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EXPLICIT FORMULA FOR THE SOLUTION OF SIMULTANEOUS PELL EQUATIONS $x^2 - (a^2-1)y^2 = 1$, $y^2 - bz^2 = 1$

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(Communicated by Matthew A. Papanikolas)

Dedicated to Professor Maurice Mignotte on the occasion of his retirement

Abstract. For b an odd integer whose square-free part has at most two prime divisors, it is shown that the equations in the title have a common solution in positive integers precisely when b divides $4a^2-1$ and the quotient is a perfect square. The proof provides an explicit formula for the common solution, known to be unique. Similar results are obtained assuming the square-free part of b is even or has three prime divisors.

1. Introduction

It is well known that the solutions to a generalised Pell equation $ax^2 - by^2 = c$, where a and b are positive integers whose product is not a perfect square, obey the principle that if there is a solution in positive integers, then there are infinitely many such solutions. The situation is somewhat unclear when one wants to count the number of positive solutions to a pair of such equations sharing a common unknown $a_1x^2 - b_1y^2 = c_1$, $a_2y^2 - bz_2^2 = c_2$: if $a_1b_2 \neq a_2b_1$, then a celebrated result of A. Thue [17] assures one that there are at most finitely many solutions.

Establishing explicit bounds on the number of solutions that are sufficiently sharp to be exploited in practice is very difficult. D. W. Masser and J. H. Rickert [15] have devised a procedure to associate to each positive integer N two integers c_1 and c_2 such that the two Pell equations $x^2 - 2z^2 = c_1$, $x^2 - 3z^2 = c_2$ have at least N common solutions. On the other hand, Y. Bugeaud, C. Levesque and M. Waldschmidt [6] have proved that the system $x^2 - b_1z^2 = c_1$, $y^2 - b_2z^2 = c_2$ ($b_1c_2 \neq b_2c_1$) has no more than $2+2^{3996(\omega(c_1c_2)+1)}$ solutions. As usual, $\omega(n)$ denotes the number of distinct prime divisors of a positive integer n.

Sharp uniform bounds for the number of solutions are available for specific classes of equations. M. A. Bennett, M. Cipu, M. Mignotte and R. Okazaki have shown in [5] that the system $x^2 - bz^2 = 1$, $y^2 - dz^2 = 1$ has at most two solutions in positive integers. The same bound has been established by Cipu and Mignotte [9] for simultaneous Pell equations of the type $ax^2 - bz^2 = 1$, $cy^2 - dz^2 = 1$, with $a > 1$, $b > 1$. Both results are optimal, as the indicated bounds are attained by

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infinite classes of systems. In [9] one also finds the best published result concerning the number of positive solutions for systems of the form

$$x^2 - by^2 = 1, \quad y^2 - dz^2 = 1 \quad (b \neq d). \tag{1.1}$$

The given bound is two. However, no example of a system of the kind is known to have non-unique solution in positive integers. The uniqueness has been established for the compatible equations

$$x^2 - (a^2 - 1)y^2 = 1, \quad y^2 - bz^2 = 1 \quad (a > 1) \tag{1.2}$$

by P. Z. Yuan (odd a) and the present author (even a) in [20] and [8], respectively. Uniqueness is also assured for any combination of ± 1 in the right hand sides of (1.1) provided that one of the parameters involved is suitably large, as shown by Bennett and Á. Pintér in [3].

Further information on the unique positive solution to (1.2) has been made available in a recent paper by N. Irmak [10] based on the fact that the solutions of any Pell equation are given by a second order linear recurrent sequence. The main aim of this paper is to show that for b odd prime, the unique positive solution to the system (1.2) appears at an even index in the sequence of solutions for the first equation. The same paper purports to give all the positive solutions for (1.2) in which one has $5 \leq a \leq 14$ and b prime, thus extending [1]. Unfortunately, the author fails both goals: the proof of the main result is based on a coprimality assumption not always fulfilled and the explicit computations miss the solution $(x, y, z) = (287, 24, 5)$ for $(a, b) = (12, 23)$.

This note grew out of an attempt to fill in the gaps in [10] and eventually succeeded to treat systems of the form (1.2) under conditions on the factorization of $sqf(b)$, the square-free part of b, defined as b divided by its largest square factor. We focus on easily verifiable criteria for solvability when $sqf(b)$ has at most two prime divisors. As in many other cases, the details are slightly different according to parity of the numbers in question.

Theorem 1.1. *Let a and b be integers greater than 1, with b not a perfect square.*

a) Assume b is odd and its square-free part has at most two prime divisors. Then the system (1.2) is solvable in positive integers if and only if b divides $4a^2 - 1$ and the quotient is a perfect square. When it exists, this solution is

$$(x, y, z) = \left(2a^2 - 1, 2a, \sqrt{(4a^2 - 1)/b}\right).$$

b) Assume $sqf(b) = 2p$ with p either prime or equal to 1. Then the system (1.2) is solvable in positive integers if and only if $(2a^2 - 1)/p$ is a perfect square and b divides $8a^2(2a^2 - 1)$. When it exists, this solution is

$$(x, y, z) = \left(4a^3 - 3a, 4a^2 - 1, \sqrt{8a^2(2a^2 - 1)/b}\right).$$

In particular, if $sqf(b) = 2p$ for some prime p congruent to ± 3 modulo 8, then the system (1.2) has no integer solutions.

The main result is proven in Section 3. Section 2 contains all published results used in the proof. In Section 4 we examine the situation when the square-free part of b has three prime divisors.

2. Preliminaries

Let $a \geq 2$ be an integer and

$$\alpha = a + \sqrt{a^2-1}, \quad \overline{\alpha} = a - \sqrt{a^2-1}, \tag{2.1}$$

For any non-negative integer k, put

$$U_k = U_k(a) = \frac{\alpha^k - \overline{\alpha}^k}{\alpha - \overline{\alpha}}, \quad V_k = V_k(a) = \frac{\alpha^k + \overline{\alpha}^k}{2}. \tag{2.2}$$

In order to partially factorise the terms of the above Lucas sequences, we also introduce the Lehmer sequences associated to

$$\xi = \frac{\sqrt{2a+2} + \sqrt{2a-2}}{2}, \quad \overline{\xi} = \frac{\sqrt{2a+2} - \sqrt{2a-2}}{2}. \tag{2.3}$$

Specifically, we consider

$$x_k = \frac{\xi^k + \overline{\xi}^k}{\xi + \overline{\xi}}, \quad y_k = \frac{\xi^k - \overline{\xi}^k}{\xi - \overline{\xi}} \quad \text{for odd } k, \tag{2.4}$$

$$x_k = \frac{\xi^k + \overline{\xi}^k}{2}, \quad y_k = \frac{\xi^k - \overline{\xi}^k}{\xi^2 - \overline{\xi}^2} \quad \text{for even } k. \tag{2.5}$$

The same numbers can be obtained from linear recurrence sequences:

$$U_0 = 0,\ U_1 = 1,\ U_{k+2} = 2aU_{k+1} - U_k,$$
$$V_0 = 1,\ V_1 = a,\ V_{k+2} = 2aV_{k+1} - V_k,$$
$$x_0 = 1,\ x_1 = 1,\ x_2 = a,\ x_3 = 2a-1,\ x_{k+4} = 2ax_{k+2} - x_k,$$
$$y_0 = 0,\ y_1 = 1,\ y_2 = 1,\ y_3 = 2a+1,\ y_{k+4} = 2ay_{k+2} - y_k.$$

These integers are interesting for their connection with solutions of generalised Pell equations. The next results are classical, so well known and frequently employed that it is very difficult to locate their first appearance in print.

Lemma 2.1. *The non-negative solutions to the Pell equation* $V^2 - (a^2-1)U^2 = 1$ *are precisely* $(V_k, U_k)_{k \geq 0}$.

Lemma 2.2. *All solutions in positive integers to the generalised Pell equation* $(a+1)X^2 - (a-1)Y^2 = 2$ *are* $(x_{2k+1}, y_{2k+1})_{k \geq 0}$.

Strong relationships existing between the sequences considered above can be explained by the identities

$$\alpha = \xi^2, \quad \overline{\alpha} = \overline{\xi}^2, \quad \alpha\overline{\alpha} = \xi\overline{\xi} = 1.$$

Thus, it can be easily seen that one has

$$x_{2k} = V_k, \quad y_{2k} = U_k \quad \text{for all } k \geq 0 \tag{2.6}$$

and more generally

$$U_k = \begin{cases} x_k y_k & \text{for odd } k, \\ 2x_k y_k & \text{for even } k, \end{cases} \tag{2.7}$$

whence

$$U_{k+1}^2 - 1 = U_{k+2}U_k = \begin{cases} x_{k+2}y_{k+2}x_k y_k & \text{for odd } k, \\ 4x_{k+2}y_{k+2}x_k y_k & \text{for even } k. \end{cases} \tag{2.8}$$

The common divisors of the factors in the right side of (2.8) have been determined in [20] for odd a and in [8] for even a.

Lemma 2.3. (i) *If k is odd, then x_k, y_k, x_{k+2}, y_{k+2} are pairwise coprime.*

(ii) *For k even one has:*

$\gcd(x_k, x_{k+2}) = \gcd(y_k, y_{k+2}) = \gcd(x_k, y_k) = 1$,

$$\gcd(x_k, y_{k+2}) = \begin{cases} 1 & \text{for } k \equiv 0 \pmod 4, \\ a & \text{for } k \equiv 2 \pmod 4, \end{cases}$$

$$\gcd(x_{k+2}, y_k) = \begin{cases} a & \text{for } k \equiv 0 \pmod 4, \\ 1 & \text{for } k \equiv 2 \pmod 4. \end{cases}$$

We shall be interested to know which terms of these Lucas / Lehmer sequences are squares. Nowadays a lot of answers to this question are available. From the wealth of accessible results, we selected the most convenient ones.

We begin with the relevant part of a theorem of Mignotte and A. Pethő.

Lemma 2.4 ([16]). *For an integer $A \geq 3$, define the sequence $(w_k(A))_{k\geq 0}$ by the second order linear recurrence $w_{k+2} = Aw_{k+1} - w_k$ with the initial conditions $w_0 = 0$, $w_1 = 1$. Then the only solutions to the equation*

$$w_n(A) = X^2, \quad A \geq 3, \quad n \geq 3,$$

are obtained for $(A, n) \in \{(3, 6), (338, 4)\}$.

The next results are phrased in terms of solutions to quartic equations. The first of them is due to W. Ljunggren.

Lemma 2.5 ([13]). *Let A, B be positive integers with A square-free and AB not a perfect square. Assume the equation $AU^2 - BV^2 = 1$ has solutions and denote by (u, v) its minimal positive integer solution. If $4Bv^2 + 3$ is not a square, then $AS^2 - BT^4 = 1$ has at most one solution in positive integers.*

Ljunggren has dealt with various other equations of the type $AX^2 - BY^4 = C$. In recent years, his work has been refined and more precise results have been obtained by many authors.

Lemma 2.6 ([4]). *Let $A > 1$ and $B > 0$ be square-free integers. Suppose $\varepsilon = u + v\sqrt{B} > 1$ is the fundamental solution of Pell's equation $S^2 - BT^2 = 1$. Define $\varepsilon^n = u_n + v_n\sqrt{B}$ for $n = 1, 2, \ldots$. If the Diophantine equation $A^2X^4 - BY^2 = 1$ is solvable, then it has at most one solution (x, y) in positive integers, which is given by $Ax^2 + y\sqrt{B} = \varepsilon^t$, where t is the least positive integer such that $u_t \equiv 0 \pmod A$.*

Lemma 2.7 ([7]). *Let the fundamental solution of the equation $S^2 - DT^2 = 1$ be $\varepsilon = s + t\sqrt{D}$. Then the only possible solutions of the equation $X^4 - DY^2 = 1$ in positive integers are given by $x^2 + y\sqrt{D} = \varepsilon$ and $x^2 + y\sqrt{D} = \varepsilon^2$; both solutions occur in only one case, $D = 1785$.*

Lemma 2.8 ([18]). *For a positive non-square integer D, let $\varepsilon = s + t\sqrt{D}$ be the minimal unit in $\mathbb{Z}[\sqrt{D}]$ of norm 1. For $n \geq 1$, define $s_n + t_n\sqrt{D} = \varepsilon^n$.*

(i) *There are at most two positive integer solutions to equation $X^2 - DY^4 = 1$. If two solutions $y_1 < y_2$ exist, then $y_1^2 = t_1$, $y_2^2 = t_2$, except only if $D = 1785$ or $D = 16 \cdot 1785$, in which case $y_1^2 = t_1$, $y_2^2 = t_4$.*

(ii) *If the positive solution is unique, it is given by $y^2 = t_n$, where $t_1 = nv^2$ for some square-free integer n, and $n = 1$, $n = 2$, or n is some prime congruent to 3 modulo 4.*

The proof of Theorem 1.1 will also require a known result on the quartic Diophantine equation

$$AX^2 - BY^4 = 2. \tag{2.9}$$

Lemma 2.9 ([14])**.** *Consider the equation* (2.9)*, where both A and B are positive odd integers for which the equation*

$$AU^2 - BV^2 = 2 \tag{2.10}$$

is solvable in positive integers. Let (u_1, v_1) *be the minimal positive solution of* (2.10)*. For odd n define integers* u_n*,* v_n *by* $u_n\sqrt{A} + v_n\sqrt{B} = 2^{(1-n)/2}(u_1\sqrt{A} + v_1\sqrt{B})^n$*. Then:*

(i) *If* v_1 *is not a square, then equation* (2.9) *has no solutions.*
(ii) *If* v_1 *is a square and* v_3 *is not a square, then* $(u_1, \sqrt{v_1})$ *is the only solution of equation* (2.9)*.*
(iii) *If* v_1 *and* v_3 *are both squares, then* $(u_1, \sqrt{v_1})$ *and* $(u_3, \sqrt{v_3})$ *are the only solutions of equation* (2.9)*.*

For the related Diophantine equation

$$AX^4 - BY^2 = 2 \tag{2.11}$$

we need a result established by Y. Li and Yuan.

Lemma 2.10 ([11])**.** *For any positive odd integers A, B the equation* (2.11) *has at most one solution in positive integers, and such a solution arises from the fundamental solution to the quadratic equation* (2.10)*.*

After these preparations we can proceed with the proof of the theorem stated in the Introduction.

3. Proof of the main result

Throughout this section, when we speak of 'solution' or 'solvability' we mean solution in positive integers. For a positive integer n, we shall denote by $sqf(n)$ its square-free part and by $\omega(n)$ the number of distinct prime divisors.

Let (x, y, z) be the unique solution of (1.2) in positive integers. Notice that one necessarily has $y \geq 2$. This in conjunction with Lemma 2.1 implies that we have $x = V_k$, $y = U_k$ for some index $k \geq 2$.

The details of the argument below are somewhat different according to $k \mod 4$, so we shall examine separately several cases.

Case I ($k = 2r$, $r \geq 1$). By (2.7) and (2.2), we have

$$bz^2 = y^2 - 1 = U_k^2 - 1 = U_{2r+1}U_{2r-1} = x_{2r+1}y_{2r+1}x_{2r-1}y_{2r-1}.$$

Having in mind Lemma 2.3, one sees that the factors in the rightmost side in this chain of equalities are pairwise coprime. Since the square-free part of b has at most two prime divisors, at least two of x_{2r+1}, y_{2r+1}, x_{2r-1}, y_{2r-1} are perfect squares.

Subcase I.1: y_{2r+1} *is a square.* Then both $(x_{2r+1}, \sqrt{y_{2r+1}})$ and $(1, 1)$ solve the equation

$$(a + 1)S^2 - (a - 1)T^4 = 2. \tag{3.1}$$

Let us examine first the situation for even a. According to Lemma 2.9, in case $2a + 1$ is not a square, $(1, 1)$ is the only solution in positive integers. This entails

$y_{2r+1} = 1$, a contradiction. If there exists an integer $c > 1$ such that $2a + 1 = c^2$, then Lemma 2.9 yields $y_{2r+1} = 2a + 1 = y_3$, whence $r = 1$ and

$$bz^2 = 4a^2 - 1.$$

This is precisely the conclusion of Theorem 1.1.

Suppose now that a is odd. Simplification in equation (3.1) results in the equation

$$\left(\frac{a+1}{2}\right) S^2 - \left(\frac{a-1}{2}\right) T^4 = 1. \tag{3.2}$$

Writing $\frac{a+1}{2} = bc^2$ with b square-free and $c > 0$, one gets $\frac{a-1}{2} = bc^2 - 1$. As the smallest solution in positive integers for $bU^2 - (bc^2 - 1)V^2 = 1$ is $(u, v) = (c, 1)$, from Lemma 2.5 it follows that $(c, 1)$ is the unique positive solution for

$$bS^2 - (bc^2 - 1)T^4 = 1. \tag{3.3}$$

Since any solution (s', t') to Eq. (3.2) gives rise to a solution $(s, t) = (cs', t')$ to Eq. (3.3), we conclude again that $y_{2r+1} = 1$, which is impossible.

Subcase I.2: *y_{2r-1} is a square.* Again Lemma 2.9 entails that for a even and $2a+1$ not a perfect square one has $y_{2r-1} = 1 = y_1$, so that $r = 1$ and, as seen in the previous subcase, the desired conclusion is obtained. Moreover, when $2a + 1 = c^2$, it is possible to have $y_{2r-1} = 2a + 1 = y_3$, whence $r = 2$ and

$$bz^2 = c^2(4a^2 - 2a - 1)(4a^2 + 2a - 1)(2a - 1). \tag{3.4}$$

Notice that from $2a+1 = c^2$ it follows, on the one hand, that $2a-1$ is not a perfect square and, on the other hand, that a is a multiple of 4. Then $4a^2 \pm 2a - 1 \equiv 7 \pmod 8$, so that none of the expressions within parentheses in (3.4) is a perfect square. Thus, $\omega(sqf(b)) \geq 3$, in contradiction to our hypothesis.

Assume a is odd. The argument used in Subcase I.1 leads to the equality $y_{2r-1} = 1$. As seen above, this entails the desired conclusion.

It remains to examine the possibility that both x_{2r+1} and x_{2r-1} are perfect squares. Then the equation $(a + 1)S^4 - (a - 1)T^2 = 2$ has the solutions $(1, 1)$ and $(\sqrt{x_{2r\pm1}}, y_{2r\pm1})$. As a quartic equation of this kind has at most two positive solutions (see [2] or [19]), it follows that one has $r = 1$. As already seen, the conclusion of Theorem 1.1 follows.

Case $(k = 4r + 1,\ r \geq 1)$. By Lemma 2.3 we obtain

$$bz^2 = 4x_{4r+2}y_{4r+2}x_{4r}y_{4r} = 4a^2 \left(\frac{V_{2r+1}}{a}\right) U_{2r+1}V_{2r} \left(\frac{U_{2r}}{a}\right)$$

and, since the factors on the right side are pairwise coprime integers, again at least two of $\frac{V_{2r+1}}{a}$, U_{2r+1}, V_{2r}, $\frac{U_{2r}}{a}$ must be perfect squares. Surely U_{2r+1} is not square (see Lemma 2.4).

Subcase II.1: *$U_{2r} = ac^2$ for some positive integer c.* In this situation (V_{2r}, c) is a solution to the Diophantine equation

$$X^2 - a^2(a^2 - 1)Y^4 = 1. \tag{3.5}$$

Note that the fundamental solution to the associated Pell equation is $(2a^2 - 1, 2)$. By Lemma 2.8, (3.5) has a unique solution, given by $c^2 = 4(2a^2 - 1)$. Therefore one has $U_{2r} = 4a(2a^2 - 1) = U_4$, whence $r = 2$ and

$$bz^2 = 4a^2c^2(16a^4 - 20a^2 + 5)(4a^2 + 2a - 1)(4a^2 - 2a - 1)(8a^4 - 8a^2 + 1).$$

We claim that no expression within parentheses can be a square. As $2a^2-1$ is a square, say, d^2, one has $8a^4-8a^2+1=2d^4-1$. A well-known result of Ljunggren [12] gives that the only solutions to $2X^4-Y^2=1$ are $(1,1)$ and $(13,239)$. Hence, $8a^4-8a^2+1$ is never square for $a>1$ satisfying $2a^2-1=d^2$. If $4a^2-2a-1=f^2$, then $4f^2=(4a-1)^2-5$, which does not hold for $a\geq 2$. The same argument employed for the other two factors establishes the claim.

Thus, the four expressions within parentheses are pairwise coprime and each contributes with a distinct factor to the square-free part of b. This means $\omega(sqf(b))\geq 4$, in contradiction with our hypothesis.

Subcase II.2: *V_{2r} is a square.* Now $(\sqrt{V_{2r}},U_{2r})$ solves the quartic equation $S^4-(a^2-1)T^2=1$. Note that the fundamental solution for the associated Pell equation $X^2-(a^2-1)Y^2=1$ is $\varepsilon=a+\sqrt{a^2-1}$. Therefore, Lemma 2.7 yields either $V_{2r}+U_{2r}=a+\sqrt{a^2-1}$ or $V_{2r}+U_{2r}=2a^2-1+2a\sqrt{a^2-1}$. Having in view that $V_1=a<V_2=2a^2-1<V_3$, the only possibility is $2r=2$. Hence,

$$bz^2=8a^2(4a^2-3)(2a^2-1)(4a^2-1). \tag{3.6}$$

This equality forces $sqf(b)$ to be even. Under the hypotheses of Theorem 1.1, $sqf(b)$ can have at most one odd prime divisor. This in turn requires at least two of $2a^2-1$, $4a^2-3$, $4a^2-1$ to be perfect squares. Since none of the last two expressions is square for $a\geq 2$, we conclude that this subcase cannot appear.

Subcase II.3: *$V_{2r+1}=ac^2$ for some positive integer c.* If this were true, write $a=a_1a_2^2$, $a^2-1=d_1d_2^2$, with a_1 and d_1 square-free positive integers. We see that the equation $a_1^2S^4-d_1T^2=1$ is solved by (a_2c,d_2U_{2r+1}) as well as by (a_2,d_2). Invoking the uniqueness asserted in Lemma 2.6, we conclude that one necessarily has $U_{2r+1}=1$, in contradiction with $r\geq 1$.

Case $(k=4r-1,\ r\geq 1)$. Now it holds

$$bz^2=4a^2V_{2r}\left(\frac{U_{2r}}{a}\right)\left(\frac{V_{2r-1}}{a}\right)U_{2r-1}$$

and again from Lemma 2.3 it follows that at least two factors in the right side are perfect squares.

Subcase III.1: *U_{2r-1} is a square.* By Lemma 2.4, this can happen only for $r=1$. Hence,

$$bz^2=8a^2(2a^2-1).$$

This is impossible for odd $sqf(b)$. For $sqf(b)=2p$ we obtain the solution

$$(x,y,z)=\left(4a^3-3a,4a^2-1,\sqrt{8a^2(2a^2-1)/b}\right)$$

for the system (1.2) under the condition that b divides $8a^2(2a^2-1)$ and $(2a^2-1)/p$ is a perfect square.

Note that if $sqf(b)=2p$ for some prime $p\equiv\pm 3 \pmod 8$, an equality of the form $2a^2-1=pc^2$ is impossible in integers.

Subcase III.2: *V_{2r} is a square.* Then $(\sqrt{V_{2r}},U_{2r})$ is a solution to the equation $S^4-(a^2-1)T^2=1$. As in Subcase II.2, from Lemma 2.7 it results that V_{2r} is either $a=V_1$ or $2a^2-1=V_2$. It follows that the only possibility is $r=1$. The arguments employed in the previous subcase lead to the desired conclusion.

Subcase III.3: *$V_{2r-1}=ac^2$ for some positive integer c.* Writing as before $a=a_1a_2^2$, $a^2-1=d_1d_2^2$, with a_1 and d_1 square-free positive integers, one sees that the equation $a_1^2S^4 - d_1T^2 = 1$ is solved by (a_2c, d_2U_{2r-1}) as well as by (a_2, d_2). From Lemma 2.6 it results that $r = 1$, and the reasoning continues with the same arguments and leads to the same conclusion as in the previous subcases.

Finally, let us examine the remaining possibility.

Subcase III.4: $U_{2r} = ac^2$ *for some positive integer* c. Then (V_{2r}, c) is a solution to the equation $S^2 - a^2(a^2-1)T^4 = 1$. As already argued in Subcase II.1, this entails $c^2 = 4(2a^2-1)$, so that $U_{2r} = 4a(2a^2-1) = U_4$. Hence, $r = 2$ and

$$bz^2 = 16a^2(8a^4 - 8a^2 + 1)(2a^2-1)(4a^2-3)(4a^2-1).$$

As seen in the previous case, such an equality is prohibited by the condition $\omega(sqf(b)) \le 2$.

The proof of Theorem 1.1 is complete.

4. Extensions

As Lemma 2.1 gives 'explicitly' the solutions to the first equation in (1.2), the solution of the system (when it exists) is of the form

$$(x, y, z) = \left(V_k, U_k, \sqrt{(U_k^2 - 1)/b}\right) \tag{4.1}$$

for some index $k \ge 2$. Such a formula is not completely satisfactory without an indication on how large could k be. The crux of Theorem 1.1 is to find a condition enforcing $k = 2$ or $k = 3$.

A short computer search provided instances of (1.2) whose solutions are given by (4.1) for some $k \ge 4$. For example, one has $k = 4$ for $(a, b) = (5, 11 \cdot 89 \cdot 109 \cdot 3^2)$ or $(a, b) = (113, 61 \cdot 227 \cdot 50849 \cdot 29^2)$, and $k = 5$ if $(a, b) = (5, 2^3 \cdot 11 \cdot 97 \cdot 7^2)$. One can have $k = 6$ for $(a, b) = (3, 29 \cdot 41 \cdot 239)$, and even $k = 7$ for $(a, b) = (5, 11 \cdot 97 \cdot 4801 \cdot 2^4)$.

These examples show that for $\omega(sqf(b)) \ge 3$ it is impossible to prove results as precise as Theorem 1.1 without some additional requirements on a and b. Such supplementary constraints are uncovered by examining the arguments given in Section 3 and understanding what changes when only one of the four known factors of $U_k^2 - 1$ is a perfect square. The next result points out several conditions that are sufficient to establish an analogue of Theorem 1.1 for systems (1.2) in which $sqf(b)$ is the product of three prime numbers.

Theorem 4.1. *Let a and b be integers greater than* 1. *Assume* $\omega(sqf(b)) = 3$.

a) If $sqf(b)$ is odd, suppose that one of the following conditions holds:

(α) *a is twice an odd number,*

(β) $a = 2c^2 - 1$ *for some integer c,*

(γ) $a \equiv 2 \pmod 6$,

(δ) $a \equiv 0 \pmod 4$ *and* $sqf(b) \not\equiv 7 \pmod 8$.

Then the system (1.2) *is solvable in positive integers if and only if b divides* $4a^2 - 1$ *and the quotient is a perfect square. When it exists, this solution is*

$$(x, y, z) = \left(2a^2 - 1, 2a, \sqrt{(4a^2-1)/b}\right).$$

b) If $sqf(b) = 2pq$ with $2 < p < q$, suppose that one of the following conditions holds:

(ε) *there is no integer c such that* $2a^2 - 1 = c^2$,

(ζ) $\{p \mod 8, q \mod 8\} \ne \{1, 3\}$.

Then the system (1.2) *is solvable in positive integers if and only if* $8a^2(2a^2-1)$ *is divisible by* b *and the quotient is a perfect square. When it exists, this solution is*

$$(x, y, z) = \left(4a^3 - 3a, 4a^2 - 1, \sqrt{8a^2(2a^2-1)/b}\right).$$

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References

[1] Xiaochuan Ai, Jianhua Chen, Silan Zhang, and Hao Hu, *Complete solutions of the simultaneous Pell equations* $x^2-24y^2=1$ *and* $y^2-pz^2=1$, J. Number Theory **147** (2015), 103–108, DOI 10.1016/j.jnt.2014.07.009. MR3276317

[2] S. Akhtari, A. Togbé, and P. G. Walsh, *Addendum on the equation* $aX^4-bY^2=2$ *[MR2388048]*, Acta Arith. **137** (2009), no. 3, 199–202, DOI 10.4064/aa137-3-1. MR2496459

[3] Michael A. Bennett and Ákos Pintér, *Intersections of recurrence sequences*, Proc. Amer. Math. Soc. **143** (2015), no. 6, 2347–2353, DOI 10.1090/S0002-9939-2015-12499-9. MR3326017

[4] Michael A. Bennett and Gary Walsh, *The Diophantine equation* $b^2X^4-dY^2=1$, Proc. Amer. Math. Soc. **127** (1999), no. 12, 3481–3491, DOI 10.1090/S0002-9939-99-05041-8. MR1625772

[5] Michael A. Bennett, Mihai Cipu, Maurice Mignotte, and Ryotaro Okazaki, *On the number of solutions of simultaneous Pell equations. II*, Acta Arith. **122** (2006), no. 4, 407–417, DOI 10.4064/aa122-4-4. MR2234424

[6] Yann Bugeaud, Claude Levesque, and Michel Waldschmidt, *Équations de Fermat–Pell–Mahler simultanées* (French, with English and French summaries), Publ. Math. Debrecen **79** (2011), no. 3-4, 357–366, DOI 10.5486/PMD.2011.5192. MR2907971

[7] J. H. E. Cohn, *The Diophantine equation* $x^4-Dy^2=1$. *II*, Acta Arith. **78** (1997), no. 4, 401–403. MR1438594

[8] Mihai Cipu, *Pairs of Pell equations having at most one common solution in positive integers*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **15** (2007), no. 1, 55–66. MR2396744

[9] Mihai Cipu and Maurice Mignotte, *On the number of solutions to systems of Pell equations*, J. Number Theory **125** (2007), no. 2, 356–392, DOI 10.1016/j.jnt.2006.09.016. MR2332594

[10] Nurettin Irmak, *On solutions of the simultaneous Pell equations* $x^2-(a^2-1)y^2=1$ *and* $y^2-pz^2=1$, Period. Math. Hungar. **73** (2016), no. 1, 130–136, DOI 10.1007/s10998-016-0137-0. MR3529822

[11] Pingzhi Yuan and Yuan Li, *Squares in Lehmer sequences and the Diophantine equation* $Ax^4-By^2=2$, Acta Arith. **139** (2009), no. 3, 275–302, DOI 10.4064/aa139-3-6. MR2545931

[12] Wilhelm Ljunggren, *Zur Theorie der Gleichung* $x^2+1=Dy^4$ (German), Avh. Norske Vid. Akad. Oslo. I. **1942** (1942), no. 5, 27. MR0016375

[13] Wilhelm Ljunggren, *Ein Satz über die diophantische Gleichung* $Ax^2-By^4=C$ $(C=1,2,4)$ (German), Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, Lunds Universitets Matematiska Inst., Lund, 1954, pp. 188–194. MR0065575

[14] Florian Luca and P. G. Walsh, *Squares in Lehmer sequences and some Diophantine applications*, Acta Arith. **100** (2001), no. 1, 47–62, DOI 10.4064/aa100-1-4. MR1864625

[15] D. W. Masser and J. H. Rickert, *Simultaneous Pell equations*, J. Number Theory **61** (1996), no. 1, 52–66, DOI 10.1006/jnth.1996.0137. MR1418319

[16] Maurice Mignotte and Attila Pethő, *Sur les carrés dans certaines suites de Lucas* (French, with English and French summaries), J. Théor. Nombres Bordeaux **5** (1993), no. 2, 333–341. MR1265909

[17] Axel Thue, *Über Annäherungswerte algebraischer Zahlen* (German), J. Reine Angew. Math. **135** (1909), 284–305, DOI 10.1515/crll.1909.135.284. MR1580770

[18] A. Togbe, P. M. Voutier, and P. G. Walsh, *Solving a family of Thue equations with an application to the equation* $x^2-Dy^4=1$, Acta Arith. **120** (2005), no. 1, 39–58, DOI 10.4064/aa120-1-3. MR2189717

[19] P. M. Voutier, *A further note on "On the equation $aX^4 - bY^2 = 2$" [MR2388048; MR2496459]*, Acta Arith. **137** (2009), no. 3, 203–206, DOI 10.4064/aa137-3-2. MR2496460
[20] Pingzhi Yuan, *On the number of solutions of $x^2 - 4m(m+1)y^2 = y^2 - bz^2 = 1$*, Proc. Amer. Math. Soc. **132** (2004), no. 6, 1561–1566, DOI 10.1090/S0002-9939-04-07418-0. MR2051114

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