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ON THE STABILITY OF THE C^∞-HYPOELLIPTICITY UNDER PERTURBATIONS

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ABSTRACT. We study the problem of perturbations of C^∞-hypoelliptic operators by lower order terms. After giving several examples which show many different possibilities, we then prove a stability result which shows that a hypoelliptic linear partial differential operator P which loses finitely many derivatives and whose formal adjoint P^* is still hypoelliptic (but with no assumption on the loss of derivatives) remains hypoelliptic with the same loss of derivatives after perturbation by a lower order linear partial differential operator (whose order depends on the loss of derivatives).

1. Introduction

In some problems of the analysis of hypoelliptic PDEs one is faced with the following natural question: *To what extent is C^∞-hypoellipticity preserved under addition of a "lower order" term?* (See Problem 2.2 below.)

As one may expect, it is basically impossible (with the present technology) to give a thorough answer to the above question. This is mainly due to the fact that we do not possess as yet general means to detect hypoellipticity, except for pdos (partial differential operators) with constant coefficients and hence for pdos of constant strength (see Hörmander [5]).

In Parmeggiani [9] a partial (but sharp in some cases) answer to the question in the case of pdos was given, and in the present paper we plan to give a small contribution by exhibiting a range (of classes) of examples of hypoelliptic pdos and ψdos (pseudodifferential operators) for which addition of "certain" lower order perturbations does destroy/does not destroy hypoellipticity (although the *loss of derivatives* can possibly be affected) and a stability result that streamlines and generalizes that of [9] (see Theorem 3.1 below).

For the sake of definiteness, we decided in this paper to deal with hypoellipticity and not with microhypoellipticity. Although, as is well known, the two notions are not equivalent (see for instance Parenti and Rodino [8]), it will be clear below that most of our considerations on hypoellipticity may be rephrased into considerations on microhypoellipticity.

We shall hence discuss the examples in the next section and in the third one we shall state and prove the stability result.

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We close this introduction by mentioning a related problem (work in progress), which in our opinion is a serious and interesting one: the study of the stability of *global hypoellipticity* for operators on a *compact boundaryless manifold* (the *global analytic hypoellipticity* case when the manifold is the flat n-torus was recently studied by Chinni and Cordaro in [3]). In such a case it is reasonable to expect an interaction between the topology of the manifold and the geometric structure of the operator.

2. The setting

Since we shall deal with hypoellipticity and not with microellipticity, a natural measure of control of the singularities of a distribution is the *singular support* and the *Sobolev singular support*, which we next recall for the reader's convenience.

If $X \subset \mathbb{R}^n$ is an open set, $u \in \mathscr{D}'(X)$ and $x_0 \in X$, we shall say that

(i) u is C^∞ at x_0 and write $u \in C^\infty(x_0)$ if for some $\varphi \in C_0^\infty(X)$ with $\varphi(x_0) \neq 0$ one has $\varphi u \in C^\infty(X)$;

(ii) u is in the s-Sobolev space H^s at x_0, $s \in \mathbb{R}$, and write $u \in H^s(x_0)$ if for some $\varphi \in C_0^\infty(X)$ with $\varphi(x_0) \neq 0$ one has $\varphi u \in H^s(\mathbb{R}^n) =: H^s$.

Obviously the sets

$$\{x_0 \in X;\ u \in C^\infty(x_0)\} \quad \text{and} \quad \{x_0 \in X;\ u \in H^s(x_0)\}$$

are open (possibly empty) subsets of X, whose complements in X are called, respectively, the *singular support* of u, denoted by $\mathrm{SS}(u)$, and the *s-singular support* of u, denoted by $\mathrm{SS}_s(u)$.

The following facts are well known:

$$(a)\qquad \mathrm{SS}_{s'}(u) \subset \mathrm{SS}_{s''}(u) \subset \mathrm{SS}(u),\ \forall u \in \mathscr{D}'(X),\ \forall s', s'' \in \mathbb{R},\ \text{with } s' < s'';$$

$$(b)\qquad \mathrm{SS}(u) = \overline{\bigcup_{s\in\mathbb{R}} \mathrm{SS}_s(u)},\ \text{the closure being taken in } X.$$

Item (a) is straightforward. As for (b), since $\mathrm{SS}(u)$ and $\mathrm{SS}_s(u)$ are closed in X, the inclusion $\overline{\bigcup_{s\in\mathbb{R}} \mathrm{SS}_s(u)} \subset \mathrm{SS}(u)$ is a consequence of (a). Now, if $x_0 \notin \overline{\bigcup_{s\in\mathbb{R}} \mathrm{SS}_s(u)}$, then one may find a neighborhood $V \subset X$ of x_0 such that $V \cap \mathrm{SS}_s(u) = \emptyset$ for all $s \in \mathbb{R}$, whence by the Sobolev embedding theorem we have $x_0 \notin \mathrm{SS}(u)$.

Furthermore, if P is a classical properly supported ψdo of order $m \in \mathbb{R}$ (from now on all the ψdos considered will be taken *properly supported*), then

$$(c)\qquad \begin{cases} \mathrm{SS}(Pu) \subset \mathrm{SS}(u), \\ \\ \mathrm{SS}_s(Pu) \subset \mathrm{SS}_{s+m}(u), \end{cases} \qquad \forall u \in \mathscr{D}'(X),\ \forall s \in \mathbb{R}.$$

One hence says that *P is (C^∞-)hypoelliptic in X if*

$$\mathrm{SS}(u) = \mathrm{SS}(Pu),\quad \forall u \in \mathscr{D}'(X), \tag{2.1}$$

and says that *P is hypoelliptic in X with a loss of $r \geq 0$ derivatives if*

$$\mathrm{SS}_{s+m-r}(u) \subset \mathrm{SS}_s(Pu),\quad \forall u \in \mathscr{D}'(X),\ \forall s \in \mathbb{R}. \tag{2.2}$$

Note that, as a consequence of (b), (2.2) implies (2.1).

In the sequel, we will consider hypoellipticity for operators which are *nonelliptic*. Hence, if (2.2) holds it must hold for some $r > 0$.

The following observation will be useful in the sequel.

Lemma 2.1. *Condition* (2.2) *above is equivalent to*

$$\mathrm{SS}_{s+m-r}(v) \subset \mathrm{SS}_s(Pv), \quad \forall v \in \mathscr{E}'(X), \ \forall s \in \mathbb{R}. \tag{2.3}$$

Proof. We have only to prove that (2.3) $\Longrightarrow$ (2.2). Let $u \in \mathscr{D}'(X)$ and suppose $x_0 \notin \mathrm{SS}_s(Pu)$. Let $\phi \in C_0^\infty(X)$, $\phi \equiv 1$ near x_0. Obviously, $x_0 \notin \mathrm{SS}_t(u - \phi u)$, for all $t \in \mathbb{R}$. Hence by (c) we get $x_0 \notin \mathrm{SS}_s(Pu - P(\phi u))$, so that $x_0 \notin \mathrm{SS}_s(P(\phi u))$, which proves the claim by virtue of (2.3). □

The problem we will be considering in this paper is the following:

Problem 2.2. Given P, a ψdo of order m which is hypoelliptic in X with a loss of $r > 0$ derivatives, what kind of ψdos Q of order $m' < m$ can we add to P so that $P + Q$ remains hypoelliptic in X? Can we estimate for $P + Q$ the possible loss of derivatives?

We next list a number of nontrivial examples that basically show that "everything is possible".

2.1. **Example 1.** This example deals with a perturbation that destroys hypoellipticity. In $X = \mathbb{R}^\nu_x \times Y$, $Y \subset \mathbb{R}^{n-\nu}_y$ an open set $(1 \le \nu < n)$, consider the differential operator with smooth coefficients

$$P = \sum_{j=1}^{\nu} \big(D_{x_j}^2 + \mu_j^2 x_j^2 |D_y|^2\big) + \sum_{j=1}^{\nu} a_j(x,y) D_{x_j} + \sum_{h=1}^{n-\nu} b_h(x,y) D_{y_h} + c(x,y), \tag{2.4}$$

where $\mu_j > 0$ are fixed constants and a_j, b_h and c are complex valued. Consider the map

$$\gamma\colon \mathbb{Z}_+^\nu \ni \alpha \longmapsto \langle \alpha, \mu\rangle = \sum_{j=1}^{\nu} \alpha_j \mu_j \in [0, +\infty),$$

and let

$$\gamma(\mathbb{Z}_+^\nu) = \{0 = \ell_0 < \ell_1 < \cdots < \ell_N \longrightarrow +\infty\}.$$

Suppose that the vector $b(0,y) = \begin{bmatrix} b_1(0,y) \\ \vdots \\ b_{n-\nu}(0,y) \end{bmatrix} \in \mathbb{R}^{n-\nu}$. If

$$|b(0,y)| < |\mu| := \sum_{j=1}^{\nu} \mu_j, \quad \forall y \in Y, \tag{2.5}$$

then P is hypoelliptic in X with a loss of 1 derivative (see Boutet de Monvel, Grigis and Helffer [2]). Next let

$$Q = \sum_{j=1}^{n-\nu} q_j(y) D_{y_j},$$

with $q = (q_1, \ldots, q_{n-\nu}) \in C^\infty(Y, \mathbb{R}^{n-\nu})$, and suppose that for some $N \ge 1$ we have

$$2\ell_{N-1} + |\mu| < |b(0,y) + q(y)| < 2\ell_N + |\mu|, \quad \forall y \in Y. \tag{2.6}$$

Then, as shown in Parenti and Parmeggiani [7], the operator $P + Q$ is no longer hypoelliptic in X, and, moreover, there is a propagation of C^∞ singularities in the characteristic set $\{x = \xi = 0\}$ (where (x, y, ξ, η) are the coordinates in T^*X) along the integral curves of the Hamilton vector field of the functions

$$\lambda_j(y, \eta) = (2\ell_j + |\mu|)|\eta| + \langle b(0, y) + q(y), \eta\rangle, \quad 0 \le j \le N-1, \ (y, \eta) \in T^*Y \setminus 0. \tag{2.7}$$

Therefore, in this case addition of Q completely destroys hypoellipticity. Note that perturbing P by an operator of order zero would not have changed hypoellipticity and the loss of derivatives.

2.2. **Example 2.** This example deals with a perturbation that yields hypoellipticity with a smaller loss of derivatives. In $X = \mathbb{R}_x^{n-1} \times Y$, with $Y \subset \mathbb{R}_y$ and open neighborhood of the origin, consider the 4th-order differential operator with smooth real coefficients

$$\begin{aligned} P &= \Big(\sum_{j=1}^{n-1} \big(D_{x_j}^2 + \mu_j^2 x_j^2 D_y^2\big)\Big)^2 + \langle A(x, y) D_x, D_x\rangle \\ &+ b(x, y) D_y^2 + \langle c(x, y), D_x\rangle + d(x, y) D_y + e(x, y) \end{aligned} \tag{2.8}$$

($\mu_j > 0$ as before). If

$$b(0, y) + |\mu|^2 > 0, \quad \forall y \in Y, \tag{2.9}$$

then P is hypoelliptic with a loss of 2 derivatives (see [2]). Now let $Q = iD_y^3$. Since

$$\Big(\sum_{j=1}^{n-1} \big(\xi_j^2 + \mu_j^2 x_j^2 \eta^2\big)\Big)^2 + i\eta^3 \ne 0, \ \forall x \in \mathbb{R}^{n-1}, \ \forall (\xi, \eta) \in \mathbb{R}^n \setminus \{0\}, \tag{2.10}$$

by Theorem 4.1 of Helffer [4], the operator $P + Q$ possesses a parametrix in the Hörmander class $S^{-3}_{\rho=3/4, \delta=1/4}(X \times \mathbb{R}^n)$, and hence $P + Q$ is hypoelliptic with a loss of 1 derivative, regardless of the coefficient b.

2.3. **Example 3.** This example deals with a perturbation that yields hypoellipticity with a bigger loss of derivatives. In $X = \mathbb{R}_x^\nu \times Y$, with $Y \subset \mathbb{R}_y^{n-\nu}$ an open neighborhood of the origin ($1 \le \nu < n - 1$), consider the 4th-order differential operator with real smooth coefficients

$$\begin{aligned} P &= \Big(\sum_{j=1}^{\nu} \big(D_{x_j}^2 + \mu_j^2 x_j^2 |D_y|^2\big)\Big)^2 + \langle B(y) D_y, D_y\rangle \\ &+ \langle c(x, y), D_x\rangle + \langle d(y), D_y\rangle + e(x, y) \end{aligned} \tag{2.11}$$

($\mu_j > 0$ as before). Consider the quadratic forms in η

$$\lambda_j(y, \eta) := \langle L_j(y)\eta, \eta\rangle, \tag{2.12}$$

where

$$L_j(y) = B(y) + (2\ell_j + |\mu|)^2 I_{n-\nu}, \quad j \ge 0.$$

If

$$\lambda_0(y, \eta) > 0, \quad \forall (y, \eta) \in T^*Y \setminus 0, \tag{2.13}$$

then, again, P is hypoelliptic with a loss of 2 derivatives. Let $Q = \langle \beta(y) D_y, D_y\rangle$, with β real, symmetric and smooth. Suppose that for $j \ge 1$ the matrices

$$L_j(y) + \beta(y) > 0, \quad \forall y \in Y,$$

whereas for $j = 0$

$$\lambda_0(y,\eta) + \langle \beta(y)\eta, \eta\rangle = \alpha_1(y)\eta_1^2 + y_1^2 \sum_{j=2}^{n-\nu} \alpha_j(y)\eta_j^2, \tag{2.14}$$

with $\alpha_j(y) > 0$, for all $y \in Y$, $1 \le j \le n-\nu$. Consider, for $y' = (y_2, \dots, y_{n-\nu}) \in Y \cap \{y_1 = 0\}$, the $(n-\nu-1) \times (n-\nu-1)$ matrix

$$\theta(y') = \operatorname{diag}\Big(\alpha_1(0,y')\alpha_2(0,y'), \dots, \alpha_1(0,y')\alpha_{n-\nu}(0,y')\Big)$$

and the vector $d_0(y') = \begin{bmatrix} d_2(0,y') \\ \vdots \\ d_{n-\nu}(0,y') \end{bmatrix} \in \mathbb{R}^{n-\nu-1}$. If

$$|\theta(y')^{-1/2} d_0(y')| < 1, \quad \forall y' \in Y \cap \{y_1 = 0\}, \tag{2.15}$$

then (see [6] and [7]) the operator $P + Q$ is hypoelliptic in X with a loss of 3 derivatives.

3. A stability result

Although the preceding examples are quite discouraging in that they show that indeed "everything is possible", taking the hint from [9] a "general" procedure can be devised. We will consider the case of a *differential operator* P of order m with smooth coefficients in an open set $X \subset \mathbb{R}^n$. Let us suppose the following:

(H1) *Both P and P^* (the formal adjoint of P) are hypoelliptic in X;*
(H2) *P is hypoelliptic with a loss of r derivatives with $0 < r < m$.*

Then we have the following result.

Theorem 3.1. *We have:*

(i) *P^* is itself hypoelliptic with the same loss r of derivatives.*
(ii) *For every differential operator Q of order $m' < m - r$, the operator $P + Q$ is still hypoelliptic with the same loss of derivatives.*

Proof. By Lemma 2.1, to prove (ii) we have to prove that

$$\mathrm{SS}_{s+m-r}(u) \subset \mathrm{SS}_s((P+Q)u), \quad \forall u \in \mathscr{E}'(X), \forall s \in \mathbb{R}. \tag{3.1}$$

We claim that it suffices to prove the following:

For any given $x_0 \in X$ there exists an open neighborhood $U \subset X$ of x_0 such that

$$\mathrm{SS}_{s+m-r}(v) \subset \mathrm{SS}_s((P+Q)v), \quad \forall v \in \mathscr{E}'(U), \ \forall s \in \mathbb{R}. \tag{3.2}$$

In fact, given $s \in \mathbb{R}$ and $u \in \mathscr{E}'(X)$ for which $x_0 \notin \mathrm{SS}_s((P+Q)u)$, take $\psi \in C_0^\infty(U)$ with $\psi \equiv 1$ near x_0. Then $x_0 \notin \mathrm{SS}_s((P+Q)(\psi u))$ by virtue of the local nature of P and Q. Since $\psi u \in \mathscr{E}'(U)$, from (3.2) we then get $x_0 \notin \mathrm{SS}_{s+m-r}(\psi u)$, and the claim follows.

To prove (3.2) we use Treves [11, Thm. 52.2]: because of the hypoellipticity of both P and P^*, given $x_0 \in X$ there exists a neighborhood $U \subset X$ of x_0 and a distribution $\mathsf{k} \in \mathscr{D}'(U \times U)$ with $\mathrm{SS}(\mathsf{k}) \subset \operatorname{diag}(U \times U)$ for which the operator K, whose Schwartz kernel is k, has the following properties:

(3.3) *K is continuous as a map $K : C_0^\infty(U) \to C^\infty(U)$ and $K : \mathscr{E}'(U) \to \mathscr{D}'(U)$,* and

$$(3.4)\qquad \begin{cases} PKf = f, & \forall f \in \mathscr{E}'(U), \\ KPu = u, & \forall u \in \mathscr{E}'(U). \end{cases}$$

From the first equality in (3.4) and hypothesis (H2) we obtain that

$$K\big(H^s \cap \mathscr{E}'(U)\big) \subset H^{s+m-r}_{\mathrm{loc}}(U), \quad \forall s \in \mathbb{R}.$$

Now, the Borel Graph Theorem (see [11, Thm. A1]), ensures that

$$(3.5)\qquad K\colon H^s \cap \mathscr{E}'(U) \longrightarrow H^{s+m-r}_{\mathrm{loc}}(U) \textit{ is continuous, } \forall s \in \mathbb{R}.$$

Next, to see (3.2), let $v \in \mathscr{E}'(U)$ and suppose that for some $s \in \mathbb{R}$ and $\bar{x} \in U$ one has $\bar{x} \notin \mathrm{SS}_s((P+Q)v)$. Pick $\psi \in C_0^\infty(U)$, $\psi \equiv 1$ near $\bar{x}$, so that

$$(P+Q)(\psi v) =: f \in H^s \cap \mathscr{E}'(U).$$

Applying K to the left, from (3.4) we get

$$(3.6)\qquad \psi v = Kf - KQ(\psi v).$$

Now, $Kf \in H^{s+m-r}_{\mathrm{loc}}(U)$, and since there exists $t \in \mathbb{R}$ such that $\psi v \in H^t \cap \mathscr{E}'(U)$, we have $KQ(\psi v) \in H^{t+\varepsilon}_{\mathrm{loc}}(U)$ with $0 < \varepsilon = m - r - m'$. Therefore

$$\psi v \in H^{\min\{s+m-r, t+\varepsilon\}} \cap \mathscr{E}'(U).$$

If $t + \varepsilon < s + m - r$, again by (3.6) we get $KQ(\psi v) \in H^{t+2\varepsilon}_{\mathrm{loc}}(U)$, so that

$$\psi v \in H^{\min\{s+m-r, t+2\varepsilon\}} \cap \mathscr{E}'(U).$$

In a finite number of steps we therefore conclude that $\psi v \in H^{s+m-r} \cap \mathscr{E}'(U)$, which thus proves (3.2) and (ii).

To prove (i), we note that (3.4) yields that K^* is a two-sided inverse of P^*. The continuity of $K\colon H^s \cap \mathscr{E}'(U) \longrightarrow H^{s+m-r}_{\mathrm{loc}}(U)$, for all $s \in \mathbb{R}$, yields the continuity of

$$K^*\colon H^{-s-(m-r)} \cap \mathscr{E}'(U) \longrightarrow H^{-s}_{\mathrm{loc}}(U), \quad \forall s \in \mathbb{R},$$

where K^* is the operator whose Schwartz kernel k^* is given by

$$\mathsf{k}^*(x,y) = \overline{\mathsf{k}(y,x)}.$$

To conclude that P^* is hypoelliptic with a loss of r derivatives it suffices to show that

$$\mathrm{SS}_{s+m-r}(v) \subset \mathrm{SS}_s(P^*v), \quad \forall v \in \mathscr{E}'(U),\ \forall s \in \mathbb{R}.$$

If $\bar{x} \in U$ is such that $\bar{x} \notin \mathrm{SS}_s(P^*v)$, then choose $\psi \in C_0^\infty(U)$, $\psi \equiv 1$ near $\bar{x}$, so that $P^*(\psi v) \in H^s \cap \mathscr{E}'(U)$ (note that P^* is also a differential operator). Hence $K^*P^*(\psi v) = \psi v \in H^{s+m-r} \cap \mathscr{E}'(U)$, which proves that $\bar{x} \notin \mathrm{SS}_{s+m-r}(v)$. This ends the proof of Theorem 3.1. ☐

Remark 3.2.

(a) We do not know whether the conclusion (ii) of Theorem 3.1 holds on assuming only that P is hypoelliptic with a loss of r derivatives ($m - r > 0$).

(b) Stein's example (see Stein [10] and also Parenti and Parmeggiani [6]) shows that when $m - r = 0$ even a perturbation by a constant may destroy hypoellipticity.

Note that (3.5) is a consequence of the fact that K is a two-sided fundamental solution of P (see (3.4)). If we *assume* the continuity property (3.5) when K is *only* a left inverse of P, then the bootstrap argument (3.6) works again. To be more precise, let P be a classical ψdo of order m on $X \subset \mathbb{R}^n$ and assume:

There exists a continuous operator $K\colon \mathscr{E}'(X) \longrightarrow \mathscr{D}'(X)$ and $r > 0$ such that

$$\mathrm{SS}_{s+m-r}(Kf) \subset \mathrm{SS}_s(f), \quad \forall f \in \mathscr{E}'(X),\ \forall s \in \mathbb{R}, \tag{3.7}$$

and

$$KP = \mathrm{id}_X - R, \tag{3.8}$$

where $R\colon \mathscr{E}'(X) \longrightarrow C^\infty(X)$ is a smoothing operator.

A first trivial consequence of these assumptions is that P is hypoelliptic with a loss of r derivatives. Furthermore, the same bootstrap argument of Theorem 3.1 gives that $P+Q$ is hypoelliptic with a loss of r derivatives for any given ψdo Q of order m' such that $m' < m - r$ (regardless of the sign of $m-r$). The core of the argument is that KQ has the property that KQf is *smoother* than f. Therefore one may dispose of the condition $m' < m - r$ provided that for some $\delta > 0$ one has

$$\mathrm{SS}_{s+\delta}(KQf) \subset \mathrm{SS}_s(f), \quad \forall f \in \mathscr{E}'(X),\ \forall s \in \mathbb{R}. \tag{3.9}$$

That this may be the case depends on *finer* properties of K and Q and *not* only on the rough balance of the orders.

A specific example is given by an operator $P \in \mathrm{OPS}^{m,k}(X,\Sigma)$, $\Sigma \subset T^*X \setminus 0$ symplectic, $k \geq 2$ an even integer (see Boutet de Monvel [1]). It is known that in this case P cannot be hypoelliptic with a loss of $r < k/2$ derivatives. Suppose that P is hypoelliptic with a loss of $k/2$ derivatives. Then from Boutet's work [1] we know that P has a left parametrix $K \in \mathrm{OPS}^{-m,-k}(X,\Sigma)$. Now, if $Q \in \mathrm{OPS}^{m-1,k-1}(X,\Sigma) \subset \mathrm{OPS}^{m-1}(X)$, then

$$KQ \in \mathrm{OPS}^{-1,-1}(X,\Sigma) \subset \mathrm{OPS}^{-1/2}_{1/2,1/2}(X),$$

so that (3.9) is satisfied with $\delta = 1/2$, whence $P+Q$ is hypoelliptic with a loss of $k/2$ derivative.

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