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TRIPLY IMPRIMITIVE REPRESENTATIONS OF GL(2)

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Abstract. We give a criterion for an irreducible, admissible, supercuspidal representation π of $\mathrm{GL}(2,K)$, where K is a p-adic field, to become a principal series representation under every quadratic base change. We determine all such π that have trivial central character and conductor 2, and explain their relevance for the theory of elliptic curves.

1. Introduction

Let K be a non-archimedean local field of characteristic zero. Let π be an irreducible, admissible, supercuspidal representation of $\mathrm{GL}(2,K)$. For a quadratic field extension L/K we denote by $\mathrm{BC}_{L/K}(\pi)$ the base change of π to L, which is an irreducible, admissible representation of $\mathrm{GL}(2,L)$; see [2] for basic properties of base change. The representation $\mathrm{BC}_{L/K}(\pi)$ may remain supercuspidal, or may be a principal series representation. In this note we investigate the following question:
(1)
Is it possible that $\mathrm{BC}_{L/K}(\pi)$ is a principal series representation for *all* quadratic extensions L ?

We reformulate this question in terms of the local parameters corresponding to the representations involved via the local Langlands correspondence (see [5] for basic properties of this correspondence). Since π is supercuspidal, its parameter is an irreducible, 2-dimensional representation (φ, V) of the Weil group $W(\bar{K}/K)$,

$$\varphi : W(\bar{K}/K) \longrightarrow \mathrm{GL}(2,V) \cong \mathrm{GL}(2,\mathbb{C}).$$

Quadratic base change corresponds to restricting φ to subgroups of index-2; such subgroups are precisely the Weil groups $W(\bar{K}/L)$ where L/K is a quadratic field extension. The restriction of φ to $W(\bar{K}/L)$ remains irreducible exactly if $\mathrm{BC}_{L/K}(\pi)$ is supercuspidal. The above question is therefore equivalent to the following:
(2)
Is it possible that $\mathrm{res}_H^{W(\bar{K}/K)}(\varphi)$ is reducible for *all* index-2 subgroups H of $W(\bar{K}/K)$?

It follows from the representation theory of $W(\bar{K}/K)$ that if $\mathrm{res}_H^{W(\bar{K}/K)}(\varphi)$ is reducible, then it is a direct sum of two 1-dimensional representations. Via the local Langlands correspondence, this direct sum corresponds to a principal series representation of $\mathrm{GL}(2,K)$.

We will show that the answer to question (1) is "no" if the residual characteristic of K is even. Assume that the residual characteristic of K is odd. For reasons to be

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explained, we call a supercuspidal π *triply imprimitive* if $\mathrm{BC}_{L/K}(\pi)$ is a principal series representation for *all* quadratic extensions L of K. In odd residual characteristic it is known that every supercuspidal π is dihedral, i.e., can be constructed via the Weil representation, as in §1 of [4]. The input for this construction is a quadratic field extension F of K and a non-Galois invariant character ξ of $F^\times$; let $\omega_{F,\xi}$ be the supercuspidal representation of $\mathrm{GL}(2,K)$ attached to this data. Then we will prove that $\omega_{F,\xi}$ is triply imprimitive if and only if ξ^2 is Galois-invariant; see Corollary 3.2.

Next we consider those supercuspidal π which have trivial central character and (exponent of the) conductor 2. Under the assumption that the residual characteristic is not 2 or 3, only such supercuspidals are relevant for the theory of elliptic curves. Our main result is Theorem 4.1 below. It states that if $q \equiv 1 \bmod 4$, then there is no triply imprimitive such π, and if $q \equiv 3 \bmod 4$, then there is a unique one; here, q is the cardinality of the residue class field.

In the final section we explain how one can easily determine from the Weierstrass equation of an elliptic curve E over K whether the associated irreducible, admissible representation of $\mathrm{GL}(2,K)$ is the triply imprimitive supercuspidal exhibited in Theorem 4.1.

2. Restricting representations to index-2 subgroups

Let G be a group, and H an index-2 subgroup. All representations of these groups are assumed to be finite-dimensional and complex. By a *character* we mean a 1-dimensional representation. We fix an element $\sigma \in G$ which is not in H, so that $G = H \sqcup \sigma H$. If ξ is a representation of H, then the *conjugate representation* ξ^σ is defined by $\xi^\sigma(h) = \xi(\sigma h \sigma^{-1})$. We denote by res^G_H and ind^G_H the restriction and induction functors. The following two lemmas are well known.

Lemma 2.1. *Let G be a group, and H an index-2 subgroup. Let χ be the unique non-trivial character of G/H. Let φ be an irreducible representation of G. Then exactly one of the following two alternatives occurs:*

(1) *$\varphi \not\cong \varphi \otimes \chi$ and $\mathrm{res}^G_H(\varphi)$ is irreducible. In this case*

$$\mathrm{ind}^G_H(\mathrm{res}^G_H(\varphi)) = \varphi \oplus (\varphi \otimes \chi).$$

(2) *$\varphi \cong \varphi \otimes \chi$ and $\mathrm{res}^G_H(\varphi) = \xi \oplus \xi^\sigma$, where ξ is an irreducible representation of H. In this case $\xi \not\cong \xi^\sigma$, and*

$$\varphi = \mathrm{ind}^G_H(\xi) = \mathrm{ind}^G_H(\xi^\sigma).$$

Lemma 2.2. *Let G be a group, and H an index-2 subgroup.*

(1) *Let ξ be a representation of H and μ a character of G. Then*

$$\mathrm{ind}^G_H(\xi) \otimes \mu \cong \mathrm{ind}^G_H(\xi \otimes \mathrm{res}^G_H(\mu)). \tag{3}$$

(2) *Let ξ_1 and ξ_2 be representations of H. Then*

$$\mathrm{ind}^G_H(\xi_1) \cong \mathrm{ind}^G_H(\xi_2) \qquad \Longleftrightarrow \qquad \big(\xi_1 \cong \xi_2 \text{ or } \xi_1 \cong \xi_2^\sigma\big). \tag{4}$$

We can now prove the following result about the restriction of 2-dimensional representations to index-2 subgroups. It is closely related to the arguments in Sect. 6 of [7].

Proposition 2.3. *Let G be a group with more than one index-2 subgroup.*

(1) *Assume that there exists an irreducible 2-dimensional representation φ of G such that $\operatorname{res}_H^G(\varphi)$ is reducible for all index-2 subgroups H. Then G has exactly three index-2 subgroups.*

(2) *Assume that G has exactly three index-2 subgroups H_1, H_2, H_3. Let ξ be a character of H_1 with $\xi \neq \xi^\sigma$; here, σ is an element of G that is not in H_1. Let $\varphi = \operatorname{ind}_{H_1}^G(\xi)$. Then*

$$\operatorname{res}_{H_i}^G(\varphi) \textit{ is reducible for } i = 1, 2, 3 \iff (\xi^2)^\sigma = \xi^2. \tag{5}$$

Proof. i) Let φ be an irreducible representation of G such that $\operatorname{res}_{H_1}^G(\varphi)$ is reducible for *some* index-2 subgroup H_1. By Lemma 2.1, there exists an irreducible representation ξ of H_1 such that $\operatorname{res}_{H_1}^G(\varphi) = \xi \oplus \xi^\sigma$. We have $\xi \ncong \xi^\sigma$ and $\varphi = \operatorname{ind}_{H_1}^G(\xi)$.

Let $\{H_i\}$ be the set of index-2 subgroups of G. Let χ_i be the non-trivial character of G that is trivial on H_i. Let σ be an element of G that is not in H_1. By Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
\operatorname{res}_{H_i}^G(\varphi) \text{ is reducible } &\iff \varphi \cong \varphi \otimes \chi_i \\
&\iff \operatorname{ind}_{H_1}^G(\xi) \cong \operatorname{ind}_{H_1}^G(\xi) \otimes \chi_i \\
&\iff \operatorname{ind}_{H_1}^G(\xi) \cong \operatorname{ind}_{H_1}^G\big(\xi \otimes \operatorname{res}_{H_1}^G(\chi_i)\big) \\
&\iff \Big(\xi \cong \xi \otimes \operatorname{res}_{H_1}^G(\chi_i) \text{ or } \xi^\sigma \cong \xi \otimes \operatorname{res}_{H_1}^G(\chi_i)\Big).
\end{aligned}$$

Assume now that $\dim(\varphi) = 2$, so that ξ is a character. Then $\xi \cong \xi \otimes \operatorname{res}_{H_1}^G(\chi_i)$ if and only if $\operatorname{res}_{H_1}^G(\chi_i) = 1$. But if $i \neq 1$, then χ_i cannot be trivial on H_1, since its kernel is H_i. Hence, for $i \neq 1$,

$$\begin{aligned}
\operatorname{res}_{H_i}^G(\varphi) \text{ is reducible } &\iff \xi^\sigma \cong \xi \otimes \operatorname{res}_{H_1}^G(\chi_i) \\
&\iff \xi^\sigma = \xi \cdot \operatorname{res}_{H_1}^G(\chi_i).
\end{aligned}$$

Assume this condition is satisfied for $i, j \neq 1$ with $i \neq j$. Then $\operatorname{res}_{H_1}^G(\chi_i) = \operatorname{res}_{H_1}^G(\chi_j)$. Hence $\chi_i\chi_j$ is a non-trivial quadratic character which is trivial on H_1. We conclude that $\chi_i\chi_j = \chi_1$. It follows that if $\operatorname{res}_{H_i}^G(\varphi)$ is reducible for *all* i, then there cannot be more than three index-2 subgroups. Note that we cannot have exactly two index-2 subgroups, since if χ_1 and χ_2 are two distinct quadratic characters of G, then $\chi_1\chi_2$ is a third such character. Hence there are exactly three index-2 subgroups.

ii) Let the notation be as in the first part of the proof. As we saw, $\chi_2\chi_3 = \chi_1$, and hence $\operatorname{res}_{H_1}^G(\chi_2) = \operatorname{res}_{H_1}^G(\chi_3)$. Let this common restriction be denoted by α. The kernel of α is $H_1 \cap H_2 = H_1 \cap H_3$, which is an index-2 subgroup of H_1. From above, we see that

$$\operatorname{res}_{H_i}^G(\varphi) \text{ is reducible for } i = 1, 2, 3 \iff \xi^\sigma = \xi \cdot \alpha. \tag{6}$$

In particular, if $\operatorname{res}_{H_i}^G(\varphi)$ is reducible for $i = 1, 2, 3$, then $(\xi^2)^\sigma = \xi^2$.

It remains to prove that if ξ is a character of H_1 with $\xi \neq \xi^\sigma$ and $(\xi^2)^\sigma = \xi^2$, then $\operatorname{res}_{H_i}^G(\varphi)$ is reducible for $i = 1, 2, 3$. Let $M \subset H_1$ be the kernel of ξ/ξ^σ. Since $(\xi/\xi^\sigma)^2 = 1$ by hypothesis, M is an index-2 subgroup of H_1. We claim that σ normalizes M. Indeed, for $m \in M$,

$$\Big(\frac{\xi}{\xi^\sigma}\Big)(\sigma m \sigma^{-1}) = \frac{\xi(\sigma m \sigma^{-1})}{\xi^\sigma(\sigma m \sigma^{-1})} = \frac{\xi(\sigma m \sigma^{-1})}{\xi(m)} = 1.$$

Since also $\sigma^2 \in M$, it follows that $M \sqcup \sigma M$ is an index-2 subgroup of G, say $M \sqcup \sigma M = H_2$. Evidently, $M = H_1 \cap H_2$. It follows that ξ/ξ^σ equals the character α appearing in (6). This concludes the proof. □

Recall that a representation φ of a group G is called *primitive* if it is not induced (from any subgroup), otherwise *imprimitive.* If the representation φ in ii) of Proposition 2.3 satisfies (5), then, by ii) of Lemma 2.1, φ is induced from any of the H_i. We will call such φ *triply imprimitive.* A similar terminology is used in [3].

3. Application to Weil groups

In this and the following sections, let K be a non-archimedean local field of characteristic zero, $\mathfrak{o}$ its ring of integers, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$, and ϖ a generator of $\mathfrak{p}$. Let q be the cardinality of the residue class field $\mathfrak{o}/\mathfrak{p}$. If L is an extension field, we denote the corresponding objects for L by $\mathfrak{o}_L$, etc. Let $K^{\rm unr}$ be the maximal unramified extension of K in $\bar{K}$.

Let $W(\bar{K}/K)$ be the Weil group of K; we refer to [10] for background. By definition,

$$W(\bar{K}/K) = \bigsqcup_{n\in\mathbb{Z}} \Phi^n I, \tag{7}$$

where $I = \mathrm{Gal}(\bar{K}/K^{\rm unr})$ is the inertia subgroup, and Φ is an *inverse* Frobenius element in $\mathrm{Gal}(\bar{K}/K)$. Inverse means that Φ induces the inverse of the map $x \mapsto x^q$ on the algebraic closure of the residue class field $\mathfrak{o}/\mathfrak{p}$. There is a topology on $W(\bar{K}/K)$ making it into a topological group, such that I is an open subset, and such that the induced topology on I coincides with the induced topology on I as a subset of $\mathrm{Gal}(\bar{K}/K)$.

Representations φ of $W(\bar{K}/K)$ are always assumed to be complex, finite-dimensional and continuous. Observe that restriction to and induction from finite index subgroups respect the continuity of a representation. We will apply Proposition 2.3 to $W(\bar{K}/K)$.

Note that the quadratic field extensions L of K correspond to the index-2 subgroups $W(\bar{K}/L)$ of $W(\bar{K}/K)$. If φ is a representation of $W(\bar{K}/K)$, we will abbreviate

$$\mathrm{res}_{L/K}(\varphi) := \mathrm{res}^{W(\bar{K}/K)}_{W(\bar{K}/L)}(\varphi),$$

and if ξ is a representation of $W(\bar{K}/L)$, we will abbreviate

$$\mathrm{ind}_{L/K}(\xi) := \mathrm{ind}^{W(\bar{K}/K)}_{W(\bar{K}/L)}(\xi).$$

The quadratic field extensions of K correspond to the non-trivial elements of the group $K^\times/K^{\times 2}$. It thus follows from Proposition II.5.7 of [6] that there are three quadratic field extensions L/K if the residual characteristic of K is odd, and more than three otherwise. From Proposition 2.3 we therefore obtain the following result.

Proposition 3.1. *Let $L_1, \ldots, L_r$ be the quadratic field extensions of K.*

(1) *Assume that there exists an irreducible 2-dimensional representation φ of $W(\bar{K}/K)$ such that $\mathrm{res}_{L_i/K}(\varphi)$ is reducible for $i = 1, \ldots, r$. Then the residual characteristic of K is odd.*

(2) *Assume that the residual characteristic of K is odd, so that $r = 3$. Let ξ be a character of $W(\bar{K}/L_1)$ with $\xi \neq \xi^\sigma$; here, σ is an element of $W(\bar{K}/K)$ that is not in $W(\bar{K}/L_1)$. Let $\varphi = \mathrm{ind}_{L_1/K}(\xi)$. Then*

$$\mathrm{res}_{L_i/K}(\varphi) \text{ is reducible for } i = 1, 2, 3 \iff (\xi^2)^\sigma = \xi^2. \tag{8}$$

Let F/K be a quadratic extension. Recall that characters of $W(\bar{K}/F)$ correspond to characters of $F^\times$ via local class field theory (this is also the local Langlands correspondence for GL(1)). We will denote both kinds of characters by the symbol ξ. Given such a $\xi : F^\times \to \mathbb{C}^\times$, there is an irreducible, admissible representation $\omega_{F,\xi}$ of $\mathrm{GL}(2, K)$ constructed via the Weil representation; see §1 of [4]. We refer to $\omega_{F,\xi}$ as a *dihedral* or a *monomial* representation. By Theorem 4.6 of [4], $\omega_{F,\xi}$ is supercuspidal if and only if ξ is not Galois invariant. In this case the representation of $W(\bar{K}/K)$ corresponding to $\omega_{F,\xi}$ via the local Langlands correspondence is nothing but $\mathrm{ind}_{F/K}(\xi)$.

If π is an irreducible, admissible, supercuspidal representation of $\mathrm{GL}(2, K)$ with corresponding 2-dimensional representation φ of $W(\bar{K}/K)$, then the base change $\mathrm{BC}_{L/K}(\pi)$ to a quadratic extension L of K corresponds to $\mathrm{res}_{L/K}(\varphi)$ (this is true for all irreducible, admissible π if one works with the Weil-Deligne group instead of the Weil group). Keeping these facts in mind, we may reformulate Proposition 3.1 as follows.

Corollary 3.2. *Let π be an irreducible, admissible, supercuspidal representation of* $\mathrm{GL}(2, K)$.

(1) *Assume that $\mathrm{BC}_{L/K}(\pi)$ is a principal series representation for all quadratic extensions L of K. Then the residual characteristic of K is odd.*
(2) *Assume that the residual characteristic of K is odd, so that π is a dihedral supercuspidal. Write $\pi = \omega_{F,\xi}$, where F/K is a quadratic extension and ξ is a non-Galois invariant character of $F^\times$. Then $\mathrm{BC}_{L/K}(\pi)$ is a principal series representation for all quadratic extensions L of K if and only if ξ^2 is Galois invariant.*

4. The case of conductor 2

Let π be an irreducible, admissible representation of $\mathrm{GL}(2, K)$. By definition, the conductor $a(\pi)$ is the smallest non-negative integer n such that π admits a non-zero vector invariant under the congruence subgroup

$$\mathrm{GL}(2, \mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & 1 + \mathfrak{p}^n \end{bmatrix}.$$

This number coincides with the conductor of the corresponding Weil-Deligne representation, as defined in §10 of [10]; for our purposes, we may take this as an alternative definition of $a(\pi)$. In this section we consider triply imprimitive supercuspidals with conductor 2 and trivial central character.

Again let F/K be a quadratic extension. Let σ be the non-trivial Galois automorphism of this extension. Let ξ be a character of $F^\times$ with $\xi \neq \xi^\sigma$, where $\xi^\sigma(x) = \xi(\sigma(x))$. Let $\omega_{F,\xi}$ be the corresponding dihedral supercuspidal. By §10 of [10], we have the conductor formula

$$a(\omega_{F,\xi}) = d(F/K) + f(F/K)a(\xi). \tag{9}$$

Here, $d(F/K)$ is the valuation of the discriminant of F/K and $f(F/K)$ is the residue class degree. The number $f(F/K)$ is 1 or 2, depending on whether F/K is ramified or unramified. Assume that the residual characteristic of K is odd. Then the number $d(F/K)$ is 0 or 1, again depending on whether F/K is ramified or unramified. Hence,

$$a(\omega_{F,\xi}) = \begin{cases} 2a(\xi) & \text{if } F/K \text{ is unramified,} \\ 1 + a(\xi) & \text{if } F/K \text{ is ramified.} \end{cases} \tag{10}$$

We are especially interested in the case of conductor 2, since this case is relevant for elliptic curves. From above, we see that

$$a(\omega_{F,\xi}) = 2 \quad \Longleftrightarrow \quad a(\xi) = 1.$$

Such ξ are *tamely ramified*, meaning their restriction to the unit group $\mathfrak{o}_F^\times$ is non-trivial, but further restriction to $1 + \mathfrak{p}_F$ is trivial. Hence, such ξ descend to a character of the multiplicative group of the residue class field $\mathfrak{o}_F/\mathfrak{p}_F$. Conversely, given $\xi : (\mathfrak{o}_F/\mathfrak{p}_F)^\times \to \mathbb{C}^\times$, we can inflate ξ to a character of $\mathfrak{o}_F^\times$, give it some value on a uniformizer ϖ_F, and thus obtain a tamely ramified character of $F^\times$.

In the following we continue to assume that the residual characteristic of K is odd and look for characters ξ of $F^\times$ satisfying the following conditions:

(11)
- (A) $\xi^\sigma \neq \xi$.
- (B) $\xi\big|_{K^\times} = \chi_{F/K}$.
- (C) $a(\xi) = 1$.
- (D) $(\xi^2)^\sigma = \xi^2$.

Condition (A) assures that $\pi := \omega_{F,\xi}$ is supercuspidal. Condition (B) is equivalent to π having trivial central character. Condition (C) is equivalent to π having conductor 2. Finally, by Proposition 3.1 ii), condition (D) means that $\mathrm{BC}_{L/K}(\pi)$ is a principal series representation for *all* quadratic field extensions L of K.

The unramified case. Assume first that F/K is the *unramified* quadratic extension of K. Then the residue class field $\mathfrak{o}_F/\mathfrak{p}_F$ is a quadratic extension of $\mathfrak{o}/\mathfrak{p}$. Assume ξ has the properties (A) – (D) in (11). By (C), ξ determines a character $\bar\xi$ of $(\mathfrak{o}_F/\mathfrak{p}_F)^\times$ with the following properties:

(12)
- ($\bar{\mathrm{A}}$) $\bar\xi^{\bar\sigma} \neq \bar\xi$.
- ($\bar{\mathrm{B}}$) The restriction of $\bar\xi$ to $(\mathfrak{o}/\mathfrak{p})^\times$ is trivial.
- ($\bar{\mathrm{D}}$) $(\bar\xi^2)^{\bar\sigma} = \bar\xi^2$.

Here, $\bar\sigma$ is the non-trivial Galois automorphism of the residue class field extension. Explicitly, $\bar\sigma$ is the Frobenius, given by $\bar\sigma(x) = x^q$.

Let g be a generator of the cyclic group $(\mathfrak{o}_F/\mathfrak{p}_F)^\times$. The order of g is $q^2 - 1$. Any character $\bar\xi$ of $(\mathfrak{o}_F/\mathfrak{p}_F)^\times$ is determined by its value on g, and this value can be any $(q^2 - 1)$-th root of unity:

$$\bar\xi(g) = e^{2\pi i \frac{k}{q^2-1}}, \qquad k = 1, 2, \ldots, q^2 - 1.$$

The conditions (12) are then equivalent to the following:

(13)
- ($\bar{\mathrm{A}}$) $k \notin (q+1)\mathbb{Z}$.
- ($\bar{\mathrm{B}}$) $k \in (q-1)\mathbb{Z}$.
- ($\bar{\mathrm{D}}$) $2k \in (q+1)\mathbb{Z}$.

Conditions $\bar{\mathrm{A}}$ and $\bar{\mathrm{D}}$ imply that

$$k = \frac{q+1}{2}(1+2m), \qquad m \in \{0, 1, \dots, q-2\}.$$

Assume that $\bar{\mathrm{B}}$ is also satisfied, i.e.,

$$\frac{q+1}{2}(1+2m) = (q-1)n$$

for some integer n. If $q \equiv 1 \bmod 4$, then the left side is odd and the right side is even, so this is impossible. Assume that $q \equiv 3 \bmod 4$. Since the integers $\frac{q+1}{4}$ and $\frac{q-1}{2}$ are relatively prime, it follows that $1+2m = j\frac{q-1}{2}$ for some $j \in \mathbb{Z}$. For reasons of size we must have $j \in \{1, 2, 3\}$. Also, j must be odd, so the only possibilities are $j = 1$ and $j = 3$. Hence the only possibilities for k are $k = \frac{q^2-1}{4}$ and $k = 3\,\frac{q^2-1}{4}$. Note that

$$q\,\frac{q^2-1}{4} \equiv 3\,\frac{q^2-1}{4} \mod q^2 - 1$$

due to our hypothesis $q \equiv 3 \bmod 4$, so that the two possible values of k lead to Galois-conjugate characters $\bar{\xi}$. We might as well fix $k = \frac{q^2-1}{4}$. Our character $\bar{\xi}$ is then given by

$$\bar{\xi}(g) = e^{2\pi i/4} = i \tag{14}$$

(its Galois-conjugate would have $g \mapsto -i$).

Conversely, assuming that $q \equiv 3 \bmod 4$, we can *define* $\bar{\xi}$ by (14). Let ξ be the inflation of $\bar{\xi}$ to $\mathfrak{o}_F^\times$. To obtain a character of $F^\times$, we also need to define the value $\xi(\varpi_F)$, where ϖ_F is a uniformizer in F. Since F/K is unramified, we can take $\varpi_F = \varpi$, where ϖ is a uniformizer in K. Condition (B) in (11) then forces us to define $\xi(\varpi) = -1$. Having defined ξ in this way, we see that all the conditions in (11) are satisfied.

The ramified case. Now assume that F/K is a *ramified* quadratic extension of K (there are two such extensions). In this case $\mathfrak{o}_F/\mathfrak{p}_F = \mathfrak{o}/\mathfrak{p}$. Assume that ξ satisfies the conditions in (11). Since $\mathfrak{o}_F^\times = \mathfrak{o}^\times(1+\mathfrak{p}_F)$, the restriction of ξ to $\mathfrak{o}_F^\times$ is determined by $\xi|_{\mathfrak{o}^\times}$. In view of (B), ξ is completely determined on $\mathfrak{o}_F^\times$. We also see that $\xi = \xi^\sigma$ on $\mathfrak{o}_F^\times$.

Choose the uniformizer ϖ of K such that $F = K(\sqrt{\varpi})$. Then $\sigma(\sqrt{\varpi}) = -\sqrt{\varpi}$, and hence

$$\xi^\sigma(\sqrt{\varpi}) = \xi(-\sqrt{\varpi}) = \chi_{F/K}(-1)\xi(\sqrt{\varpi}).$$

In order to satisfy (A), we must have $\chi_{F/K}(-1) = -1$; this holds exactly if $q \equiv 3 \bmod 4$. Assume this is the case, so that $\xi^\sigma(\sqrt{\varpi}) = -\xi(\sqrt{\varpi})$. We have

$$\xi(\sqrt{\varpi})^2 = \xi(\varpi) = \chi_{F/K}(\varpi) = \chi_{F/K}(-1)\chi_{F/K}(-\varpi) = \chi_{F/K}(-1) = -1,$$

since $-\varpi$ is a norm. It follows that $\xi(\sqrt{\varpi}) = \pm i$, and up to Galois conjugation we may assume $\xi(\sqrt{\varpi}) = i$. We proved that ξ is unique up to Galois conjugation. Conversely, we see how to construct a character ξ with the properties (A) – (D) provided that $q \equiv 3 \bmod 4$.

Summary. Assume that the residual characteristic of K is odd. Recall that an irreducible, admissible, supercuspidal representation π of $\mathrm{GL}(2,K)$ is called triply imprimitive if $\mathrm{BC}_{L/K}(\pi)$ is a principal series representation for every quadratic field extension L of K. The following theorem summarizes the results of this section.

Theorem 4.1. *Assume that the residual characteristic of K is odd. Consider irreducible, admissible, supercuspidal representations π of $\mathrm{GL}(2,K)$ with the following properties:*

- *π has trivial central character.*
- *$a(\pi)=2$.*
- *π is triply imprimitive.*

If $q\equiv 1 \bmod 4$, then there exists no such representation. If $q\equiv 3 \bmod 4$, then there exists a unique such representation π, given in any one of the following two ways:

(1) *Let F/K be the unramified quadratic extension. Let g be a generator of $(\mathfrak{o}_F/\mathfrak{p}_F)^\times$, and define a character $\bar\xi$ of this group by $\bar\xi(g)=i$. Inflate $\bar\xi$ to a character ξ of $\mathfrak{o}_F^\times$, and extend ξ to a character of $F^\times$ by setting $\xi(\varpi)=-1$. Then $\pi=\omega_{F,\xi}$.*

(2) *Let F/K be a ramified quadratic extension. Let ξ be the character of $\mathfrak{o}_F^\times=\mathfrak{o}^\times(1+\mathfrak{p}_F)$ that is trivial on $1+\mathfrak{p}_F$ and coincides with $\chi_{F/K}$ on $\mathfrak{o}^\times$. Extend ξ to a character of $F^\times$ by setting $\xi(\sqrt{\varpi})=i$; here, ϖ is a uniformizer of K such that $F=K(\sqrt{\varpi})$. Then $\pi=\omega_{F,\xi}$.*

We remark that the image of the representation $W(\bar K/K)\to\mathrm{GL}(2,\mathbb{C})$ corresponding to π as in Theorem 4.1 is the quaternion group Q. Our result is thus compatible with the fact that K admits a unique Galois extension E with Galois group $G(E/K)\cong Q$ if $q\equiv 3 \bmod 4$, and no such extension if $q\equiv 1 \bmod 4$. We would like to thank David Roberts for pointing this out.

5. The relevance for elliptic curves

We continue to let K be a non-archimedean local field of characteristic zero. Let E/K be an elliptic curve. Then there is an irreducible, admissible representation π of $\mathrm{GL}(2,K)$ attached to E/K via the following procedure:

- Choose a prime ℓ different from the residual characteristic of K.
- The Galois group $\mathrm{Gal}(\bar K/K)$ acts on the Tate module $T_\ell(E)$, yielding a 2-dimensional ℓ-adic representation $\varphi_\ell:\mathrm{Gal}(\bar K/K)\to\mathrm{GL}(2,\mathbb{Q}_\ell)$.
- Via the procedure outlined in §4 of [10], φ_ℓ can be converted to a *complex* representation $\varphi: W(\bar K/K)\to\mathrm{GL}(2,\mathbb{C})$. The isomorphism class of this representation is independent of the choice of ℓ.
- After a twist, we may assume that φ has image in $\mathrm{SL}(2,\mathbb{C})$.
- Via the local Langlands correspondence (see [5]), φ corresponds to an irreducible, admissible representation π of $\mathrm{GL}(2,K)$. Since the image of φ is contained in $\mathrm{SL}(2,\mathbb{C})$, this π has trivial central character.

The correspondence between E and π is such that $L(E,s)=L(s-1/2,\pi)$. Note that these are *local* L-factors, not global L-functions; in particular, $L(s,\pi)$ does not necessarily characterize π. The shift in s is a consequence of the fact that we normalized π to have trivial central character. In other words, $L(E,s)$ is given in *arithmetic* normalization, and $L(s,\pi)$ in *analytic* normalization.

Another feature of the correspondence between E and π is that the conductors coincide, $a(E) = a(\pi)$. This is by definition, since the conductors of both E and π are defined as the conductor of the Weil-Deligne representation φ. Assume that the residual characteristic of K is not 2 or 3. Then, as is well known, $a(E)$ can only take the values 0, 1 or 2. We have $a(E) = 0$ if E/K has good reduction, $a(E) = 1$ if E/K has multiplicative reduction, and $a(E) = 2$ if E/K has additive reduction.

A natural question is this: Given E/K (by some Weierstrass equation), determine π. A uniform answer is possible in the case of *potential multiplicative reduction*. Recall that E/K has potential multiplicative reduction if and only if its j-invariant is not contained in $\mathfrak{o}$. In this case the γ-invariant $\gamma(E/K) := -c_4/c_6$ is a well-defined element of $K^\times/K^{\times 2}$; see Lemma 5.2 in Sect. V.5 of [11]. Here, c_4 and c_6 are the usual quantities derived from a Weierstrass equation for E over K. Using the arguments in §15 of [10], one can show that

$$\pi = (\gamma(E/K), \cdot)\mathrm{St}_{\mathrm{GL}(2)}. \tag{15}$$

Here, $\mathrm{St}_{\mathrm{GL}(2)}$ denotes the Steinberg representation of $\mathrm{GL}(2, K)$; the symbol $(\cdot, \cdot)$ is the quadratic Hilbert symbol over K; and the notation in (15) means the twist of $\mathrm{St}_{\mathrm{GL}(2)}$ by the quadratic character $x \mapsto (\gamma(E/K), x)$ of $K^\times$. Note that this character is trivial if and only if E has split multiplicative reduction over K, and is the unique non-trivial unramified quadratic character if and only if E has non-split multiplicative reduction over K. Note also that the formula (15) holds in any residual characteristic.

We will now assume that E has *potential good reduction*, i.e., that the j-invariant of E is contained in $\mathfrak{o}$. In this case π is either a principal series representation or supercuspidal. The first case occurs if the Weil-Deligne representation φ is a direct sum of two 1-dimensionals, and the second case occurs if φ is irreducible. Assuming that the residual characteristic is ≥ 5, an easy-to-apply criterion to distinguish between the two cases is given in Proposition 2 of [9]:

$$\pi \text{ is a principal series representation} \quad \Longleftrightarrow \quad (q-1)v(\Delta) \equiv 0 \mod 12. \tag{16}$$

Here, Δ is the discriminant of E/K, for any Weierstrass equation with integral coefficients, and v is the normalized valuation on K. (In [9], the criterion is formulated for $K = \mathbb{Q}_p$, but the generalization is straightforward.) Equation (16) is a good example for determining a property of the representation π directly from the Weierstrass equation. For related results, including the more complicated cases $p = 2$ and $p = 3$, see [1].

Let L/K be a field extension, and let E_L be the base change of E to L. Let π be the representation of $\mathrm{GL}(2, K)$ attached to E, and let π_L be the representation of $\mathrm{GL}(2, L)$ attached to E_L. It is easy to see from the definitions that

$$\pi_L = \mathrm{BC}_{L/K}(\pi). \tag{17}$$

In other words, base change for elliptic curves corresponds to base change for the associated irreducible, admissible representations. Using these facts, it is easy to determine from the Weierstrass equation whether π is the triply imprimitive supercuspidal from Theorem 4.1:

Proposition 5.1. *Assume that the residual characteristic of K is ≥ 5. Let E/K be an elliptic curve with discriminant Δ. Assume that E has bad,but potential good* *reduction. Let π be the irreducible, admissible representation of* $\mathrm{GL}(2, K)$ *attached to E/K. Then the following are equivalent:*

(1) *π is supercuspidal, and for every quadratic extension L of K, the irreducible, admissible representation of* $\mathrm{GL}(2, L)$ *attached to E_L is a principal series representation.*
(2) *π is the triply imprimitive supercuspidal representation from Theorem* 4.1.
(3) *The following conditions are satisfied:*
   - $(q-1)v(\Delta) \not\equiv 0 \mod 12$.
   - $2(q-1)v(\Delta) \equiv 0 \mod 12$.

   Here, v is the normalized valuation on K.

Proof. i) and ii) are equivalent by (17). Assume these conditions are satisfied. Then $(q-1)v(\Delta) \not\equiv 0 \mod 12$ by (16). Let L/K be a ramified quadratic extension. Since the representation attached to E_L is a principal series by assumption, we have

$$(q_L - 1)v_L(\Delta) \equiv 0 \mod 12 \tag{18}$$

by (16). But $q_L = q$ and $v_L(\Delta) = 2v(\Delta)$, so that $2(q-1)v(\Delta) \equiv 0 \mod 12$.

Conversely, assume iii) is satisfied. Then π is supercuspidal by (16). Also, we claim that (18) is satisfied for every quadratic extension L of K. For if L/K is ramified, then (18) holds since $q_L = q$ and $v_L(\Delta) = 2v(\Delta)$, and if L/K is unramified, then (18) holds since $q_L = q^2$. Thus, by (16), the representation attached to E_L is a principal series representation for every quadratic extension L/K. □

In the situation of Proposition 5.1, assume that i), ii) and iii) are satisfied. While the representation $\pi_L = \mathrm{BC}_{L/K}(\pi)$ attached to E_L is a principal series representation, it is not unramified. One way to see this is to note that if $\pi = \omega_{F,\xi}$ with ξ as in i) or ii) of Theorem 4.1, then the parameter of π_L is $\xi \oplus \xi^\sigma$. Since ξ is ramified, it follows that π_L is a ramified principal series representation. As a consequence, E does not acquire good reduction over any quadratic extension. (This can also be seen from the conditions on $v(\Delta)$ in iii).)

For related results about the local and global representations attached to elliptic curves, see [8].

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