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GLOBAL ANALYTIC SOLUTIONS AND TRAVELING WAVE SOLUTIONS OF THE CAUCHY PROBLEM FOR THE NOVIKOV EQUATION

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ABSTRACT. In this paper, we mainly study the existence and uniqueness of the analytic solutions for the Novikov equation. We first investigate whether the equation has analytic solutions which exist globally in time, provided the initial data satisfies certain sign conditions. We also get the analyticity of the Cauchy problem for a family of nonlinear wave equations. Finally, we prove that the Novikov equation has a family of traveling wave solutions.

1. Introduction

This paper is devoted to the analyticity and traveling wave solution for the following partial differential equation (PDE):

$$\begin{cases} u_t - u_{txx} + 4u^2u_x = & 3uu_xu_{xx} + u^2u_{xxx}, \\ & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

Eq.(1.1) arises as a zero curvature equation $F_t - G_x + [F,G] = 0$, which is the compatibility condition for the linear system [23],

$$\begin{cases} \Psi_x = F\Psi, \\ \Psi_t = G\Psi, \end{cases}$$

where $y = u - u_{xx}$,

$$F = \begin{pmatrix} 0 & y\lambda & 1 \\ 0 & 0 & y\lambda \\ 1 & 0 & 0 \end{pmatrix}, G = \begin{pmatrix} \frac{1}{3\lambda^2} - uu_x & \frac{u_x}{\lambda} - u^2y\lambda & u_x^2 \\ \frac{u}{\lambda} & -\frac{2}{3\lambda^2} & -\frac{u_x}{\lambda} - u^2y\lambda \\ -u^2 & \frac{u}{\lambda} & \frac{1}{3\lambda^2} + uu_x \end{pmatrix}.$$

Eq.(1.1) was discovered very recently by Novikov in a symmetry classification of nonlocal PDEs with cubic nonlinearity [32]. The perturbative symmetry approach [30] yields necessary conditions for a PDE to admit infinitely many symmetries. Using this approach, Novikov is able to isolate Eq.(1.1) and find its first few symmetries. He subsequently finds a scalar Lax pair for it, and also proves that the equation is integrable. By defining a new dependent variable y, Eq.(1.1) can be written as

$$y_t + u^2y_x + 3uu_xy = 0, \qquad y = u - u_{xx}.$$

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In 1993, Camassa and Holm [2] obtained the equation

$$y_t + uy_x + 2u_x y = 0, \qquad y = u - u_{xx}, \tag{1.2}$$

from an asymptotic approximation to the Hamiltonian for the Green–Naghdi equations in shallow water theory, which approximates to the incompressible Euler equation at the next order beyond the KdV equation [9]. The Camassa–Holm equation models the unidirectional propagation of shallow water waves over a flat bottom [2], and also is a model for the propagation of axially symmetric waves in hyperelastic rods [12]. It has a bi-Hamiltonian structure [18], a Lax pair based on a linear spectral problem of second order, and is completely integrable [5]. Moreover, Eq.(1.2) has peakon solitons [3], which are orbital stable [11]. The Camassa–Holm equation has attracted a lot of interest in the past seventeen years for various reasons; cf. [1, 6, 7].

The Camassa–Holm equation is not the only integrable PDE of its kind, being a shallow water equation whose dispersionless version has weak solitons. In 1999, in order to isolate integrable third order equations using an asymptotic integrability approach, Degasperis and Procesi [15] derived the Degasperis–Procesi equation

$$y_t + uy_x + 3u_x y = 0, \qquad y = u - u_{xx}. \tag{1.3}$$

The Degasperis–Procesi equation can be regarded as a model for nonlinear shallow water dynamics [22]. Degasperis, Holm and Hone [14] study the formal integrability of Eq.(1.3) by constructing a Lax pair. They also show [14] that it has a bi-Hamiltonian structure and an infinite sequence of conserved quantities, and admits exact peakon solutions.

Despite the similarity of the Degasperis–Procesi equation to the Camassa–Holm equation, it should be emphasized that these two equations are truly different, for example, the conservation laws of Eq.(1.3) are weaker than those of Eq.(1.2) [16]. One of the important features of Eq.(1.3) is that it has not only peakon solitons [14], i.e. solutions of the form $u(t,x) = ce^{-|x-ct|}$ and periodic peakon solitons [42], but also has shock peakons [4, 28] which are given by

$$u(t,x) = -\frac{1}{t+k}\operatorname{sgn}(x)e^{-|x|}, \qquad k > 0,$$

and periodic shock peakons [17],

$$u_c(t,x) = \begin{cases} \left(\frac{\cosh\frac{1}{2}}{\sinh\frac{1}{2}}t + c\right)^{-1} \frac{\sinh(x-[x]-\frac{1}{2})}{\sinh\frac{1}{2}}, & x \in \mathbb{R}/\mathbb{Z},\ c > 0, \\ 0, & x \in \mathbb{Z}. \end{cases}$$

Analogous to the Camassa–Holm equation, the Novikov equation has a bi-Hamiltonian structure and an infinite sequence of conserved quantities, and admits exact peakon solutions [23, 38], i.e. solutions of the form

$$u(t,x) = \pm\sqrt{c}e^{-|x-ct-x_0|}, \quad c > 0, \quad x_0 \text{ is constant}.$$

Moreover, Eq.(1.1) also has n-peakon solutions [21, 23, 37],

$$u(t,x) = \sum_{j=1}^{n} p_j(t)\exp(-|x - q_j(t)|),$$

where the positions q_j and amplitudes $p_j,\ j=1,\cdots,n$, satisfy the system of ODEs,

$$\begin{cases} \dot{q_j} = \sum_{k,l=1}^{n} p_k p_l \exp(-|q_j-q_k|-|q_j-q_l|), \\ \dot{p_j} = p_j \sum_{k,l=1}^{n} p_k q_l \mathrm{sgn}(q_j-q_k)\exp(-|q_j-q_k|-|q_j-q_l|). \end{cases}$$

The Novikov equation is similar to the Degasperis–Procesi equation in form, although, the H^1-norm of Eq.(1.1) is a conservation law, which is the same as the Camassa–Holm equation. Eq.(1.1) has cubic nonlinearity, rather than the quadratic nonlinearity of Eqs.(1.2) and (1.3). Moreover, Eq.(1.1) itself is not symmetrical, i.e. $(u,x) \nrightarrow (-u,-x)$; some results of the Novikov equation are truly different from the Camassa–Holm and Degasperis–Procesi equation [21,23,37,38]. Recently, the Cauchy problem of the Novikov equation on the line and on the circle was investigated in [36,38,40], and the global weak solution was obtained as the initial potential $y_0(x)$ satisfies certain sign conditions [39].

The analyticity of solutions for water wave equations has been studied extensively [8, 20, 24]. In this paper, we establish existence and uniqueness of local analytic solutions to Eq.(1.1) where the initial datum is analyticity. Moreover, the global analyticity in time is investigated as the initial potential satisfies certain sign conditions.

The remainder of the paper is organized as follows. In Section 2, we recall several definitions and theorems, which come from [35]. As in [27], we first prove the existence and uniqueness of analytic solutions to Eq.(1.1) with the analytic initial datum. Then we show the existence and uniqueness of global analytic solutions in time to Eq.(1.1), provided the initial potential satisfies certain sign conditions. In Section 3, similarly, we get the analyticity of the Cauchy problem for a family of nonlinear wave equations. In Section 4, we prove that Eq.(1.1) has a family of traveling wave solutions.

2. Global existence of analytic solutions

2.1. **Local analytic solutions (in time).** First, applying the operator $(1-\partial_x^2)^{-1}$ on both sides of Eq.(1.1), we can rewrite it as follows:

$$\begin{cases} u_t + u^2 u_x + (1-\partial_x^2)^{-1}[\partial_x(\tfrac{3}{2}uu_x^2 + u^3) + \tfrac{1}{2}u_x^3] = 0,\ t>0, x\in\mathbb{R}, \\ u(0,x) = u_0(x),\ x\in\mathbb{R}. \end{cases} \tag{2.1}$$

In this subsection, we will establish the existence and uniqueness of analytic solutions to Eq.(1.1) in finite time. For the convenience of the readers, we first recall the following useful results.

Definition 2.1. $H^{0,\rho}$ is the Banach space of all the functions $f(x)$ such that
(i) f is analytic in $D(\rho) = \mathbb{R}\times(-\rho,\rho) = \{x\in\mathbb{C} : \tilde{\tau}x \in (-\rho,\rho)\}$;
(ii) $f\in L^2(\Gamma(\tilde{\tau}x))$ for $\tilde{\tau}x\in(-\rho,\rho)$; i.e. if $\tilde{\tau}x\in(-\rho,\rho)$, then $f(\tilde{v}x + i\tilde{\tau}x)$ is a square integrable function of $\tilde{v}x$, where $\Gamma(b) = \{x\in\mathbb{C} : \tilde{\tau}x = b\}$;
(iii) $|f|_\rho = \sup_{\tilde{\tau}x\in(-\rho,\rho)} ||f(\cdot + i\tilde{\tau}x)||_{L^2(\Gamma(\tilde{\tau}x))} < \infty$.

Definition 2.2. $H^{k,\rho}$ is the Banach space of all the functions $f(x)$ such that
(i) $\partial_x^j f \in H^{0,\rho}$ for all $0\le j\le k$;
(ii) $||f||_{k,\rho} = \sum_{0\le j\le k} |\partial_x^j f|_\rho < \infty$.

In view of the Fourier transform, the norm of $H^{k,\rho}$ is equivalent to

$$||f||_{k,\rho} = \left(\int_{\mathbb{R}} e^{2\rho|\xi|}(1+|\xi|^2)^k |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}. \tag{2.2}$$

It is not difficult to check the following properties:
(1) For any $0 < \rho_1 \le \rho_2$, $H^{k,\rho_2} \subseteq H^{k,\rho_1}$ and $||\cdot||_{k,\rho_1} \le ||\cdot||_{k,\rho_2}$;
(2) $H^{1,\rho} \subset H^k$, $\forall k \ge 0$ and $||\partial_x(u-v)||_{1,\rho} \le c\frac{||u-v||_{1,\rho_1}}{\rho_1-\rho}$, $0 < \rho < \rho_1$;
(3) If $u, v \in H^{1,\rho_1}, \rho < \rho_1$, then $||u\partial_x u - v\partial_x v||_{1,\rho} \le c\frac{||u-v||_{1,\rho_1}}{\rho_1-\rho}$.
Now, we state the main result in this subsection.

Theorem 2.1. *Assume $u_0(x) \in H^{1,\rho_0}$. Then there exists $\beta > 0$, such that for any $0 < \rho < \rho_0$ and for a unique continuously differential (w.r.t. time) solution with the initial datum $u_0(x)$ to Eq.(1.1). Moreover, we have*

$$u(t,\cdot) \in H^{1,\rho} \quad \textit{and}\, \partial_t u(t,\cdot) \in H^{1,\rho}, \ \textit{if} \quad t \in \left[0, \frac{\rho_0-\rho}{\beta}\right].$$

Let $F(t,u) = u_0 - \int_0^t \left[u^2 u_x + (1-\partial_x^2)^{-1}(\partial_x(\frac{3}{2}uu_x^2 + u^3) + \frac{1}{2}u_x^3)\right](s,x)ds$. We can transform Eq.(2.1) into the following form:

$$\begin{cases} u = F(t,u), \quad t > 0, \\ u(0,x) = u_0(x). \end{cases} \tag{2.3}$$

We now state the Abstract Cauchy Kovalevskaya (ACK) Theorem in another form as given by Safonov in [35]. Then Theorem 2.1 is a straightforward consequence of the following result.

Proposition 2.1 ([27,35])**.** *Consider the problem: $u = F(t,u)$. Let $\exists R > 0, \rho_0 > 0$ and $\beta_0 > 0$ such that if $0 < t \le \frac{\rho_0}{\beta_0}$, $F(t,u)$ satisfies the following conditions:*
(i) For any $0 < \rho_1 < \rho \le \rho_0$ and $u \in \{u \in X_\rho : \sup_{0\le t\le T}|u(t)|_\rho \le R\}$, the function $F(t,u) : [0,T] \mapsto X_{\rho_1}$ is continuous.
(ii) For any $0 < \rho < \rho_0$, the function

$$F(t,u) : [0, \rho_0/\beta_0] \mapsto \{u \in X_\rho : \sup_{0\le t\le T} |u(t)|_\rho \le R\}$$

is continuous and satisfies

$$|F(t,0)|_\rho \le R_0 < R.$$

(iii) For any $0 < \rho_1 < \rho(s) < \rho_0$ and $u, v \in \{u \in X_\rho : \sup_{0\le t\le T}|u(t)|_{\rho-\beta_0 t} \le R\}$,

$$|F(t,u) - F(t,v)|_{\rho_1} \le C \int_0^t \frac{|u-v|_{\rho(s)}}{\rho(s)-\rho_1} ds.$$

Then $\exists \beta > \beta_0$ such that for any $0 < \rho < \rho_0$, $u = F(t,u)$ has a unique solution $u(t) \in X_\rho$ with $0 \le t \le (\rho_0-\rho)/\beta$. Moreover,

$$\sup_{\rho<\rho_0-\beta t} |u(t)|_\rho \le R.$$

It is easy to check the conditions (i) and (ii) to Eq.(2.3). To get the result of Theorem 2.1, we only need to show the following result.

Lemma 2.1. *Let $R > 0$. For any $0 < \rho_1 < \rho(s) < \rho_0$ and $u, v \in \{u \in H^{1,\rho} : \sup_{0\le t\le T} ||u(t)||_{1,\rho-\beta_0 t} \le R\}$, there exists a constant $C > 0$ such that*

$$||F(t,u) - F(t,v)||_{1,\rho_1} \le C\int_0^t \frac{||u-v||_{1,\rho(s)}}{\rho(s)-\rho_1}ds,$$

where the constant C depends on $||u||_{1,\rho}$, $||v||_{1,\rho}$.

Proof. Using the equivalent norm of $H^{1,\rho}$ (2.2), for any $0 < \rho_1 < \rho$, we have

$$||\partial_x(u^3 - v^3)||_{1,\rho_1} \le c||\partial_x(u-v)||_{1,\rho_1} \le \frac{c||u-v||_{1,\rho}}{\rho-\rho_1},$$

$$||u^3 - v^3||_{1,\rho_1} \le c||u-v||_{1,\rho_1} \le \frac{c||u-v||_{1,\rho}}{\rho-\rho_1},$$

$$||u_x^3 - v_x^3||_{1,\rho_1} \le c||\partial_x(u-v)||_{1,\rho_1} \le \frac{c||u-v||_{1,\rho}}{\rho-\rho_1},$$

$$||uu_x^2 - vv_x^2||_{1,\rho_1} \le c(||u-v||_{1,\rho_1} + ||u_x^2 - v_x^2||_{1,\rho_1}) \le \frac{c||u-v||_{1,\rho}}{\rho-\rho_1}.$$

In view of the above relations and Proposition 2.1, one has that

$$\begin{aligned}
&||F(t,u) - F(t,v)||_{1,\rho_1} \\
&\le \int_0^t \frac{1}{3}\left(||\partial_x(u^3-v^3)||_{1,\rho_1} + \frac{1}{2}||(1-\partial_x^2)^{-1}(u_x^3 - v_x^3)||_{1,\rho_1}\right)ds \\
&+ \int_0^t \left\|(1-\partial_x^2)^{-1}\partial_x\left(\frac{3}{2}(uu_x^2 - vv_x^2) + (u^3-v^3)\right)\right\|_{1,\rho_1} ds \\
&\le C\int_0^t \frac{||u-v||_{1,\rho(s)}}{\rho(s)-\rho_1}ds.
\end{aligned}$$

This completes the proof of Lemma 2.1. □

Remark 2.1. Similar to the proof of Theorem 2.1, we also obtain the kind of analytic solutions of the Cauchy problem for the periodic Novikov equation.

2.2. **Global analytic solutions (in time).** Based on the local existence of analytic solutions in Section 2.1, in this subsection we will establish the global analytic solution of Eq.(1.1), by dealing with the space $H^{s,\rho}$, $s > \frac{3}{2}$, $\rho > 0$, with the norm (2.2). Consider G^1 of the Gevrey function space of index 1 [25]; we have $G^1 = \bigcup_{\rho>0} H^{s,\rho}$.

In order to obtain the global analytic solution, we first present the following useful lemmas.

Lemma 2.2 ([38])**.** *Assume that $u_0(x) \in H^s(\mathbb{R}), s > \frac{3}{2}$. If the initial potential $y_0 = (1-\Delta)u_0$ does not change sign on $\mathbb{R}$, then Eq.(1.1) is globally well-posed in $C(\mathbb{R}_+; H^s(\mathbb{R})) \cap C^1(\mathbb{R}_+; H^{s-1}(\mathbb{R}))$. Moreover, we have*

$$\|u_x\|_{L^\infty} \le \|u\|_{L^\infty} \le \frac{\sqrt{2}}{2}\|u\|_{H^1} = \frac{\sqrt{2}}{2}\|u_0\|_{H^1}.$$

Lemma 2.3. *Let $u \in D(T_1) = \{u : T_1 u \in L^2(\mathbb{R})\}$, $s > \frac{3}{2}$ and $\|u_x\|_{L^\infty} \le \|u\|_{L^\infty}$. Then we obtain*

$$(a)\ |\langle T(u^2u_x), Tu\rangle| \le c\|u\|_{H^s}^2(\|u\|_{H^s}^2 + 2\rho\|u\|_{s+\frac{1}{2},\rho}^2);$$

$$(b)\ |\langle T(1-\Delta)^{-1}u_x^3, Tu\rangle| \le c\|u\|_{H^s}^2(\|u\|_{H^s}^2 + 2\rho\|u\|_{s+\frac{1}{2},\rho}^2);$$

$$(c)\quad \left|\left\langle T(1-\Delta)^{-1}\partial_x\left(\frac{3}{2}uu_x^2+u^3\right), Tu\right\rangle\right| \le c\|u\|_{H^s}^2(\|u\|_{H^s}^2+2\rho\|u\|_{s+\frac{1}{2},\rho}^2),$$

where the operator $Tu=(1-\Delta)^{\frac{s}{2}}e^{\rho\sqrt{-\Delta}}u$ *and* $T_1u=(1-\Delta)^{\frac{s}{2}+\frac{1}{4}}e^{\rho\sqrt{-\Delta}}u$.

Proof. First, we prove (a). Using the definition of T and Fourier transform, we have

$$\begin{aligned}
|\langle T(u^2u_x), Tu\rangle| &= |\langle(1-\Delta)^{\frac{s}{2}}e^{\rho\sqrt{-\Delta}}(u^2u_x), (1-\Delta)^{\frac{s}{2}}e^{\rho\sqrt{-\Delta}}u\rangle| \\
&= |\langle u^2u_x, (1-\Delta)^s e^{2\rho\sqrt{-\Delta}}u\rangle| \\
&\le \|u\|_{L^\infty}^2|\langle Tu, Tu\rangle| \\
&\le c\|u\|_{H^s}^2|\langle(1+|\xi|^2)^{\frac{s}{2}}e^{\rho|\xi|}\hat{u}, (1+|\xi|^2)^{\frac{s}{2}}e^{\rho|\xi|}\hat{u}\rangle| \\
&= c\|u\|_{H^s}^2\int_{\mathbb{R}}(1+|\xi|^2)^s e^{2\rho|\xi|}|\hat{u}|^2d\xi \\
&\le c\|u\|_{H^s}^2\int_{\mathbb{R}}(1+|\xi|^2)^s(1+2\rho|\xi|e^{2\rho|\xi|})|\hat{u}|^2d\xi \\
&\le c\|u\|_{H^s}^2\left(\int_{\mathbb{R}}(1+|\xi|^2)^s|\hat{u}|^2+2\rho(1+|\xi|^2)^{s+\frac{1}{2}}e^{2\rho|\xi|}|\hat{u}|^2d\xi\right) \\
&= c\|u\|_{H^s}^2\left(\|u\|_{H^s}^2+2\rho\|u\|_{s+\frac{1}{2},\rho}^2\right),
\end{aligned} \tag{2.4}$$

where we have applied the fact that $e^x \le (1+xe^x)$, as $x\ge 0$.

Similarly, we can get the result of (b). Finally, we will show (c) as

$$\begin{aligned}
&\left|\left\langle T(1-\Delta)^{-1}\partial_x\left(\frac{3}{2}uu_x^2+u^3\right), Tu\right\rangle\right| \\
&\le c\|u\|_{H^s}^2|\langle u, (1-\Delta)^{s-1}\partial_x e^{2\rho\sqrt{-\Delta}}u\rangle| \\
&\le c\|u\|_{H^s}^2\int_{\mathbb{R}}(1+|\xi|^2)^s e^{2\rho|\xi|}|\hat{u}|^2d\xi \\
&\le c\|u\|_{H^s}^2\int_{\mathbb{R}}(1+|\xi|^2)^s(1+2\rho|\xi|e^{2\rho|\xi|})|\hat{u}|^2d\xi \\
&\le c\|u\|_{H^s}^2\left(\int_{\mathbb{R}}(1+|\xi|^2)^s|\hat{u}|^2+2\rho(1+|\xi|^2)^{s+\frac{1}{2}}e^{2\rho|\xi|}|\hat{u}|^2d\xi\right) \\
&\le c\|u\|_{H^s}^2(\|u\|_{H^s}^2+2\rho\|u\|_{s+\frac{1}{2},\rho}^2).
\end{aligned} \tag{2.5}$$

Therefore, the results of Lemma 2.3 are derived. □

Next, we will establish the global analytic solutions for Eq.(1.1).

Theorem 2.2. *Let* $u_0(x)\in H^{s,\rho},\ s>\frac{3}{2}, \rho>0$. *If the initial potential* $(1-\Delta)u_0$ *does not change sign on* $\mathbb{R}$ *and the function* $\rho(t)$ *satisfies*

$$\rho(t)\le\rho(0)\exp(-6\|u\|_{H^s}^2t),$$

then the solution $u(t,x)$ *of Eq.(1.1) with the initial datum* $u_0(x)$ *belongs to* G^1 *globally in time.*

Proof. By virtue of the definition of operator T, the norm of $H^{s,\rho}$ (2.2), Eq.(2.1), and Lemma 2.3, one has that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|_{s,\rho}^2 &= \frac{1}{2}\frac{d}{dt}\langle Tu, Tu\rangle \\
&= \langle \partial_t Tu, Tu\rangle = \left\langle \partial_t[(1-\Delta)^{\frac{s}{2}}e^{\rho\sqrt{-\Delta}}u], Tu\right\rangle \\
&= \dot{\rho}\langle T\sqrt{-\Delta}u, Tu\rangle - \langle T(u^2u_x), Tu\rangle - \frac{1}{2}\langle T(1-\Delta)^{-1}u_x^3, Tu\rangle \\
&\quad - \left\langle T(1-\Delta)^{-1}\partial_x\left(\frac{3}{2}uu_x^2+u^3\right), Tu\right\rangle \\
&\le \dot{\rho}\|u\|_{s+\frac{1}{2},\rho}^2 + 3c\|u\|_{H^s}^2\left(\|u\|_{H^s}^2 + 2\rho\|u\|_{s+\frac{1}{2},\rho}^2\right) \\
&= c(\dot{\rho} + 6\rho\|u\|_{H^s}^2)\|u\|_{s+\frac{1}{2},\rho}^2 + 3c\|u\|_{H^s}^4,
\end{aligned}
\tag{2.6}
$$

where $Tu = (1-\Delta)^{\frac{s}{2}}e^{\rho\sqrt{-\Delta}}u$ and $\dot{\rho}$ denotes the time derivative of ρ.

Due to $\rho(t) \le \rho(0)\exp(-6\|u\|_{H^s}^2 t)$, we obtain

$$\dot{\rho} + 6\rho\|u\|_{H^s}^2 \le 0. \tag{2.7}$$

In view of (2.6) and (2.7), it follows that

$$\frac{d}{dt}\|u\|_{s,\rho}^2 \le 6c\|u\|_{H^s}^4. \tag{2.8}$$

By virtue of Lemma 2.2, there exists a function $\mathcal{R}(t)$, with $\|u\|_{H^s} \le \mathcal{R}(t)$. Moreover, for any T_{u_0}, there exists $\mathcal{M}(T_{u_0}) > 0$ such that

$$\int_0^t \mathcal{R}^4(s)ds \le \mathcal{M}(T_{u_0}), \quad \forall t \in [0, T_{u_0}].$$

Solving inequality (2.8) yields that

$$\|u\|_{s,\rho}^2 \le \|u_0\|_{s,\rho}^2 + 6c\mathcal{M}(T_{u_0}),$$

which concludes the proof of Theorem 2.2. □

3. Analytic solutions for some other equations

In this section, we study the analytic solutions to the following family of third order dispersive PDE conservation laws[15]:

$$u_t + c_0u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = (c_1u^2 + c_2u_x^2 + c_3uu_{xx})_x,\ t>0,\ x\in\mathbb{R}, \tag{3.1}$$

where α, c_0, c_1, c_2, and c_3 are real constants and subscripts denote partial derivatives.

When $c_1 = -\frac{3}{2}$, $c_3 = 2c_2 = \alpha^2$, Eq.(3.1) becomes the D–G–H equation [13],

$$
\begin{cases}
u_t - \alpha^2u_{txx} + c_0u_x + 3uu_x + \gamma u_{xxx} = & \alpha^2(2uu_{xx} + uu_{xxx}), \\
 & t>0,\ x\in\mathbb{R}, \\
u(0,x) = u_0(x), & x\in\mathbb{R},
\end{cases}
\tag{3.2}
$$

where the constant α^2 and γ/c_0 are squares of length scales, and the constant $c_0 = \sqrt{gh} > 0$ is the critical shallow water speed for undisturbed water at rest at spatial infinity, where h is the mean fluid depth and $g = 9.8\ m/s^2$ is the gravitation constant.

Since $\alpha^2 > 0$, we can rewrite Eq.(3.2) in the following form:

$$
(3.3) \qquad \begin{cases} u_t + uu_x - \frac{\gamma}{\alpha^2} u_x = & -\partial_x (1-\partial_x^2)^{-1} \left(u^2 + \frac{\alpha^2}{2} u_x^2 + \left(c_0 + \frac{\gamma}{\alpha^2}\right) u \right), \\ & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}
$$

where $u_0(x) \in H^{1,\rho_0},\ \rho_0 > 0$.

Similarly to Section 2.1, it is not difficult to obtain the unique local analytic solutions in time of Eq.(3.3). In view of the following lemma, analogous to Section 2.2, we also can get the global analytic solutions in time of Eq.(3.3).

Lemma 3.1 ([26])**.** *Let $\gamma = -c_0\alpha^2$. Assume the initial datum $u_0 \in H^s(\mathbb{R}),\ s > \frac{3}{2}$, and $y_0(x) = u_0(x) - \alpha^2 u_{0,xx}(x)$ satisfies $y_0(x) \le 0$ for $x \in (-\infty, x_0]$, but $y_0(x) \ge 0$ for $x \in [x_0, \infty)$ for some point $x_0 \in \mathbb{R}$, and y_0 changes sign. Then there exists a unique global solution $u(t,x)$ with the corresponding initial datum $u_0(x)$ to Eq.*(3.3) *that satisfies $u(t,x) \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^s(\mathbb{R}))$. Moreover,*

$$
u_x(t,x) \ge -\frac{1}{|\alpha|}|u(t,x)|.
$$

When $c_0 = \gamma = 0,\ \alpha = c_3 = 1$ and $c_1 + c_2 = -1,\ 2c_2 + 1 = b$, Eq.(3.1) becomes the b-family of equations

$$
(3.4) \qquad u_t - u_{txx} + c_0 u_x + (b+1) u u_x = b u_x u_{xx} + u u_{xxx},\ t > 0,\ x \in \mathbb{R},
$$

where b is a balance or bifurcation parameter.

Note that Eq.(3.4) is equivalent to the following form:

$$
(3.5) \qquad \begin{cases} u_t + uu_x + \partial_x (1-\partial_x^2)^{-1} \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 \right) = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}
$$

where $u_0(x) \in H^{1,\rho_0},\ \rho_0 > 0$.

Similar to the method used in Section 2, by virtue of the following lemma, it is not difficult to derive the global analytic solutions in time of Eq.(3.5).

Lemma 3.2 ([19])**.** *Let $b \ge 1$. Suppose that $u_0(x) \in H^s(\mathbb{R}),\ s > \frac{3}{2}$, and $y_0(x) = u_0(x) - u_{0,xx}(x)$ does not change sign. Then the corresponding solution $u(t,x)$ to Eq.*(3.5) *exists globally.*

Remark 3.1*.* The b-family equations (3.4) can be considered as the family of asymptotically equivalent shallow water wave equations that emerge at quadratic order accuracy for any $b \neq 1$ by an appropriate Kodama transformation. Two classical water wave equations as the special cases of Eq.(3.4) are as follows:

(a) If $b = 2$, Eq.(3.4) becomes the Camassa–Holm equation; the analytic solutions were obtained in [20, 27].

(b) If $b = 3$, Eq.(3.4) becomes the Degasperis–Procesi equation.

4. Traveling wave solutions

In this section, we will prove that Eq.(1.1) has a family of traveling wave solutions. First, we present two definitions.

Definition 4.1. A solution $u(t,x)$ to Eq.(1.1) is x-symmetric if there exists a function $b(t) \in C^1(\mathbb{R}_+)$ such that

$$
u(t,x) = u(t, 2b(t) - x), \forall t \in [0, \infty),
$$

for almost every $x \in \mathbb{R}$. We say that $b(t)$ is the symmetric axis of $u(t,x)$.

Definition 4.2. Assume that $u(t,x) \in X(\mathbb{R})$ and satisfies

$$\iint_{\mathbb{R}_+\times\mathbb{R}} \left[u(1-\partial_x^2)\varphi_t - \frac{1}{2}u_x^3\varphi + (\frac{4}{3}u^3 + \frac{1}{2}uu_x^2)\varphi_x - \frac{u^3}{3}\varphi_{xxx} \right] dtdx = 0, \tag{4.1}$$

for all $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$. Then $u(t,x)$ is a weak solution to the Novikov equation, where $X(\mathbb{R}) = \{u : u \in C(\mathbb{R}_+, H^1(\mathbb{R})),\ u_x \in L^3_{loc}(\mathbb{R})\}$.

Remark 4.1. Since $C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$ is dense in $C_0^1(\mathbb{R}_+, C_0^3(\mathbb{R}))$, by the density argument, we can consider the test functions belonging to $C_0^1(\mathbb{R}_+, C_0^3(\mathbb{R}))$. Using the $\langle \cdot, \cdot \rangle$ notation for distributions, we can rewrite (4.1) as follows:

$$\langle u, (1-\partial_x^2)\varphi_t \rangle - \left\langle \frac{u_x^3}{2}, \varphi \right\rangle + \left\langle \frac{4}{3}u^3 + \frac{1}{2}uu_x^2, \varphi_x \right\rangle - \left\langle \frac{u^3}{3}, \varphi_{xxx} \right\rangle = 0. \tag{4.2}$$

Lemma 4.1. *Assume that $U(x) \in X(\mathbb{R})$ and satisfies*

$$\int_{\mathbb{R}} \left[-cU(1-\partial_x^2)\phi_x - \frac{1}{2}U_x^3\phi + (\frac{4}{3}U^3 + \frac{1}{2}UU_x^2)\phi_x - \frac{U^3}{3}\phi_{xxx} \right] dx = 0, \tag{4.3}$$

for all $\phi \in C_0^\infty(\mathbb{R})$. Then u, given by

$$u(t,x) = U(x - c(t-t_0)), \tag{4.4}$$

is a weak solution of Eq.(1.1)*, for any fixed $t_0 \in \mathbb{R}$.*

Proof. Without loss of generality, we can assume $t_0 = 0$. For all $\eta \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$, let $\eta_c = \eta(t, x+ct)$; it follows that

$$\begin{cases} \partial_x(\eta_c) = (\eta_x)_c, \\ \partial_t(\eta_c) = (\eta_t)_c + c(\eta_x)_c. \end{cases} \tag{4.5}$$

Assume $u(t,x) = U(x - c(t-t_0))$. One can easily check that

$$\begin{cases} \langle u, \eta \rangle = \langle U, \eta_c \rangle,\ \langle u^3, \eta \rangle = \langle U^3, \eta_c \rangle, \\ \langle uu_x^2, \eta \rangle = \langle UU_x^2, \eta_c \rangle,\ \langle u_x^3, \eta \rangle = \langle U_x^3, \eta_c \rangle, \end{cases} \tag{4.6}$$

where $U = U(x)$. In view of (4.5) and (4.6), we have

$$\begin{aligned} &\langle u, (1-\partial_x^2)\eta_t \rangle - \left\langle \frac{1}{2}u_x^3, \eta \right\rangle = \langle U, ((1-\partial_x^2)\eta_t)_c \rangle - \left\langle \frac{1}{2}U_x^3, \eta_c \right\rangle \\ &= \langle U, (1-\partial_x^2)(\partial_t\eta_c - c\partial_x\eta_c) \rangle - \left\langle \frac{1}{2}U_x^3, \eta_c \right\rangle \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &\left\langle \frac{4}{3}u^3 + \frac{1}{2}uu_x^2, \eta_x \right\rangle - \left\langle \frac{u^3}{3}, \eta_{xxx} \right\rangle = \left\langle \frac{4}{3}U^3 + \frac{1}{2}UU_x^2, (\eta_x)_c \right\rangle - \left\langle \frac{U^3}{3}, (\eta_{xxx})_c \right\rangle \\ &= \left\langle \frac{4}{3}U^3 + \frac{1}{2}UU_x^2, \partial_x\eta_c \right\rangle - \left\langle \frac{U^3}{3}, \partial_x^3\eta_c \right\rangle. \end{aligned} \tag{4.8}$$

Note that U is independent of time, and for T large enough such that it does not belong to the support of η_c, we deduce that

$$\begin{aligned}\langle U,(1-\partial_x^2)\partial_t\eta_c\rangle &= \int_{\mathbb{R}} U(x)\int_{\mathbb{R}_+}\partial_t(1-\partial_x^2)\eta_c dtdx \\ &= \int_{\mathbb{R}} U(x)[(1-\partial_x^2)\eta_c(T,x)-(1-\partial_x^2)\eta_c(0,x)]dx = 0.\end{aligned} \tag{4.9}$$

Combining (4.7), (4.8) with (4.9), it follows that

$$\begin{aligned}&\langle u,(1-\partial_x^2)\eta_t\rangle - \left\langle \frac{u_x^3}{2},\eta\right\rangle + \left\langle \frac{4}{3}u^3+\frac{1}{2}uu_x^2,\eta_x\right\rangle - \left\langle \frac{u^3}{3},\eta_{xxx}\right\rangle \\ &= \langle U,-c(1-\partial_x^2)\partial_x\eta_c\rangle - \left\langle \frac{U_x^3}{2},\eta_c\right\rangle + \left\langle \frac{4}{3}U^3+\frac{1}{2}UU_x^2,\partial_x\eta_c\right\rangle - \left\langle \frac{U^3}{3},\partial_x^3\eta_c\right\rangle \\ &= \int_{\mathbb{R}_+}\int_{\mathbb{R}}\left[-cU(1-\partial_x^2)\partial_x\eta_c - \frac{U_x^3}{2}\eta_c + (\frac{4}{3}U^3+\frac{1}{2}UU_x^2)\partial_x\eta_c - \frac{U^3}{3}\partial_x^3\eta_c\right]dxdt \\ &= 0,\end{aligned}$$

where we have used (4.3) with $\phi(x)=\eta_c(t,x)$, which belongs to $C_0^\infty(\mathbb{R})$, for every given $t\geq 0$. This concludes the proof of Lemma 4.1. □

Finally, for the x-symmetric solutions of Eq.(1.1), the following theorem holds.

Theorem 4.1. *Let $u(t,x)$ be x-symmetric. If $u(t,x)$ is a unique weak solution of Eq.*(1.1)*, then $u(t,x)$ is a traveling wave solution.*

Proof. In view of Remark 4.1, we can assume that $\varphi\in C_0^1(\mathbb{R}_+,C_0^3(\mathbb{R}))$. Let

$$\varphi_b(t,x)=\varphi(t,2b(t)-x), \qquad b(t)\in C^1(\mathbb{R}).$$

Then we obtain that $(\varphi_b)_b=\varphi$ and

$$\begin{cases} \partial_x u_b = -(\partial_x u)_b,\ \partial_x\varphi_b = -(\partial_x\varphi)_b, \\ \partial_t\varphi_b = (\partial_t\varphi)_b + 2\dot{b}(\partial_x\varphi)_b. \end{cases} \tag{4.10}$$

Moreover,

$$\begin{cases} \langle u_b,\varphi\rangle = \langle u,\varphi_b\rangle,\ \langle u_b^3,\varphi\rangle = \langle u^3,\varphi_b\rangle, \\ \langle u_b(\partial_x u_b)^2,\varphi\rangle = \langle u(\partial_x u)^2,\varphi_b\rangle,\ \langle(\partial_x u_b)^3,\varphi\rangle = \langle -(\partial_x u)^3,\varphi_b\rangle, \end{cases} \tag{4.11}$$

where $\dot{b}$ denotes the time derivative of b.

Since u is x-symmetric, by virtue of (4.10) and (4.11), one has that

$$\begin{aligned}&\langle u,(1-\partial_x^2)\partial_t\varphi\rangle - \left\langle \frac{(\partial_x u)^3}{2},\varphi\right\rangle = \langle u,((1-\partial_x^2)\partial_t\varphi)_b\rangle + \left\langle \frac{(\partial_x u)^3}{2},\varphi_b\right\rangle \\ &= \langle u,(1-\partial_x^2)(\partial_t\varphi_b+2\dot{b}\partial_x\varphi_b)\rangle + \left\langle \frac{(\partial_x u)^3}{2},\varphi_b\right\rangle\end{aligned}$$

and

$$\begin{aligned}&\left\langle \frac{4}{3}u^3+\frac{u}{2}(\partial_x u)^2,\partial_x\varphi\right\rangle - \left\langle \frac{u^3}{3},\partial_x^3\varphi\right\rangle \\ &= \left\langle \frac{4}{3}u^3+\frac{u}{2}(\partial_x u)^2,-\partial_x\varphi_b\right\rangle - \left\langle \frac{u^3}{3},-\partial_x^3\varphi_b\right\rangle.\end{aligned}$$

Then (4.2) and the above relations yield

$$\begin{aligned}
&\langle u,(1-\partial_x^2)\varphi_t\rangle-\left\langle \frac{u_x^3}{2},\varphi\right\rangle+\left\langle \frac{4}{3}u^3+\frac{1}{2}uu_x^2,\varphi_x\right\rangle-\left\langle \frac{u^3}{3},\varphi_{xxx}\right\rangle\\
&=\langle u,(1-\partial_x^2)(\partial_t\varphi_b+2\dot{b}\partial_x\varphi_b)\rangle+\left\langle \frac{(\partial_x u)^3}{2},\varphi_b\right\rangle\\
&\quad+\left\langle \frac{4}{3}u^3+\frac{u}{2}(\partial_x u)^2,-\partial_x\varphi_b\right\rangle+\left\langle \frac{u^3}{3},\partial_x^3\varphi_b\right\rangle=0.
\end{aligned}\tag{4.12}$$

Thus, using φ in place of φ_b in (4.12), as $(\varphi_b)_b=\varphi$, we can get

$$\begin{aligned}
&\langle u,(1-\partial_x^2)(\partial_t\varphi+2\dot{b}\partial_x\varphi)\rangle+\left\langle \frac{(\partial_x u)^3}{2},\varphi\right\rangle\\
&\qquad+\left\langle \frac{4}{3}u^3+\frac{u}{2}(\partial_x u)^2,-\partial_x\varphi\right\rangle+\left\langle \frac{u^3}{3},\partial_x^3\varphi\right\rangle=0.
\end{aligned}\tag{4.13}$$

Subtracting (4.13) from (4.2), we obtain

$$\langle u,-2\dot{b}(1-\partial_x^2)\partial_x\varphi\rangle-\langle(\partial_x u)^3,\varphi\rangle+\left\langle \frac{8}{3}u^3+u(\partial_x u)^2,\partial_x\varphi\right\rangle-\left\langle \frac{2u^3}{3},\partial_x^3\varphi\right\rangle=0.\tag{4.14}$$

For any $\phi\in C_0^\infty(\mathbb{R})$, let $\varphi_\varepsilon(t,x)=\phi(x)\rho_\varepsilon(t)$, where $\rho_\varepsilon\in C_0^\infty(\mathbb{R}_+)$ is a mollifier with the property that $\rho_\varepsilon\to\delta(t-t_0)$, the Dirac mass at t_0, as $\varepsilon\to 0$. From (4.14), by using the test function $\varphi_\varepsilon(t,x)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\left(-2(1-\partial_x^2)\partial_x\phi\int_{\mathbb{R}_+}\dot{b}u\rho_\varepsilon(t)dt\right)dx-\int_{\mathbb{R}}\left(\phi\int_{\mathbb{R}_+}(\partial_x u)^3\rho_\varepsilon(t)dt\right)dx\\
&+\int_{\mathbb{R}}\left(\partial_x\phi\int_{\mathbb{R}_+}\left(\frac{8}{3}u^3+u(\partial_x u)^2\right)\rho_\varepsilon(t)dt\right)dx\\
&-\int_{\mathbb{R}}\left(\frac{2}{3}\partial_x^3\phi\int_{\mathbb{R}_+}u^3\rho_\varepsilon(t)dt\right)dx=0.
\end{aligned}\tag{4.15}$$

Note that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+}\dot{b}u\rho_\varepsilon(t)dt=\dot{b}(t_0)u(t_0,x)\quad\text{in } L^2(\mathbb{R}),$$

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+}((\partial_x u)^3\rho_\varepsilon(t)+u^3\rho_\varepsilon(t))dt=(\partial_x u(t_0,x))^3+u^3(t_0,x),$$

and

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+}\left(\frac{8}{3}u^3+u(\partial_x u)^2\right)\rho_\varepsilon(t)dt=\frac{8}{3}u^3(t_0,x)+u(\partial_x u)^2(t_0,x),$$

in $L^1(\mathbb{R})$. Therefore, letting $\varepsilon\to 0$, (4.15) implies that

$$\begin{aligned}
&\int_{\mathbb{R}}\left(-\dot{b}(t_0)u(t_0,x)(1-\partial_x^2)\partial_x\phi-\frac{1}{2}(\partial_x u(t_0,x))^3\phi\right)dx\\
&+\int_{\mathbb{R}}\left((\frac{4}{3}u^3(t_0,x)+(u(\partial_x u)^2)(t_0,x))\partial_x\phi-\frac{1}{3}u^3(t_0,x)\partial_x^3\phi\right)dx=0.
\end{aligned}\tag{4.16}$$

Thus, we deduce that $u(t_0,x)$ satisfies (4.3) for $c=\dot{b}(t_0)$. Applying Lemma 4.1, we can get that $\tilde{u}(t,x)=u(t_0,x-\dot{b}(t_0)(t-t_0))$ is a traveling wave solution of Eq.(1.1).

Since $\tilde{u}(t_0, x) = u(t_0, x)$, by the uniqueness of the solution of Eq.(1.1), we obtain $\tilde{u}(t, x) = u(t, x)$, for all time t. This completes the proof of Theorem 4.1. $\square$

Remark 4.1. Theorem 2.1 in [38] and Theorem 4.1 in [39] ensure the existence and uniqueness of the solution $u(t, x)$ to Eq.(1.1). In both cases, one can consider the traveling wave solution. Moreover, we get the result of Theorem 4.1.

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