that the rank of $\mathcal{S}$ is not finite (see also [7], [12] and [13]).

A quotient module $\mathcal{Q}$ of $H^2(\mathbb{D}^2)$ is *doubly commuting* if $C_z C_w^* = C_w^* C_z$, where $C_z = P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$ and $C_w = P_{\mathcal{Q}} M_w|_{\mathcal{Q}}$. A submodule $\mathcal{S}$ of $H^2(\mathbb{D}^2)$ is *co-doubly commuting* if the quotient module $\mathcal{S}^\perp (\cong H^2(\mathbb{D}^2)/\mathcal{S})$ is doubly commuting.

The following useful characterization of co-doubly commuting submodules is essential for our study (see [9, 11]): If $\mathcal{Q}$ is a quotient module of $H^2(\mathbb{D}^2)$, then $\mathcal{Q}$ is a doubly commuting quotient module if and only if

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \mathcal{Q}_2,$$

for some quotient modules $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of $H^2(\mathbb{D})$.

Let $\mathcal{S} = (\mathcal{Q}_1 \otimes \mathcal{Q}_2)^\perp$ be a non-zero co-doubly commuting submodule. If $\mathcal{Q}_j = H^2(\mathbb{D})$, for some $j = 1, 2$, then it is easy to see that

$$\text{rank } \mathcal{S} = 1.$$

Now let both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be non-trivial quotient modules of $H^2(\mathbb{D})$, that is, $\mathcal{Q}_j \neq \{0\}, H^2(\mathbb{D})$, $j = 1, 2$. Then there exist inner functions $\varphi, \psi \in H^\infty(\mathbb{D})$ such that $\mathcal{Q}_1 = \mathcal{Q}_\varphi$ and $\mathcal{Q}_2 = \mathcal{Q}_\psi$. The main purpose of the present paper is to prove that (see Theorem 2.1)

$$\text{rank } (\mathcal{Q}_\varphi \otimes \mathcal{Q}_\psi)^\perp = 2.$$

As a consequence of this, we give a complete and affirmative answer to a conjecture of Douglas and Yang (see page 220 [4]): If $\mathcal{S}$ is a rank one co-doubly commuting submodule, then $\mathcal{S} = \Phi H^2(\mathbb{D}^2)$ for some one variable inner function $\Phi \in H^\infty(\mathbb{D})$.

2. Proof of the main result

We begin with a simple but crucial observation on the rank of a joint semi-invariant subspace of a commuting tuple of operators.

Lemma 2.1. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting operators on a Hilbert space $\mathcal{H}$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two joint T-invariant subspaces of $\mathcal{H}$ and $\mathcal{S}_2 \subseteq \mathcal{S}_1$. If $\mathcal{S} = \mathcal{S}_1 \ominus \mathcal{S}_2$, then*

$$\textit{rank } (P_{\mathcal{S}} T_1|_{\mathcal{S}}, \ldots, P_{\mathcal{S}} T_n|_{\mathcal{S}}) \leq \textit{rank } (T_1|_{\mathcal{S}_1}, \ldots, T_n|_{\mathcal{S}_1}).$$

Proof. Let $m \in \mathbb{N}$ be the right side of the above inequality. Let $\{f_j\}_{j=1}^m \subseteq \mathcal{S}_1$ be a generating set for $(T_1|_{\mathcal{S}_1}, \ldots, T_n|_{\mathcal{S}_1})$. Clearly, $P_{\mathcal{S}} T_j P_{\mathcal{S}} = P_{\mathcal{S}} T_j|_{\mathcal{S}_1}$ for all $j = 1, \ldots, n$. This yields

$$(P_{\mathcal{S}} T_i P_{\mathcal{S}})(P_{\mathcal{S}} T_j P_{\mathcal{S}}) = P_{\mathcal{S}} (T_i T_j)|_{\mathcal{S}_1} \qquad (i, j = 1, \ldots, n).$$

It hence follows that $\{P_{\mathcal{S}} f_j\}_{j=1}^m$ is a generating set for $(P_{\mathcal{S}} T_1|_{\mathcal{S}}, \ldots, P_{\mathcal{S}} T_n|_{\mathcal{S}})$. This completes the proof. □

We now prove the main result of this paper.

Theorem 2.1. *Let $\varphi, \psi \in H^\infty(\mathbb{D})$ be two inner functions. If*

$$\mathcal{S} = (\mathcal{Q}_\varphi \otimes \mathcal{Q}_\psi)^\perp,$$

then rank $\mathcal{S} = 2$.

Proof. Let $X = I_{H^2(\mathbb{D}^2)} - (I_{H^2(\mathbb{D}^2)} - M_\varphi M_\varphi^* \otimes I_{H^2(\mathbb{D})})(I_{H^2(\mathbb{D}^2)} - I_{H^2(\mathbb{D})} \otimes M_\psi M_\psi^*)$. Since

$$\mathcal{S} = \text{ran} X,$$

and

$$X = (M_\varphi M_\varphi^* \otimes (I_{H^2(\mathbb{D})} - M_\psi M_\psi^*)) \oplus (I_{H^2(\mathbb{D})} \otimes M_\psi M_\psi^*),$$

it follows that

$$\mathcal{S} = (\mathcal{S}_\varphi \otimes \mathcal{Q}_\psi) \oplus \left(H^2(\mathbb{D}) \otimes \mathcal{S}_\psi\right).$$

Since by Theorem 6.2 of [1], rank $\mathcal{S} \leq 2$, we only need to show that rank $\mathcal{S} \geq 2$. Set

$$\mathcal{E} = \mathcal{S} \ominus (\mathcal{S}_\varphi \otimes \mathcal{S}_\psi).$$

It follows that

$$\mathcal{E} = (\mathcal{S}_\varphi \otimes \mathcal{Q}_\psi) \oplus (\mathcal{Q}_\varphi \otimes \mathcal{S}_\psi).$$

Since $\mathcal{S}_\varphi \otimes \mathcal{S}_\psi \subseteq \mathcal{S}$ is a submodule of $H^2(\mathbb{D}^2)$, by Lemma 2.1, it follows that

$$\text{rank}(P_\mathcal{E} M_z|_\mathcal{E}, P_\mathcal{E} M_w|_\mathcal{E}) \leq \text{rank}(M_z|_\mathcal{S}, M_w|_\mathcal{S}) = \text{rank}(\mathcal{S}). \tag{2.1}$$

Note that

$$P_\mathcal{E} = (P_{\mathcal{S}_\varphi} \otimes P_{\mathcal{Q}_\psi}) \oplus (P_{\mathcal{Q}_\varphi} \otimes P_{\mathcal{S}_\psi}),$$

and hence, an easy calculation yields

$$P_\mathcal{E} M_z|_\mathcal{E} = (M_z|_{\mathcal{S}_\varphi} \otimes P_{\mathcal{Q}_\psi}) \oplus (P_{\mathcal{Q}_\varphi} M_z|_{\mathcal{Q}_\varphi} \otimes P_{\mathcal{S}_\psi}),$$

and

$$P_\mathcal{E} M_w|_\mathcal{E} = (P_{\mathcal{S}_\varphi} \otimes P_{\mathcal{Q}_\psi} M_w|_{\mathcal{Q}_\psi}) \oplus (P_{\mathcal{Q}_\varphi} \otimes M_w|_{\mathcal{S}_\psi}).$$

Therefore it follows from the above equalities that $(\mathcal{S}_{\varphi^2} \otimes \mathcal{Q}_\psi) \oplus (\mathcal{Q}_\varphi \otimes \mathcal{S}_{\psi^2})$ is a joint $(P_\mathcal{E} M_z|_\mathcal{E}, P_\mathcal{E} M_w|_\mathcal{E})$-invariant subspace of $\mathcal{E}$. Set

$$\tilde{\mathcal{E}} = \mathcal{E} \ominus ((\mathcal{S}_{\varphi^2} \otimes \mathcal{Q}_\psi) \oplus (\mathcal{Q}_\varphi \otimes \mathcal{S}_{\psi^2})).$$

Notice that for any inner function $\theta \in H^\infty(\mathbb{D})$, we have

$$\mathcal{S}_\theta \ominus \mathcal{S}_{\theta^2} = \theta \mathcal{Q}_\theta.$$

From this and the representation of $\mathcal{E} = (\mathcal{S}_\varphi \otimes \mathcal{Q}_\psi) \oplus (\mathcal{Q}_\varphi \otimes \mathcal{S}_\psi)$ it follows that

$$\begin{aligned}\tilde{\mathcal{E}} &= ((\mathcal{S}_\varphi \otimes \mathcal{Q}_\psi) \oplus (\mathcal{Q}_\varphi \otimes \mathcal{S}_\psi)) \ominus ((\mathcal{S}_{\varphi^2} \otimes \mathcal{Q}_\psi) \oplus (\mathcal{Q}_\varphi \otimes \mathcal{S}_{\psi^2}))\\ &= (\varphi \mathcal{Q}_\varphi \otimes \mathcal{Q}_\psi) \oplus (\mathcal{Q}_\varphi \otimes \psi \mathcal{Q}_\psi).\end{aligned}$$

Then Lemma 2.1 and (2.1) implies that

$$\text{rank}(P_{\tilde{\mathcal{E}}} M_z|_{\tilde{\mathcal{E}}}, P_{\tilde{\mathcal{E}}} M_w|_{\tilde{\mathcal{E}}}) \leq \text{rank}(P_\mathcal{E} M_z|_\mathcal{E}, P_\mathcal{E} M_w|_\mathcal{E}) \leq \text{rank}(\mathcal{S}) \leq 2.$$

To finish the proof of the theorem it is now enough to prove the following:

$$\text{rank}(P_{\tilde{\mathcal{E}}} M_z|_{\tilde{\mathcal{E}}}, P_{\tilde{\mathcal{E}}} M_w|_{\tilde{\mathcal{E}}}) > 1.$$

Equivalently, it is enough to prove that the set $\{\xi\}$, for any $\xi \in \tilde{\mathcal{E}}$, is not a generating set corresponding to $(P_{\tilde{\mathcal{E}}} M_z|_{\tilde{\mathcal{E}}}, P_{\tilde{\mathcal{E}}} M_w|_{\tilde{\mathcal{E}}})$. Equivalently, given $\xi \in \tilde{\mathcal{E}}$, we show that there exists $\eta_\xi (\neq 0) \in \tilde{\mathcal{E}}$ such that

$$\langle (z^p \otimes w^q)\xi, \eta_\xi \rangle = 0 \qquad (p, q \in \mathbb{N}).$$

To this end, let $\{f_i\}$ and $\{g_j\}$ be orthonormal bases of $\mathcal{Q}_\varphi$ and $\mathcal{Q}_\psi$, respectively, and let $\xi \in \tilde{\mathcal{E}}$ where

$$\xi = (\sum_{k,l} a_{kl} \varphi f_k \otimes g_l) \oplus (\sum_{k,l} b_{kl} f_k \otimes \psi g_l),$$

$\{a_{kl}\}, \{b_{kl}\} \subseteq \mathbb{C}$, and

$$\sum_{k,l} |a_{kl}|^2, \sum_{k,l} |b_{kl}|^2 < \infty.$$

Again we observe that for any inner function $\theta \in H^\infty(\mathbb{D})$ and $f = \sum_{m \geq 0} c_m z^m \in \mathcal{Q}_\theta$ we have

$$M_z^*(\theta \bar{f}) \in \mathcal{Q}_\theta,$$

where $\bar{f} = \sum_{m \geq 0} \bar{c}_m e^{-imt} \in L^2(\mathbb{T})$. This follows from the fact that θ is a bounded holomorphic function on $\mathbb{D}$ and $M_z^*(\theta \bar{f}) \perp z^m$ for all $m < 0$ (which gives that $M_z^*(\theta \bar{f}) \in H^2(\mathbb{D})$), and then $M_z^*(\theta \bar{f}) \perp \theta z^m$ in $L^2(\mathbb{T})$ for all $m \geq 0$ (which gives that $M_z^*(\theta \bar{f}) \in \mathcal{Q}_\theta$). It should be noted that $M_z^*(\theta \bar{f}) = \theta \overline{zf} = C_\theta(f)$, where the conjugation map $C_\theta : \mathcal{Q}_\theta \to \mathcal{Q}_\theta$, $f \mapsto M_z^*(\theta \bar{f})$, is called a C-symmetry and it is used extensively in the study of Toeplitz operators on model spaces (for more details see [5]).

Coming back to our context, this immediately yields that

$$M_z^*(\varphi\overline{f_k}) \otimes M_w^*(\psi\overline{g_l}) \in \mathcal{Q}_\varphi \otimes \mathcal{Q}_\psi \qquad (k, l \geq 0),$$

and hence $s_0 \otimes s_1, t_0 \otimes t_1 \in \mathcal{Q}_\varphi \otimes \mathcal{Q}_\psi$, where

$$s_0 \otimes s_1 := -\sum_{k,l} \bar{a}_{kl} M_z^*(\varphi \overline{f}_k) \otimes M_w^*(\psi \overline{g}_l) = -(M_z^* \otimes M_w^*)(\varphi \otimes \psi)(\sum_{k,l} \bar{a}_{kl} \bar{f}_k \otimes \bar{g}_l)$$

and

$$t_0 \otimes t_1 := \sum_{k,l} \bar{b}_{kl} M_z^*(\varphi \overline{f}_k) \otimes M_w^*(\psi \overline{g}_l) = (M_z^* \otimes M_w^*)(\varphi \otimes \psi)(\sum_{k,l} \bar{b}_{kl} \bar{f}_k \otimes \bar{g}_l).$$

Set

$$\eta_\xi = (\varphi t_0 \otimes t_1) \oplus (s_0 \otimes \psi s_1) \in \tilde{\mathcal{E}}.$$

Then $\eta_\xi \neq 0$ and for every $p, q \in \mathbb{N}$ we have

$$\begin{aligned}
&\langle (z^p \otimes w^q)\xi, \eta_\xi \rangle \\
&\quad = \langle (z^p \otimes w^q)((\sum_{k,l} a_{kl} \varphi f_k \otimes g_l) \oplus (\sum_{k,l} b_{kl} f_k \otimes \psi g_l)), (\varphi t_0 \otimes t_1) \oplus (s_0 \otimes \psi s_1) \rangle \\
&\quad = \langle (z^p \otimes w^q)(\sum_{k,l} a_{kl} \varphi f_k \otimes g_l), \varphi t_0 \otimes t_1 \rangle \\
&\qquad\qquad + \langle (z^p \otimes w^q)(\sum_{k,l} b_{kl} f_k \otimes \psi g_l), s_0 \otimes \psi s_1 \rangle \\
&\quad = \langle (z^p \otimes w^q)(\sum_{k,l} a_{kl} f_k \otimes g_l), t_0 \otimes t_1 \rangle + \langle (z^p \otimes w^q)(\sum_{k,l} b_{kl} f_k \otimes g_l), s_0 \otimes s_1 \rangle \\
&\quad = \langle (z^{p+1} \otimes w^{q+1})(\sum_{k,l} a_{kl} \ f_k \otimes g_l), (\varphi \otimes \psi)(\sum_{k,l=1}^{\infty} \bar{b}_{kl} \bar{f}_k \otimes \bar{g}_l) \rangle \\
&\qquad\qquad - \langle (z^{p+1} \otimes w^{q+1})(\sum_{k,l} b_{kl} f_k \otimes g_l), (\varphi \otimes \psi)(\sum_{k,l} \bar{a}_{kl} \bar{f}_k \otimes \bar{g}_l) \rangle \\
&\quad = 0.
\end{aligned}$$

We have thus shown that $\{\xi\}$ is not a minimal generating subset of $\tilde{\mathcal{E}}$ with respect to $(P_{\tilde{\mathcal{E}}} M_z|_{\tilde{\mathcal{E}}}, P_{\tilde{\mathcal{E}}} M_w|_{\tilde{\mathcal{E}}})$ as desired. □

As a consequence of the above theorem we have the following corollary which provides an affirmative answer to the question raised by Douglas and Yang [4].

Corollary 2.2. *Let $\mathcal{S}$ be a co-doubly commuting submodule of $H^2(\mathbb{D}^2)$. Then rank $(\mathcal{S}) = 1$ if and only if $\mathcal{S} = \Theta H^2(\mathbb{D}^2)$ for some one variable inner function $\Theta \in H^\infty(\mathbb{D})$.*

Proof. If $\mathcal{S} = \Theta H^2(\mathbb{D}^2)$ for some one variable inner function $\Theta \in H^\infty(\mathbb{D})$, then $\mathcal{S} \cong H^2(\mathbb{D}^2)$ and hence rank $\mathcal{S} = 1$. To prove the sufficient part let $\mathcal{S}$ be a rank one co-doubly commuting submodule of $H^2(\mathbb{D}^2)$. Then there exist quotient modules $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of $H^2(\mathbb{D})$ such that (see [9, 11])

$$\mathcal{S} = (\mathcal{Q}_1 \otimes \mathcal{Q}_2)^\perp.$$

Since rank $(\mathcal{S}) = 1$, it follows from Theorem 2.1 that $\mathcal{Q}_j = H^2(\mathbb{D})$, for some $j = 1, 2$. This shows that

$$\mathcal{S} = \mathcal{S}_\varphi \otimes H^2(\mathbb{D}), \quad \text{or} \qquad \mathcal{S} = H^2(\mathbb{D}) \otimes \mathcal{S}_\psi,$$

for some inner functions $\varphi, \psi \in H^\infty(\mathbb{D})$. This concludes the proof of the corollary. $\square$

There is now the following interesting and natural question: Let $m \geq 2$ and let $\{\varphi_j\}_{j=1}^m \subseteq H^\infty(\mathbb{D})$ be inner functions. Is then

$$\text{rank } (\mathcal{Q}_{\varphi_1} \otimes \cdots \otimes \mathcal{Q}_{\varphi_m})^\perp = m?$$

Our present approach does not seem to work for the $m > 2$ case.

Acknowledgments

The first-named author acknowledges Fulbright-Nehru Postdoctoral Research Fellowship (Award No. 2164/FNPDR/2016) and the University of New Mexico for their warm hospitality. The second author's research work was supported by DST-INSPIRE Faculty Fellowship No. DST/INSPIRE/04/2015/001094. The research of the third author was supported in part by NBHM (National Board of Higher Mathematics, India) Research Grant NBHM/R.P.64/2014.

References

[1] Arup Chattopadhyay, B. Krishna Das, and Jaydeb Sarkar, *Tensor product of quotient Hilbert modules*, J. Math. Anal. Appl. **424** (2015), no. 1, 727–747, DOI 10.1016/j.jmaa.2014.11.038. MR3286591

[2] Arup Chattopadhyay, B. Krishna Das, and Jaydeb Sarkar, *Star-generating vectors of Rudin's quotient modules*, J. Funct. Anal. **267** (2014), no. 11, 4341–4360, DOI 10.1016/j.jfa.2014.09.024. MR3269879

[3] Ronald G. Douglas and Vern I. Paulsen, *Hilbert modules over function algebras*, Pitman Research Notes in Mathematics Series, vol. 217, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989. MR1028546

[4] Ronald G. Douglas and Rongwei Yang, *Operator theory in the Hardy space over the bidisk. I*, Integral Equations Operator Theory **38** (2000), no. 2, 207–221, DOI 10.1007/BF01200124. MR1791052

[5] Stephan Ramon Garcia, *Conjugation and Clark operators*, Recent advances in operator-related function theory, Contemp. Math., vol. 393, Amer. Math. Soc., Providence, RI, 2006, pp. 67–111, DOI 10.1090/conm/393/07372. MR2198373

[6] Kei Ji Izuchi, Kou Hei Izuchi, and Yuko Izuchi, *Blaschke products and the rank of backward shift invariant subspaces over the bidisk*, J. Funct. Anal. **261** (2011), no. 6, 1457–1468, DOI 10.1016/j.jfa.2011.05.009. MR2813478

[7] Kei Ji Izuchi, Kou Hei Izuchi, and Yuko Izuchi, *Ranks of invariant subspaces of the Hardy space over the bidisk*, J. Reine Angew. Math. **659** (2011), 101–139, DOI 10.1515/CRELLE.2011.069. MR2837012

[8] Kei Ji Izuchi, Kou Hei Izuchi, and Yuko Izuchi, *Ranks of backward shift invariant subspaces of the Hardy space over the bidisk*, Math. Z. **274** (2013), no. 3-4, 885–903, DOI 10.1007/s00209-012-1100-2. MR3078251

[9] Keiji Izuchi, Takahiko Nakazi, and Michio Seto, *Backward shift invariant subspaces in the bidisc. II*, J. Operator Theory **51** (2004), no. 2, 361–376. MR2074186

[10] Walter Rudin, *Function theory in polydiscs*, W. A. Benjamin, Inc., New York-Amsterdam, 1969. MR0255841

[11] Jaydeb Sarkar, *Jordan blocks of $H^2(\mathbb{D}^n)$*, J. Operator Theory **72** (2014), no. 2, 371–385, DOI 10.7900/jot.2013mar16.1980. MR3272037

[12] Michio Seto, *Infinite sequences of inner functions and submodules in $H^2(\mathbb{D}^2)$*, J. Operator Theory **61** (2009), no. 1, 75–86. MR2481804

[13] Michio Seto and Rongwei Yang, *Inner sequence based invariant subspaces in* $H^2(D^2)$, Proc. Amer. Math. Soc. **135** (2007), no. 8, 2519–2526, DOI 10.1090/S0002-9939-07-08745-X. MR2302572

[14] Rongwei Yang, *Operator theory in the Hardy space over the bidisk. III*, J. Funct. Anal. **186** (2001), no. 2, 521–545, DOI 10.1006/jfan.2001.3799. MR1864831

Department of Mathematics, Indian Institute of Technology Guwahati, Amingaon Post, Guwahati 781039 Assam, India

E-mail address: arupchatt@iitg.ernet.in

E-mail address: 2003arupchattopadhyay@gmail.com

Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India

E-mail address: dasb@math.iitb.ac.in

E-mail address: bata436@gmail.com

Statistics and Mathematics Unit, Indian Statistical Institute, 8th Mile, Mysore Road, Bangalore 560059, India

E-mail address: jay@isibang.ac.in

E-mail address: jaydeb@gmail.com