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SELF-ACCESSIBLE STATES FOR LINEAR SYSTEMS ON TIME SCALES

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Abstract. In this paper, we are concerned with linear control systems on time scales. We show that, under appropriate hypotheses, the self-accessible trajectories have diameter greater than or equal to a certain fixed positive number.

1. Introduction

The theory of time scales was introduced in the literature by Stefan Hilger in 1988, and since then it has shown great potential to represent applications in several fields of knowledge. See, for instance, [1, 8–11, 14, 16, 21, 25] and the references therein.

This theory allows us also to describe continuous-discrete hybrid processes, which have several applications in economics, biology, engineering, and physics, among others. For instance, it is a known fact that certain economically important phenomena do not possess either only continuous property or only discrete aspects. However they contain aspects which have elements of both the continuous and the discrete phenomena. For instance, the continuous-discrete hybrid processes might be used to investigate option-pricing and stock dynamics in finance, the frequency of markets and duration of market trading in economics, large-scale models of DNA dynamics, gene mutation fixation, electric circuits, population models, among others. For these applications, we refer the reader to [4, 13, 14, 22, 25].

We point out that the control systems on time scales have been attracting the attention of several researchers, since they encompass discrete, continuous and hybrid control systems, allowing more general analysis and results. See, for instance, [4, 9–11, 15, 17, 24]. On the other hand, the self-accessibility property for classical humped control systems has been studied by several authors. For linear systems, the property was considered by Boltyanskii [12] to establish sufficient conditions of optimality in relation to the maximum principle. Later, this property was studied in [2, 5] for nonlinear systems and in [3] for multivalued systems. In addition, the characterization of a distributed control system was studied in [19]. Furthermore, it

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is worth mentioning that the self-accessibility property is related to the stabilization of systems, which was clarified in [20].

As something superficial, the property ensures that an attainable state from an initial state x can be returned to x using an admissible control function. Hence, if we assume that x is the operation point of the system, then for self-accessible systems the operation point has a stability property under disturbances. However, rather surprisingly, it has been shown in [18, 23] that trajectories which start and end in the same state are quite large, in a sense that will be specified later. Motivated by this fact, we focus our attention on the study of control systems on time scales. More specially, in the present paper, our goal is to extend the mentioned geometric property of self-accessible trajectories to control systems on time scales.

This paper is organized as follows. In Section 2, we recall some basic aspects of dynamic systems on time scales and finite dimensional spaces. In Section 3, we study dynamic systems on Banach spaces and we prove some properties needed to establish our main results. Finally, in Section 4, we study self-accessible states of control systems on time scales.

2. Preliminaries

In this section, we recall some basic concepts and results concerning the theory of dynamic equations on time scales. For more details, we refer to [6, 7].

Let $\mathbb{T}$ be a time scale, that is, a closed and nonempty subset of $\mathbb{R}$. We assume $\mathbb{T}$ has the topology that it inherits from the real numbers with standard topology.

Definition 2.1. For every $t \in \mathbb{T}$, we define the *forward and backward jump operator* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, respectively, by $\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, where $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$, by convention.

If $\sigma(t) > t$, then t is *right-scattered*. Otherwise, t is *right-dense*. Similarly, if $\rho(t) < t$, then t is *left-scattered*, whereas if $\rho(t) = t$, then t is *left-dense*.

Definition 2.2. The *graininess function* $\mu : \mathbb{T} \to \mathbb{R}^+$ is given by $\mu(t) = \sigma(t) - t$.

Definition 2.3. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if its left-sided limits exist at all left-dense points in $\mathbb{T}$ and are continuous at right-dense points of $\mathbb{T}$. If the function $f : \mathbb{T} \to \mathbb{R}$ is continuous at each right-dense point and each left-dense point, then the function f is called *continuous* on $\mathbb{T}$.

Throughout the paper, given a pair of numbers a, $b \in \mathbb{T}$, the symbol $[a, b]_{\mathbb{T}}$ will be used to denote a closed interval in $\mathbb{T}$. On the other hand, $[a, b]$ is the usual closed interval on the real line.

We define the set $\mathbb{T}^\kappa$ which is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2.4. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, we define the *delta-derivative* of f at t to be the number $f^\Delta(t)$ (if it exists) with the following property: given $\varepsilon > 0$, there exists a neighborhood U of t for the relative topology such that $|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$, for all $s \in U$. In this case, $f^\Delta(t)$ denotes the *delta-derivative* of f at t.

In what follows, we present some properties of delta-differentiable functions.

Theorem 2.5 (See [6, Theorem 1.20])**.** *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are Δ-differentiable at $t \in \mathbb{T}^\kappa$. Then:*

(i) *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at t with $(f+g)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$.*

(ii) *For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at t with $(\alpha f)^\Delta(t) = \alpha f^\Delta(t)$.*

(iii) *The product $fg : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at t with*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).$$

In the sequel, we present the definition of a partition of $[a, b]_\mathbb{T}$. See [7].

Definition 2.6. A *partition* of $[a, b]_\mathbb{T}$ is a finite sequence of points $\{t_0, t_1, \dots, t_m\} \subset [a, b]_\mathbb{T}$, where $a = t_0 < t_1 < \cdots < t_m = b$.

A *tagged partition* consists of a partition and a sequence of tags $\{\xi_1, \dots, \xi_m\}$ such that $\xi_i \in [t_{i-1}, t_i)$ for every $i \in \{1, \dots, m\}$. If $\delta > 0$, then $D_\delta(a, b)$ denotes the set of all tagged partitions of $[a, b]_\mathbb{T}$ such that for every $i \in \{1, \dots, m\}$, either $\Delta t_i \le \delta$ or $\Delta t_i = t_i - t_{i-1} > \delta$ and $\sigma(t_{i-1}) = t_i$.

Definition 2.7. We say that f is *Riemann Δ-integrable* on $[a, b]_\mathbb{T}$ if there exists a number I with the following property: for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \sum_i f(\xi_i)(t_i - t_{i-1}) - I \right| < \varepsilon,$$

for every $P \in D_\delta(a, b)$ independent of $\xi_i \in [t_{i-1}, t_i)_\mathbb{T}$ for $1 \le i \le n$. It is clear that the number I is unique, and I is the Riemann Δ-integral of f from a to b.

The next results contain some important properties of Riemann Δ-integrable functions.

Theorem 2.8 (Fundamental Theorem of Calculus, [7, Theorem 5.34])**.** *Let g be a continuous function on $[a, b]_\mathbb{T}$ such that g is Δ-differentiable on $[a, b)$. If g^Δ is Δ-integrable from a to b, then $\int_a^b g^\Delta(t)\Delta t = g(b) - g(a)$.*

In the sequel, we present some basic results concerning the theory.

Theorem 2.9 (See [7, Theorems 5.12, 5.26 and 5.29])**.** *Let f and g be Δ-integrable functions on $[a, b]_\mathbb{T}$ and let $c \in \mathbb{R}$. Then:*

(i) *cf is Δ-integrable and $\int_a^b (cf)\Delta t = c\int_a^b f(t)\Delta t$.*

(ii) *$f + g$ is Δ-integrable and $\int_a^b (f+g)(t)\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$.*

(iii) *If $f(t) \le g(t)$ for every $t \in [a, b)_\mathbb{T}$, then $\int_a^b f(t)\Delta t \le \int_a^b g(t)\Delta t$.*

(iv) *If f is a constant function, then $f : \mathbb{T} \to \mathbb{R}$ is Δ-integrable from a to b and $\int_a^b K\Delta t = K(b-a)$.*

As usual, a function $f : [a, b]_\mathbb{T} \to \mathbb{R}^n$, $f(t) = (f_1(t), \dots, f_n(t))$, is said to be Δ-integrable if each component f_i is integrable. In this case, we define

$$\int_a^b f(t)\Delta t = \left(\int_a^b f_1(t)\Delta t, \dots, \int_a^b f_n(t)\Delta t \right).$$

As a consequence of Theorem 2.9, the following property follows immediately.

Lemma 2.10. *Let $x^* \in (\mathbb{R}^n)^*$. Then*

$$\left\langle x^*, \int_a^b f(t)\Delta t \right\rangle = \int_a^b \langle x^*, f(t)\rangle \Delta t.$$

Proof. Assume that $x^* = (a_1, \dots, a_n)$. Applying Theorem 2.9, we can write

$$\left\langle x^*, \int_a^b f(t)\Delta t \right\rangle = \sum_{i=1}^n a_i \int_a^b f_i(t)\Delta t = \int_a^b \sum_{i=1}^n a_i f_i(t)\Delta t = \int_a^b \langle x^*, f(t)\rangle \Delta t,$$

obtaining the desired result. □

In the sequel, we present some important concepts which will be fundamental to our purposes (see [6]).

Definition 2.11. We say that a function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* provided $1 + \mu(t)p(t) \neq 0$, for all $t \in \mathbb{T}^\kappa$ holds. The set of all regressive and rd-continuous functions will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 2.12. If $p \in \mathcal{R}$, then the *generalized exponential function* is given by $e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)$ for $s, t \in \mathbb{T}$, where the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ is given by $\xi_h(z) = \frac{1}{h}\mathrm{Log}(1+zh)$, where Log is the principal logarithm function. For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$.

Let A be an $m \times n$ matrix-valued function on $\mathbb{T}$. A is called *rd-continuous* on $\mathbb{T}$ if each entry of A is rd-continuous on $\mathbb{T}$. On the other hand, A is *delta-differentiable* at $\mathbb{T}$ if each entry of A is delta-differentiable on $\mathbb{T}$.

Definition 2.13. An $n \times n$ matrix-valued function A on a time scale $\mathbb{T}$ is called *regressive* (with respect to $\mathbb{T}$) provided $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^\kappa$, and the class of all such regressive rd-continuous matrices is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$.

Definition 2.14 (Matrix exponential function)**.** Let $t_0 \in \mathbb{T}$ and $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$. The unique matrix-valued solution of the IVP (Initial Value Problem) $Y^\Delta(t) = AY(t), Y(t_0) = I$, where I denotes as usual the $n \times n$-identity matrix, is called the *matrix exponential function* at t_0 and is denoted by $e_A(\cdot, t_0)$.

Next, we state a result which describes the properties of the matrix exponential function.

Theorem 2.15 (See [6, Theorem 5.21])**.** *If $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$, then*

(i) $e_0(t,s) \equiv I$ *and* $e_A(t,t) \equiv I$*,*
(ii) $e_A(\sigma(t), s) = (I + \mu(t)A)e_A(t,s)$*,*
(iii) $e_A(t,s) = e_A^{-1}(s,t)$ *and* $e_A(t,s)e_A(s,r) = e_A(t,r)$*.*

Using these notions, one can obtain the following result, which corresponds to the variation of constants formula on time scales, whose proof can be found in [6, Theorem 5.24].

Theorem 2.16 (Variation of constants formula)**.** *Let $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$, $f : \mathbb{T} \to \mathbb{R}^n$ be rd-continuous, $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then, the IVP*

$$\begin{cases} y^\Delta = Ay + f(t), \\ y(t_0) = y_0 \end{cases}$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$. *Moreover, this solution is given by* $y(t) = e_A(t, t_0)y_0 + \int_{t_0}^t e_A(t, \sigma(\tau))f(\tau)\Delta\tau$.

As an immediate consequence, we obtain the next important result.

Corollary 2.17. *Let* $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ *and* $t_0, t_1 \in \mathbb{T}$. *Then* $\mathrm{Im}(I - e_A(t_1, t_0)) \subseteq \mathrm{Im(A)}$.

Proof. We consider the homogeneous system

$$\begin{cases} y^\Delta = Ay, \\ y(t_0) = y_0. \end{cases}$$

It follows from Theorems 2.8 and 2.16 that

$$y(t_1) - y(t_0) = (e_A(t_1, t_0) - I)y_0 = A \int_{t_0}^{t_1} y(t)\Delta t.$$

Since $y_0 \in \mathbb{R}^n$ is an arbitrary element, this implies the assertion. □

3. Vector functions

In this section, we study vector functions from $\mathbb{T}$ into Banach spaces. The concepts and basic results of continuity, Δ-differentiability and Δ-integrability established in Section 2 can be generalized to vector functions. In what follows, X, Y denote Banach spaces provided with a norm $\|\cdot\|$.

In the sequel, we mention a few properties needed to establish our results. For Banach spaces X, Y, we denote by $\mathcal{B}(X, Y)$ the Banach space consisting of bounded linear maps from X into Y endowed with the norm of operators. When $X = Y$, we abbreviate this notation by $\mathcal{B}(X)$. Moreover, X^* denotes the topological dual space of X, and we use the notation $x^*(x) = \langle x^*, x\rangle$ for $x^* \in X^*$ and $x \in X$.

The following properties are a direct consequence of Definition 2.7.

Proposition 3.1. Let $f : [a, b]_\mathbb{T} \to X$ be an rd-continuous function. Then

$$\left\|\int_a^b f(t)\Delta t\right\| \le \int_a^b \|f(t)\|\Delta t.$$

Proposition 3.2. Let $f : [a, b]_\mathbb{T} \to X$ be a Δ-integrable function, and let $A \in \mathcal{B}(X, Y)$. Then $Af : [a, b]_\mathbb{T} \to Y$ is a Δ-integrable function, and

$$\int_a^b Af(t)\Delta t = A \int_a^b f(t)\Delta t.$$

Theorem 3.3 (Fundamental Theorem of Calculus, first version)**.** *Let* $g : [a, b]_\mathbb{T} \to X$ *be an rd-continuous function on* $[a, b]_\mathbb{T}$ *such that* g *is* Δ*-differentiable on* $[a, b)_\mathbb{T}$. *If* g^Δ *is* Δ*-integrable from* a *to* b, *then*

$$\int_a^b g^\Delta(t)\Delta t = g(b) - g(a).$$

Proof. Let $x^* \in X^*$. Then $\langle x^*, g\rangle$ satisfies the condition of Theorem 2.8. Combining this with Proposition 3.2, we have

$$\int_a^b \langle x^*, g\rangle^\Delta(t)\Delta t = \langle x^*, g(b)\rangle - \langle x^*, g(a)\rangle = \langle x^*, g(b) - g(a)\rangle = \left\langle x^*, \int_a^b g^\Delta(t)\Delta t\right\rangle.$$

Since $x^* \in X^*$ was arbitrarily chosen, the assertion is a consequence of the Hahn-Banach Theorem. □

In this case, we also have the following version.

Theorem 3.4 (Fundamental Theorem of Calculus, second version)**.** *Let $f : [a,b]_{\mathbb{T}} \to X$ be an rd-continuous function. Let $F : [a,b]_{\mathbb{T}} \to X$ be given by $F(t) = \int_a^t f(s)\Delta s$. Then F is Δ-differentiable on $[a,b)_{\mathbb{T}}$ and $F^{\Delta}(t) = f(t)$ for $t \in [a,b)_{\mathbb{T}}$.*

We omit the proof of this result because it is essentially the same as that performed in [7, Theorem 5.36].

For vector functions, we can establish the following mean value theorem. We denote by $c(S)$ the convex hull of the set S.

Theorem 3.5. *Let $f : [a,b]_{\mathbb{T}} \to X$ be a Δ-integrable function. Then*

$$\int_a^b f(t)\Delta t \in (b-a)\overline{c(\mathrm{Im}(f))}.$$

Proof. Let $x = \int_a^b f(t)\Delta t$, and assume that $x \notin (b-a)\overline{c(\mathrm{Im}(f))}$. Inasmuch as $(b-a)\overline{c(\mathrm{Im}(f))}$ is a closed convex set, applying the Hahn-Banach Theorem, we deduce the existence of a linear functional $x^* \in X^*$ and a constant $\alpha \in \mathbb{R}$ such that

$$\mathrm{Re}\left\langle x^*, \frac{1}{b-a}x \right\rangle > \alpha > \sup_{t\in\mathbb{T}} \mathrm{Re}\langle x^*, f(t)\rangle.$$

On the other hand, using Theorem 2.9 and Lemma 2.10, we now deduce that

$$\begin{aligned}
\mathrm{Re}\left\langle x^*, \frac{1}{b-a}x \right\rangle &= \frac{1}{b-a}\mathrm{Re}\left\langle x^*, \int_a^b f(t)\Delta t \right\rangle \\
&= \frac{1}{b-a}\int_a^b \mathrm{Re}\langle x^*, f(t)\rangle \Delta t \\
&\le \frac{1}{b-a}\int_a^b \alpha \Delta t \\
&= \alpha,
\end{aligned}$$

which is a contradiction. This completes the proof. □

We will now study the abstract Cauchy problem (abbreviated, ACP) in the space X, but first we establish the following property, which is an immediate consequence of Definition 2.7.

Lemma 3.6. *Let $a, b \in \mathbb{T}$ with $a < b$ and $k \in \mathbb{N}$. Then*

$$\int_a^b s^k \Delta s \le \frac{1}{k+1}(b^{k+1} - a^{k+1}).$$

Proof. For every $\varepsilon > 0$, there exist $\delta > 0$ and a partition $P \in D_\delta(a,b)$ consisting of points $a = t_0 < t_1 \ldots < t_m = b$ such that

$$\begin{aligned}\int_a^b s^k \Delta s &\leq \sum_{i=1}^m t_{i-1}^k (t_i - t_{i-1}) + \varepsilon \\ &\leq \frac{1}{k+1} \sum_{i=1}^m (t_i - t_{i-1}) \sum_{j=0}^k t_{i-1}^j t_i^{k-j} + \varepsilon \\ &= \frac{1}{k+1} \sum_{i=1}^m (t_i^{k+1} - t_{i-1}^{k+1}) + \varepsilon \\ &= \frac{1}{k+1} (b^{k+1} - a^{k+1}) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily chosen, this shows the assertion. □

Theorem 3.7. *Let $A \in \mathcal{B}(X)$, $t_0 \in \mathbb{T}$, and $f : [t_0, \infty)_\mathbb{T} \to X$ be an rd-continuous function. Then, the ACP*

$$\left\{\begin{array}{l} x^\Delta(t) = Ax(t) + f(t), \quad t \in \mathbb{T},\ t \geq t_0, \\ x(t_0) = x_0 \end{array}\right. \tag{3.1}$$

has a unique solution $x(\cdot) : [t_0, \infty)_\mathbb{T} \to X$. Moreover, if we denote by $x(\cdot, t_0; x_0)$ the solution corresponding to $f = 0$, then for every $t, t_0 \in \mathbb{T}$, $x(t, t_0; \cdot) : X \to X$ is a bounded linear map.

Proof. We fix $a \in \mathbb{T}$, $t_0 < a$. We define the map $\Gamma : C_{rd}([t_0, a]_\mathbb{T}, X) \to C_{rd}([t_0, a]_\mathbb{T}, X)$ by

$$\Gamma x(t) = x_0 + A \int_{t_0}^t x(s) \Delta s + \int_{t_0}^t f(s) \Delta s. \tag{3.2}$$

It is clear that

$$\|\Gamma x(t) - \Gamma y(t)\| \leq \|A\| \int_{t_0}^t \|x(s) - y(s)\| \Delta s.$$

Then, we have

$$\|\Gamma x(t) - \Gamma y(t)\| \leq \|A\| (t - t_0) \sup_{t_0 \leq s \leq t} \|x(s) - y(s)\|.$$

Combining this estimate with Lemma 3.6 and proceeding inductively, we can establish as usual that

$$\|\Gamma^k x - \Gamma^k y\| \leq \frac{1}{k!} \|A\|^k (t - t_0)^k \|x - y\|,$$

which shows that there exists $n \in \mathbb{N}$ sufficiently large such that Γ^n is a contraction. Consequently, by the Banach Fixed-Point Theorem, there is a unique fixed point $x(\cdot)$ of Γ. As a consequence of Theorem 3.4, we obtain that $x(\cdot)$ is a solution of problem (3.1). Now, we abbreviate $x(t) = x(t, t_0; x_0)$. Using the uniqueness of the solution, we obtain easily that $x(t, t_0; \cdot) : X \to X$ is a linear map. Indeed, since

$$x(t) = x_0 + A \int_{t_0}^t x(s) \Delta s,$$

we can estimate

$$\begin{aligned} \|x(t)\| &\leq \|x_0\| + \|A\| \int_{t_0}^{t} \|x(s)\| \Delta s \\ &\leq \|x_0\| + \|A\|(t-t_0) \sup_{t_0 \leq s \leq t} \|x(s)\|. \end{aligned}$$

Repeating this argument and again using Lemma 3.6, we infer that

$$\begin{aligned} \|x(t)\| &= \|\Gamma^n x(t)\| \\ &\leq \sum_{i=0}^{n-1} \frac{1}{i!} \|A\|^i (t-t_0)^i \|x_0\| + \frac{1}{n!} \|A\|^n (t-t_0)^n \sup_{t_0 \leq s \leq t} \|x(s)\| \\ &\leq \sum_{i=0}^{n-1} \frac{1}{i!} \|A\|^i (a-t_0)^i \|x_0\| + \frac{1}{n!} \|A\|^n (a-t_0)^n \sup_{t_0 \leq s \leq a} \|x(s)\|. \end{aligned}$$

Selecting $n \in \mathbb{N}$ such that $\alpha = \frac{1}{n!} \|A\|^n (a-t_0)^n < 1$, we obtain

$$\|x\| \leq \frac{1}{1-\alpha} \sum_{i=0}^{n-1} \frac{1}{i!} \|A\|^i (a-t_0)^i \|x_0\|,$$

which shows that $x(t, t_0; \cdot) : X \to X$ is a bounded linear map. □

In what follows, we denote by $e_A(t, t_0)$ the bounded linear map $x(t, t_0; \cdot)$ involved in Theorem 3.7. It follows from the previous estimate that there exists a constant $M_a \geq 0$ such that $\|e_A(t, t_0)\| \leq M_a$ for all $t_0 \leq t$ with $t - t_0 \leq a$. The next proposition abridges a few properties of $e_A(t, t_0)$.

Proposition 3.8. The following properties are fulfilled.

(i) $e_0(t, s) = I$ and $e_A(t, t) = I$.
(ii) Let $r, s, t \in \mathbb{T}$, $r \leq s \leq t$; then $e_A(t, s) e_A(s, r) = e_A(t, r)$.
(iii) Let $t_0, t \in \mathbb{T}$, $t - t_0 \leq a$; then $\|e_A(t, t_0) - I\| \leq M_a \|A\| (t - t_0)$.
(iv) Let $t_0 \in \mathbb{T}$. The operator valued map $[t_0, \infty)_{\mathbb{T}} \to \mathcal{B}(X)$, $t \mapsto e_A(t, t_0)$, is Δ-differentiable.
(iv) Let $t_0, t \in \mathbb{T}$, $t_0 < t$. The operator valued map $[t_0, t]_{\mathbb{T}} \to \mathcal{B}(X)$, $s \mapsto e_A(t, s)$, is Δ-differentiable.

Proof. The assertion (i) is immediate. The assertion (ii) is a consequence of the uniqueness of solutions of problem (3.1). To prove the assertion (iii), we note that

$$(e_A(t, t_0) - I)x_0 = A \int_{t_0}^{t} e_A(s, t_0) x_0 \Delta s,$$

which implies that

$$\begin{aligned} \|(e_A(t, t_0) - I)x_0\| &\leq \|A\| \int_{t_0}^{t} M_a \|x_0\| \Delta s \\ &= \|A\| M_a (t - t_0) \|x_0\| \end{aligned}$$

for all $x_0 \in X$.

Finally, assertions (iv) and (v) are immediate consequences of (ii) and (iii). □

It is worth pointing out that $e_A(t, t_0)$ is defined only for $t \geq t_0$. However, to simplify the writing of our statements in the following result, we consider $e_A(t, s) = I$

for $s \geq t$. Using this convention and the fact that the function $\sigma(\cdot)$ is rd-continuous ([6, Theorem 1.60]), we can reobtain the variation of constants formula.

Theorem 3.9 (Variation of constants formula)**.** *Let $A \in \mathcal{B}(X)$ and suppose $f : \mathbb{T} \to X$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $x_0 \in X$. Then the solution $x(\cdot)$ of problem* (3.1) *is given by*

$$x(t) = e_A(t, t_0)x_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Proof. We define

$$y(t) = e_A(t, t_0)x_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Define the map $\Gamma : C_{rd}(\mathbb{T}, X) \to C_{rd}(\mathbb{T}, X)$ by

$$\Gamma x(t) = x_0 + A\int_{t_0}^{t} x(s)\Delta s + \int_{t_0}^{t} f(s)\Delta s.$$

Then, combining the properties of $e_A(\cdot, \cdot)$ with the results in [8], we infer that

$$\begin{aligned}
\Gamma y(t) &= x_0 + A\int_{t_0}^{t} y(s)\Delta s + \int_{t_0}^{t} f(s)\Delta s \\
&= x_0 + A\int_{t_0}^{t} e_A(s, t_0)x_0\Delta s + A\int_{t_0}^{t}\int_{t_0}^{s} e_A(s, \sigma(\tau))f(\tau)\Delta\tau\Delta s + \int_{t_0}^{t} f(s)\Delta s \\
&= x_0 + \int_{t_0}^{t} [e_A(s, t_0)x_0]^{\Delta}\Delta s + A\int_{t_0}^{t}\int_{\sigma(\tau)}^{t} e_A(s, \sigma(\tau))f(\tau)\Delta s\Delta\tau + \int_{t_0}^{t} f(s)\Delta s \\
&= e_A(t, t_0)x_0 + \int_{t_0}^{t} f(s)\Delta s + \int_{t_0}^{t}\int_{\sigma(\tau)}^{t} Ae_A(s, \sigma(\tau))f(\tau)\Delta s\Delta\tau \\
&= e_A(t, t_0)x_0 + \int_{t_0}^{t} f(s)\Delta s + \int_{t_0}^{t} [e_A(t, \sigma(\tau)) - e_A(\sigma(\tau), \sigma(\tau))]f(\tau)\Delta\tau \\
&= e_A(t, t_0)x_0 + \int_{t_0}^{t} f(s)\Delta s + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau)\Delta\tau - \int_{t_0}^{t} f(\tau)\Delta\tau \\
&= y(t),
\end{aligned}$$

which implies that $y(\cdot)$ is a fixed point of Γ. This implies that $y(t) = x(t)$. □

4. Self-accessible states

In this section, we focus our attention on control systems on time scales described by

$$\begin{cases} x^{\Delta}(t) = Ax(t) + Bu(t),\ t \in \mathbb{T},\ t \geq t_0, \\ x(t_0) = z \end{cases} \tag{4.1}$$

where $t_0 \in \mathbb{T}$, the states $x(t) \in X$ and controls $u(t) \in U$ such that X and U are Banach spaces. Throughout this section, we keep the notation and assumptions introduced in Section 3 to ensure the existence of solutions of the system (4.1). Moreover, we assume that $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(U, X)$.

We study the system (4.1) on the interval $[t_0, t_1]_{\mathbb{T}}$, where $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. In order to do that, we restrict ourselves to considering as admissible control the functions $u \in C_{rd}([t_0, t_1]_{\mathbb{T}}, U)$. We denote by $x(\cdot; z, u)$ the solution of (4.1), which is the solution of (3.1) with $Bu(t)$ instead of $f(t)$.

Let Y be a closed subspace of X. We denote

$$E_Y = \{x \in X : Ax + Bu \in Y, \text{ for some } u \in U\}.$$

It is clear that E_Y is a subspace of X.

Lemma 4.1. *Assume the solution $x(t) = x(t; z, u)$ of (4.1) satisfies $x(t_1) - z \in Y$. Then $y = \int_{t_0}^{t_1} x(t)\Delta t \in E_Y$.*

Proof. It follows from Theorem 3.3 that

$$\begin{aligned} x(t_1) - z &= A\int_{t_0}^{t_1} x(s)\Delta s + \int_{t_0}^{t_1} Bu(s)\Delta s \\ &= A\int_{t_0}^{t_1} x(s)\Delta s + B\int_{t_0}^{t_1} u(s)\Delta s \in Y, \end{aligned}$$

which implies $y \in E_Y$. □

We are now in a position to establish the following geometric property of admissible trajectories of control systems.

Theorem 4.2. *Assume the solution $x(t) = x(t; z, u)$ of (4.1) satisfies $x(t_1) - z \in Y$. Let $x_0 \in X$. Then*

$$\sup_{t \in [t_0, t_1]_{\mathbb{T}}} \|x_0 - x(t)\| \geq d(x_0, E_Y).$$

Proof. We define $y_0 = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} x(t)\Delta t$. From Lemma 4.1, we can affirm that $y_0 \in E_Y$. Moreover, it follows from Theorem 3.5 that $y_0 \in \overline{c(\text{Im}(x))}$. Consequently, we get

$$\begin{aligned} d(x_0, E_Y) &= \inf\{\|x_0 - \widetilde{y}\| : \widetilde{y} \in E_Y\} \\ &\leq \|x_0 - y_0\| \\ &\leq \sup\{\|x_0 - y\| : y \in \overline{c(\text{Im}(x))}\} \\ &= \sup\{\|x_0 - x(t)\| : t \in [t_0, t_1]_{\mathbb{T}}\}, \end{aligned}$$

and the proof is finished. □

In what follows, let us investigate a particular case.

Definition 4.3. A state $z \in X$ is said to be *self-accessible* for system (4.1) on $[t_0, t_1]_{\mathbb{T}}$ if there exists an admissible control function $u(\cdot)$ such that $x(t_0; z, u) = x(t_1; z, u) = z$. The system (4.1) is said to be *self-accessible* on $[t_0, t_1]_{\mathbb{T}}$ if every state $z \in X$ is self-accessible on $[t_0, t_1]_{\mathbb{T}}$.

We take $Y = \{0\}$. The space

$$E = \{x \in X : Ax \in \text{Im}(B)\}$$

is called *space of stationary states* whenever the following condition is satisfied: if $z \in E$ and $Az + Bu = 0$, then the solution of system (4.1) for the control function $u(t) = u$ is given by $x(t; z, u) = z$ for all $t \in [t_0, t_1]_{\mathbb{T}}$. Moreover, if Im(B) is a closed subspace, then E is also a closed subspace of X.

The next result follows as an immediate consequence of Lemma 4.1 for the case when $Y = \{0\}$. Therefore, we omit its proof.

Theorem 4.4. *Assume the solution* $x(t) = x(t; z, u)$ *of* (4.1) *satisfies* $x(t_1) = z$. *Then* $y = \int_{t_0}^{t_1} x(t)\Delta t \in E$.

Proof. Since x is a solution of (4.1), we get

$$x(t) = e_A(t, t_0)x(t_0) + \int_{t_0}^{t} e_A(t, \sigma(\tau))Bu(\tau)\Delta\tau,$$

which implies that

$$\int_{t_0}^{t_1} x(t)\Delta t = \int_{t_0}^{t_1} e_A(t, t_0)x(t_0)\Delta t + \int_{t_0}^{t_1} \int_{t_0}^{t} e_A(t, \sigma(\tau))Bu(\tau)\Delta\tau\Delta t.$$

Thus, we have

$$y = \int_{t_0}^{t_1} e_A(t, t_0)z\Delta t + \int_{t_0}^{t_1} \int_{t_0}^{t} e_A(t, \sigma(\tau))Bu(\tau)\Delta\tau\Delta t$$

for some admissible control $u(\cdot)$. Moreover,

$$A\int_{t_0}^{t_1} e_A(t, t_0)z\Delta t = e_A(t_1, t_0)z - z.$$

Hence, we obtain

$$A\int_{t_0}^{t_1} \int_{t_0}^{t} e_A(t, \sigma(\tau))Bu(\tau)\Delta\tau\Delta t = A\int_{t_0}^{t_1} \int_{\sigma(\tau)}^{t_1} e_A(t, \sigma(\tau))Bu(\tau)\Delta t\Delta\tau,$$

which implies that

$$A\left[y - \int_{t_0}^{t_1} e_A(t, t_0)z\Delta t\right] = \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau))Bu(\tau)\Delta\tau - \int_{t_0}^{t_1} e_A(\sigma(\tau), \sigma(\tau))Bu(\tau)\Delta\tau.$$

Therefore,

$$Ay - e_A(t_1, t_0)z + z = x(t_1) - e_A(t_1, t_0)x(t_0) - \int_{t_0}^{t_1} Bu(\tau)\Delta\tau.$$

By the fact that $x(t_1) = z = x(t_0)$, we obtain

$$Ay = -\int_{t_0}^{t_1} Bu(\tau)\Delta\tau,$$

which implies that $y \in E$. □

Corollary 4.5. *Let* z *be a self-accessible state of system* (4.1), *and assume that the solution* $x(t) = x(t; z, u)$ *of* (4.1) *satisfies* $x(t_1) = z$. *Then*

$$\sup_{t\in[t_0,t_1]_{\mathbb{T}}} \|z - x(t)\| \geq d(z, E).$$

Proof. It follows analogously as the proof of Theorem 4.2. □

When Im(B) is a closed subspace and $z \notin E$, under the assumptions of Corollary 4.5 we infer that $\sup_{t\in[t_0,t_1]_{\mathbb{T}}} \|z - x(t)\| > 0$, which shows that the self-accessible trajectories are quite large. This occurs in particular for lumped systems. In this case, we can also establish a sufficient condition for the system to be self-accessible.

Theorem 4.6. *If* $\mathrm{Im}(A) \subseteq \mathrm{Im}[B, AB, \ldots, A^{n-1}B]$, *then the system* (4.1) *is self-accessible on* $[t_0, t_1]_{\mathbb{T}}$.

Proof. It follows from Corollary 2.17 that $z - e_A(t_1, t_0)z \in \mathrm{Im}(A)$. Using our hypothesis and the characterization of controllability in [24], we infer that $z - e_A(t_1, t_0)z$ is a reachable state. Hence, there exists an admissible control $u(\cdot)$ such that

$$z - e_A(t_1, t_0)z = \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau))Bu(\tau)\Delta\tau,$$

which implies that z is a self-accessible state on $[t_0, t_1]_{\mathbb{T}}$. □

It is easy to see that the converse assertion does not hold.

Example 4.7. Let $\mathbb{T} = \mathbb{Z}$ and $A = -2I$, where I is the $n \times n$ identity, and $B = 0$. The solution of system (4.1) on $[0, 2]_{\mathbb{T}}$ is given by $x(2) = (I + A)^2 z = z$, for all $z \in \mathbb{R}^n$. This shows that system (4.1) is self-accessible on $[0, 2]_{\mathbb{T}}$. However, it is clear that $\mathrm{Im}(A) = \mathbb{R}^n \subsetneq \mathrm{Im}[B, AB, \ldots, A^{n-1}B] = \{0\}$.

Example 4.8. Let $\mathbb{T} = 2^{\mathbb{N}_0}$ and $A = -\frac{3}{4}I$, where I is the $n \times n$ identity, and $B = 0$. Note that

$$x^{\Delta}(2) = \frac{x(\sigma(2)) - x(2)}{\mu(2)} = \frac{x(4) - x(2)}{4 - 2}.$$

Since x is the solution of (4.1) on $[2, 8]_{\mathbb{T}}$, we get

$$(I + 2A)x(2) = x(4).$$

On the other hand, by the same procedure, we obtain

$$(I + 4A)x(4) = x(8).$$

Inductively, we have

$$(I + 4A)(I + 2A)x(2) = x(8),$$

which implies that

$$x(8) = (I + 4A)(I + 2A)z = z$$

for all $z \in \mathbb{R}^n$, where $x(2) = z$. This shows that the system (4.1) is self-accessible on $[2, 8]_{\mathbb{T}}$. However, it is clear that $\mathrm{Im}(A) = \mathbb{R}^n \subsetneq \mathrm{Im}[B, AB, \ldots, A^{n-1}B] = \{0\}$.

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