PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 4, April 2018, Pages 1417–1430
http://dx.doi.org/10.1090/proc/13754
Article electronically published on December 26, 2017

GENERALIZING SERRE'S SPLITTING THEOREM AND BASS'S CANCELLATION THEOREM VIA FREE-BASIC ELEMENTS

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(Communicated by Irena Peeva)

Abstract. We give new proofs of two results of Stafford, which generalize two famous Theorems of Serre and Bass regarding projective modules. Our techniques are inspired by the theory of basic elements. Using these methods we further generalize Serre's Splitting Theorem by imposing a condition to the splitting maps, which has an application to the case of Cartier algebras.

1. Introduction

Throughout this section, R denotes a commutative Noetherian ring with unity. The existence of unimodular elements has always played a crucial role in algebraic K-theory. An element of a projective module is called unimodular if it generates a free summand. Note that unimodular elements are basic elements, a notion introduced by Swan [Swa67]: an element of a module is called basic if it remains a minimal generator after localization at every prime of the ring. In a fundamental paper from 1973, Eisenbud and Evans make use of basic elements for modules that are not necessarily projective. Using basic element theory, Eisenbud and Evans generalize and greatly improve several theorems about projective modules, as well as about general finitely generated modules [EE73]. Two such theorems are Serre's Splitting Theorem and Bass's Cancellation Theorem on projective modules.

Theorem 1.1 ([Ser58, Theorem 1] and [Bas64, Theorem 8.2])**.** *Let R be a commutative Noetherian ring of Krull dimension d. Let P be a finitely generated projective R-module whose rank at each localization at a prime ideal is at least $d+1$. Then P contains a free R-summand.*

Theorem 1.2 ([Bas64, Theorem 9.1])**.** *Let R be a commutative Noetherian ring of Krull dimension d, and P be a finitely generated projective R-module whose rank at each localization at a prime ideal is at least $d+1$. Let Q be a finitely generated projective R-module, and assume that $Q \oplus P \cong Q \oplus N$ for some R-module N. Then $P \cong N$.*

Given a projective module P, an equivalent way of saying that $P_{\mathfrak{p}}$ has rank at least $d+1$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ is to say that, locally at each prime, P contains a free summand of rank at least $d+1$. In this article, we present new methods to extend

Received by the editors September 20, 2016 and, in revised form, March 30, 2017.
2010 *Mathematics Subject Classification.* Primary 13C10; Secondary 13D15.
Key words and phrases. Free-basic elements, local and global free summands, projective modules.

Theorems 1.1 and 1.2 to all finitely generated modules, not necessarily projective, that contain a free summand of large enough rank at each localization at a prime.

Theorem A. *Let R be a commutative Noetherian ring, and M a finitely generated R-module. Assume that $M_{\mathfrak{p}}$ contains a free $R_{\mathfrak{p}}$ summand of rank at least* $\dim(R/\mathfrak{p})+1$ *for each $\mathfrak{p} \in \operatorname{Spec}(R)$. Then M contains a free R-summand.*

Theorem B. *Let R be a commutative Noetherian ring and M a finitely generated R-module such that, for each $\mathfrak{p} \in \operatorname{Spec}(R)$, $M_{\mathfrak{p}}$ contains a free $R_{\mathfrak{p}}$ summand of rank at least* $\dim(R/\mathfrak{p})+1$. *For any finitely generated projective R-module Q and any finitely generated R-module N, if $Q \oplus M \cong Q \oplus N$, then $M \cong N$.*

Stafford reached analogous conclusions in [Sta81, Corollaries 5.9 and 5.11], where he used the notion of r-rank. Stafford's results are actually very general, in that he proved analogues of Theorem A and Theorem B for rings that are not necessarily commutative. In the assumptions of Theorem A and Theorem B, the r-rank of a module M carries essentially the same information that is recorded by the rank of a local free summand of M, that we consider instead. However, in Section 3, using our approach we prove stronger results under more general assumptions.

Let M be a finitely generated R-module. Observe that the conclusion of Theorem A can be viewed as a statement about the existence of a surjective homomorphism inside $\operatorname{Hom}_R(M,R)$. For several applications, it is useful to restrict the selection of homomorphisms $M \to R$ to those belonging to a given R-submodule $\mathscr{E}$ of $\operatorname{Hom}_R(M,R)$. This is the scenario arising, for instance, from the study of the F-signature of Cartier subalgebras of $\mathscr{C}^R = \bigoplus_e \operatorname{Hom}_R(F^e_* R, R)$, where R is an F-finite local ring of prime characteristic [BST12]. Our main achievement in this direction is the following generalization of Theorem A.

Theorem C. *Let R be a commutative Noetherian ring, M a finitely generated R-module, and $\mathscr{E}$ an R-submodule of* $\operatorname{Hom}_R(M,R)$. *Assume that, for each $\mathfrak{p} \in \operatorname{Spec}(R)$, $M_{\mathfrak{p}}$ contains a free $\mathscr{E}_{\mathfrak{p}}$-summand of rank at least* $\dim(R/\mathfrak{p})+1$. *Then M contains a free $\mathscr{E}$-summand.*

The terminology "free $\mathscr{E}_{\mathfrak{p}}$-summand" of Theorem C will be explained in detail in Section 3. As the main application of Theorem C, these authors establish in [DSPY16] the existence of a global F-signature with respect to Cartier subalgebras of $\mathscr{C}^R$, where R is an F-finite ring of prime characteristic, not necessarily local.

The statements of Theorems A, B and C can actually be strengthened by just requiring that $M_{\mathfrak{p}}$ has a free summand (free $\mathscr{E}_{\mathfrak{p}}$-summand for Theorem C) of rank at least $1 + \dim_{j\text{-}\operatorname{Spec}(R)}(\mathfrak{p})$, for each $\mathfrak{p} \in j\text{-}\operatorname{Spec}(R)$. Here, j-$\operatorname{Spec}(R)$ is the set of all primes that can be written as an intersection of maximal ideals, while $\dim_{j\text{-}\operatorname{Spec}(R)}(\mathfrak{p})$ denotes the supremum over the lengths of increasing chains of primes in j-$\operatorname{Spec}(R)$ that start with $\mathfrak{p}$. In addition, we can get even stronger conclusions, provided some additional assumptions hold. Namely, if there exists a fixed R-submodule N of M and a positive integer i such that, for all $\mathfrak{p} \in j$-$\operatorname{Spec}(R)$, $M_{\mathfrak{p}}$ has a free $R_{\mathfrak{p}}$-summand (free $\mathscr{E}_{\mathfrak{p}}$-summand for Theorem C) of rank at least $\dim_{j\text{-}\operatorname{Spec}(R)}(\mathfrak{p}) + i$ that is contained in $N_{\mathfrak{p}}$, then the global free R-summand of M can be realized to be of rank at least i, and inside N.

This article is structured as follows: in Section 2, we present a version of Theorem A via rather direct and elementary techniques. However, the methods we employ are not effective enough to prove the result in its full generality. In Section 3, we introduce the notion of free-basic element, which allows us to prove Theorems A, B and C in their full generality. We believe that free-basic elements are interesting and worth exploring on their own, as they share many good properties both with basic elements and unimodular elements.

2. An elementary approach to Theorem A

Throughout this section, R is a commutative Noetherian ring with identity, and of finite Krull dimension d. Given an ideal I of R, we denote by $V(I)$ the closed set of $\mathrm{Spec}(R)$ consisting of all prime ideals that contain I. Given $s \in R$, we denote by $D(s)$ the open set consisting of all prime ideals that do not contain s. Recall that $\{D(s) \mid s \in R\}$ is a basis for the Zariski topology on $\mathrm{Spec}(R)$, and that $D(s)$ can be identified with $\mathrm{Spec}(R_s)$. We present an elementary approach to Theorem A, which will lead to a proof in the case when R has infinite residue fields. We first need to recall some facts about the symmetric algebra of a module. If M is a non-zero finitely generated R-module, we denote by $\mathrm{Sym}_R(M)$ the symmetric algebra of M over R. Furthermore, we denote by $\mu_R(M)$ the minimal number of generators of M as an R-module. Given a presentation

$$R^{n_2} \xrightarrow{(a_{ij})} R^{n_1} \longrightarrow M \longrightarrow 0$$

then we can describe $\mathrm{Sym}_R(M)$, as an R-algebra, in the following way:

$$\mathrm{Sym}_R(M) \cong R[x_1, x_2, \dots, x_{n_1}]/J, \quad \text{where } J = \left(\sum_{i=1}^{n_1} a_{ij} x_i \mid 1 \leqslant j \leqslant n_2 \right).$$

When M is free of rank n, then $\mathrm{Sym}_R(M)$ is just a polynomial ring over R, in n indeterminates. Huneke and Rossi give a formula for the Krull dimension of a symmetric algebra.

Theorem 2.1 ([HR86, Theorem 2.6 (ii)])**.** *Let R be a ring which is the homomorphic image of a Noetherian universally catenary domain of finite dimension. Let M be a finitely generated R-module; then* $\dim(\mathrm{Sym}_R(M)) = \max_{\mathfrak{p} \in \mathrm{Spec}(R)} \{\dim(R/\mathfrak{p}) + \mu_{R_\mathfrak{p}}(M_\mathfrak{p})\}$.

Now, suppose that P is a finitely generated projective R-module, and suppose we have a surjection $R^n \to P \to 0$. Let Q be its kernel. Applying $(-)^* := \mathrm{Hom}_R(-, R)$ to the split short exact sequence $0 \to Q \to R^n \to P \to 0$ gives another short exact sequence

$$0 \longrightarrow P^* \longrightarrow (R^n)^* \cong R^n \longrightarrow Q^* \longrightarrow 0.$$

This provides a natural way to view $\mathrm{Sym}_R(Q^*)$ as the homomorphic image of $R[x_1, \dots, x_n] \cong \mathrm{Sym}_R(R^n)$. Moreover, as Q is projective, if $\mathfrak{m} \in \mathrm{Max}(R)$ is a maximal ideal, then $(Q/\mathfrak{m}Q)^* \cong Q^* \otimes_R R/\mathfrak{m}$, where the first $(-)^*$ is over $R/\mathfrak{m}$ while the second is over R. In addition, we have $\mathrm{Sym}_{R/\mathfrak{m}}(Q^* \otimes_R R/\mathfrak{m}) \cong \mathrm{Sym}_R(Q^*) \otimes_R R/\mathfrak{m}$. In fact, for any finitely generated R-module M, we have

$$\mathrm{Sym}_{R/\mathfrak{m}}(M \otimes_R R/\mathfrak{m}) \cong \mathrm{Sym}_R(M) \otimes_R R/\mathfrak{m}.$$

To see this, one only needs to notice that if $R^{n_2} \to R^{n_1} \to M \to 0$ is a presentation of M as an R-module, then applying $- \otimes_R R/\mathfrak{m}$ gives a presentation of $M \otimes_R R/\mathfrak{m}$ as an $R/\mathfrak{m}$-vector space. The following lemma shows how the symmetric algebra can detect whether an element of a projective module is locally a minimal generator or not.

Lemma 2.2. *Let R be a commutative Noetherian ring, P be a finitely generated projective R-module with generators $\eta_1, \ldots, \eta_n$, and Q be the kernel of the surjective map $R^n \to P$ which sends $e_i \mapsto \eta_i$. For any $r_1, \ldots, r_n \in R$ and $\mathfrak{m} \in \operatorname{Spec}(R)$, we have $r_1\eta_1 + \cdots + r_n\eta_n \in \mathfrak{m}P_\mathfrak{m}$ if and only if $(\mathfrak{m}, x_1 - r_1, \ldots, x_n - r_n)$ is a maximal ideal of $\operatorname{Sym}_R(Q^*)$.*

Proof. Checking that $r_1\eta_1+\cdots+r_n\eta_n \in \mathfrak{m}P_\mathfrak{m}$ is equivalent to checking that $r_1\eta_1 + \cdots + r_n\eta_n = 0$ in $P_\mathfrak{m}/\mathfrak{m}P_\mathfrak{m} \cong P/\mathfrak{m}P$. Let $\kappa := R/\mathfrak{m}$. As P is projective, tensoring with κ gives a short exact sequence $0 \to Q/\mathfrak{m}Q \to \kappa^n \to P/\mathfrak{m}P \to 0$ of κ-vector spaces. Let $t = \dim_\kappa(P/\mathfrak{m}P)$, and choose a basis for $P/\mathfrak{m}P \cong \kappa^t$. Let (a_{ij}) be the $t \times n$ matrix representing the onto map $\kappa^n \to P/\mathfrak{m}P$. After dualizing, we see that $\operatorname{Sym}_\kappa((Q/\mathfrak{m}Q)^*) \cong \kappa[x_1, \ldots, x_n]/J$ where $J = (\sum_{i=1}^n a_{ji}x_i \mid 1 \leqslant j \leqslant t)$. Furthermore, observe that the maximal ideal $(x_1 - r_1, \ldots, x_n - r_n) \in \operatorname{Max}(\kappa[x_1, \ldots, x_n])$ contains J if and only if $\sum_{i=1}^n a_{ji}r_i = 0$ for each $1 \leqslant j \leqslant t$. This happens if and only if $(r_1, \ldots, r_n)(a_{ij})^T = 0$, which in turn is equivalent to $(a_{ij})(r_1, \ldots, r_n)^T = 0$. Finally, since (a_{ij}) is the matrix representing $\kappa^n \to P/\mathfrak{m}P$, this happens if and only if $r_1\eta_1 + \cdots + r_n\eta_n = 0$ in $P/\mathfrak{m}P$. Therefore, $r_1\eta_1+\ldots+r_n\eta_n \in \mathfrak{m}P_\mathfrak{m}$ if and only if $(x_1-r_1, \ldots, x_n-r_n) \in \operatorname{Max}(\operatorname{Sym}_\kappa(Q/\mathfrak{m}Q)^*) = \operatorname{Max}(\operatorname{Sym}_R(Q^*) \otimes R/\mathfrak{m})$, if and only if $(\mathfrak{m}, x_1 - r_1, \ldots, x_n - r_n) \in \operatorname{Max}(\operatorname{Sym}_R(Q^*))$. $\square$

Using Lemma 2.2, we can give a proof of Serre's Splitting Theorem 1.1 [Ser58, Theorem 1] for algebras of finite type over an algebraically closed field. The main purpose of showing this argument here is for the reader to get more familiar with the techniques that will be employed and generalized later in this section. We thank Mohan Kumar for suggesting this proof to us, and for pointing out how to generalize it to any finitely generated module in the case when R is an affine algebra over an algebraically closed field.

Theorem 2.3. *Let R be a d-dimensional ring of finite type over an algebraically closed field k, and P be a finitely generated projective R-module whose rank at each localization at a prime ideal is at least $d + 1$. Then P contains a free R-summand.*

Proof. Let $\eta_1, \ldots, \eta_n$ be generators of P and Q be the kernel of the natural surjection $R^n \to P$. Then $\dim(\operatorname{Sym}_R(Q^*)) = \max_{\mathfrak{p}\in\operatorname{Spec}(R)}\{\dim(R/\mathfrak{p}) + \mu_{R_\mathfrak{p}}((Q^*)_\mathfrak{p})\}$ by Theorem 2.1. Observe that Q is a locally free R-module of rank equal to $n - \operatorname{rank}(P)$. Therefore Q^* is locally free of the same rank, and thus $\dim(\operatorname{Sym}_R(Q^*)) \leqslant d + n - d - 1 = n - 1$. Let I be the ideal of $k[x_1, \ldots, x_n]$ defining the kernel of the composition $k[x_1, \ldots, x_n] \subseteq R[x_1, \ldots, x_n] \to \operatorname{Sym}_R(Q^*)$. As $\dim(k[x_1, \ldots, x_n]) = n > \dim(\operatorname{Sym}_R(Q^*))$ it must be the case that $I \neq 0$ and, by the Nullstellensatz, there must be a maximal ideal $(x_1 - r_1, \ldots, x_n - r_n)$ of $k[x_1, \ldots, x_n]$ not containing I. In particular, for each $\mathfrak{m} \in \operatorname{Max}(R)$, the ideal $(\mathfrak{m}, x_1 - r_1, \ldots, x_n - r_n)$ cannot be a maximal ideal of $\operatorname{Sym}_R(Q^*)$. By Lemma 2.2, $r_1\eta_1 + \ldots + r_n\eta_n$ is then a minimal generator of $P_\mathfrak{m}$ for each $\mathfrak{m} \in \operatorname{Max}(R)$. Therefore, the map $R \to P$ sending $1 \mapsto r_1\eta_1+\cdots+r_n\eta_n$ splits locally at each $\mathfrak{m} \in \operatorname{Max}(R)$, and hence it splits globally. $\square$

We now present a generalization of the techniques just employed. This allows us to prove Theorem A for rings R of which all the residue fields $R/\mathfrak{m}$, for $\mathfrak{m} \in \operatorname{Max}(R)$, are infinite. This includes rings that contain an infinite field. We first prove an auxiliary lemma.

Lemma 2.4. *Let R be a commutative Noetherian ring of finite dimension d such that all residue fields $R/\mathfrak{m}$, for $\mathfrak{m} \in \operatorname{Max}(R)$, are infinite. Let $R[x_1, \ldots, x_n]$ be a polynomial algebra over R, and let $J \subseteq R[x_1, \ldots, x_n]$ be an ideal with $\operatorname{ht}(J) \geqslant d+1$. For all integers $0 \leqslant \ell \leqslant n$, there exist $r_1, \ldots, r_\ell \in R$ such that either $J+(x_1-r_1, \ldots, x_\ell-r_\ell) = R[x_1, \ldots, x_n]$, or $\operatorname{ht}(J+(x_1-r_1, \ldots, x_\ell-r_\ell)) \geqslant d+\ell+1$ (when $\ell = 0$, the reader should think of $J + (x_1 - r_1, \ldots, x_\ell - r_\ell)$ as being equal to J).*

Proof. We proceed by induction on ℓ, where the case $\ell = 0$ follows from our assumptions. Let $1 \leqslant \ell \leqslant n$. By inductive hypothesis, we can find $r_1, \ldots, r_{\ell-1}$ such that either $J + (x_1 - r_1, \ldots, x_{\ell-1} - r_{\ell-1}) = R[x_1, \ldots, x_n]$, or

$$\operatorname{ht}(J + (x_1 - r_1, \ldots, x_{\ell-1} - r_{\ell-1})) \geqslant d + \ell.$$

In the first case, any choice of r_ℓ will yield $J+(x_1-r_1, \ldots, x_\ell-r_\ell) = R[x_1, \ldots, x_n]$. In the second case, we have that $J + (x_1 - r_1, \ldots, x_{\ell-1} - r_{\ell-1})$ is a proper ideal of height at least $d+\ell$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the minimal primes over $J+(x_1-r_1, \ldots, x_{\ell-1}-r_{\ell-1})$. For each $1 \leqslant j \leqslant t$, let $\mathfrak{q}_j = \mathfrak{p}_j \cap R$, and let $H_j := \{r \in R \mid x_\ell - r \in \mathfrak{p}_j\}$. We want to show that $\bigcup_{j=1}^t H_j \neq R$. For this matter, from now on we can deal only with those indices $1 \leqslant j \leqslant t$ for which $H_j \neq \emptyset$. In this case, we see that $H_j = a_j + \mathfrak{q}_j$, i.e., H_j is a coset determined by some $a_j \in R$. For each $\mathfrak{q}_j$, choose a maximal ideal $\mathfrak{m}_j$ of R such that $\mathfrak{m}_j$ contains $\mathfrak{q}_j$. We then have a finite set $Y = \{\mathfrak{m}_j \in \operatorname{Max}(R) \mid H_j \neq \emptyset\}$ of maximal ideals of R. For each $\mathfrak{m} \in Y$, since $R/\mathfrak{m}$ is infinite, there exists $b_\mathfrak{m} \in R$ such that $b_\mathfrak{m} + \mathfrak{m} \notin \{a_j + \mathfrak{m}_j \mid \mathfrak{m}_j = \mathfrak{m}, 1 \leqslant j \leqslant t\}$. By the Chinese Remainder Theorem, there exists $r_\ell \in R$ such that $r_\ell+\mathfrak{m} = b_\mathfrak{m}+\mathfrak{m}$ for all $\mathfrak{m} \in Y$. In particular, r_ℓ avoids all cosets H_j, for $1 \leqslant j \leqslant t$. By definition of H_j, it follows that $x_\ell - r_\ell \notin \bigcup_{j=1}^t \mathfrak{p}_j$, and this implies either that $J + (x_1 - r_1, \ldots, x_\ell - r_\ell) = R[x_1, \ldots, x_n]$, or that $\operatorname{ht}(J + (x_1 - r_1, \ldots, x_\ell - r_\ell)) \geqslant d + \ell + 1$, as desired. □

Remark 2.5. In Lemma 2.4, if R contains an infinite field k, we can actually choose $r_1, \ldots, r_\ell$ to be any generic elements inside k. In fact, given any proper ideal I of $R[x_1, \ldots, x_n]$ and any fixed $f \in R[x_1, \ldots, x_n]$, no two distinct $r, s \in k \subseteq R$ can be such that both $f - r \in I$ and $f - s \in I$, otherwise the unit $r - s$ would be inside I. Therefore, with the same notation as in the inductive step in the proof of the lemma, a generic choice $r_\ell \in k \subseteq R$ is such that $x_\ell - r_\ell$ avoids all the minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ over $J + (x_1 - r_1, \ldots, x_{\ell-1} - r_{\ell-1})$.

We are now ready to present the main result of this section: Theorem A in the case when R has infinite residue fields. As already pointed out in the introduction, we will prove Theorem A in its full generality in Section 3, using different methods.

Theorem 2.6. *Let R be a commutative Noetherian ring of dimension d such that all residue fields $R/\mathfrak{m}$, where $\mathfrak{m} \in \operatorname{Max}(R)$, are infinite. Let M be a finitely generated R-module such that $M_\mathfrak{p}$ has a free summand of rank at least $d+1$ locally at every $\mathfrak{p} \in \operatorname{Spec}(R)$. Then M has a free summand.*

Proof. Suppose that M can be generated by n elements, and fix a surjective map $\pi : R^n \to M$. Our assumptions about local number of free summands of M guarantee that we can cover $\mathrm{Spec}(R)$ with open sets $D(s_1),\dots,D(s_t)$, such that for all $1 \leqslant i \leqslant t$ the module M_{s_i} has an R_{s_i}-free summand of rank at least $d+1$. In particular, for all i, we can find maps $M_{s_i} \to R_{s_i}^{d+1}$ such that the compositions $R_{s_i}^n \to M_{s_i} \to R_{s_i}^{d+1}$ are split surjections. Let Q_i be the kernel of $R_{s_i}^n \to R_{s_i}^{d+1} \to 0$, so that we have a surjection $R_{s_i}[x_1,\dots,x_n] \to \mathrm{Sym}_{R_{s_i}}(Q_i^*) \to 0$ at the level of symmetric algebras. For each i, consider the composition $R[x_1,\dots,x_n] \to R_{s_i}[x_1,\dots,x_n] \to \mathrm{Sym}_{R_{s_i}}(Q_i^*)$, and let $\mathfrak{a}_i \subseteq R[x_1,\dots,x_n]$ be its kernel. We claim that $\mathfrak{a}_i$ is an ideal of height at least $d+1$. In fact, let $\mathrm{Sym}_{R_{s_i}}(Q_i^*) \cong R_{s_i}[x_1,\dots,x_n]/J_i$ be a presentation of the symmetric algebra of Q_i^* over R_{s_i}. Then J_i can be generated by $d+1$ linear forms that, possibly after a change of variables, can be regarded as $d+1$ distinct variables. Therefore J_i has height $d+1$ in $R_{s_i}[x_1,\dots,x_n]$. Let $I_i \subseteq R[x_1,\dots,x_n]$ be any ideal such that $(I_i)_{s_i} = J_i$. Note that the ideal $\mathfrak{a}_i$ is just the saturation $I_i : s_i^\infty = \{f \in R[x_1,\dots,x_n] \mid s_i^m f \in I_i \text{ for some } m \in \mathbb{N}\}$, and this implies that $\mathrm{ht}(\mathfrak{a}_i) \geqslant \mathrm{ht}(J_i) = d+1$, since no minimal primes of $\mathfrak{a}_i$ contain s_i.

Now consider the intersection $\mathfrak{a} = \mathfrak{a}_1 \cap \ldots \cap \mathfrak{a}_n$, which is then an ideal of height at least $d+1$ inside $R[x_1,\dots,x_n]$. Applying Lemma 2.4 to the ideal $\mathfrak{a}$ with $\ell = n$, we obtain elements $r_1,\dots,r_n \in R$ such that either $\mathfrak{a} + (x_1 - r_1,\dots,x_n - r_n) = R[x_1,\dots,x_n]$, or $\mathrm{ht}(\mathfrak{a}+(x_1-r_1,\dots,x_n-r_n)) \geqslant d+n+1$. Since $\dim(R[x_1,\dots,x_n]) = d+n$, the latter cannot happen. This implies that the ideals $\mathfrak{a}_i+(x_1-r_1,\dots,x_n-r_n)$ cannot be contained in $(\mathfrak{m}, x_1 - r_1,\dots,x_n - r_n)$, for any $1 \leqslant i \leqslant t$ and any $\mathfrak{m} \in \mathrm{Max}(R) \cap D(s_i)$. As a consequence, $(\mathfrak{m}, x_1 - r_1,\dots,x_n - r_n)$ is not a maximal ideal of $\mathrm{Sym}_{R_{s_i}}(Q_i^*)$ for any $\mathfrak{m} \in \mathrm{Max}(R) \cap D(s_i)$, for $1 \leqslant i \leqslant t$. Now, Lemma 2.2 implies that the image of $(r_1,\dots,r_n)$ under $R^n \to M$ generates a non-trivial free R_{s_i}-summand locally on $D(s_i)$, for each i. Since the open sets $D(s_i)$ cover $\mathrm{Spec}(R)$, it generates a global free summand of M. ☐

We point out, once again, that we cannot deduce Theorem A in its generality from Theorem 2.6, even after making the residue fields of R infinite under a faithfully flat extension $R \to S$. In fact, given an R-module M, detecting the splittings of $M \otimes_R S$ over S does not necessarily allow one to keep track of the splittings of M over R.

3. Free-basic elements

Throughout this section, R denotes a commutative Noetherian ring with identity. The purpose of this section is to prove Theorems A, B and C in full generality. We first recall the notion of a basic set, while the reader is referred to [EG85] for a more general and detailed treatment of basic sets and basic elements. Also, some of the arguments that we present have a similar flavor as the treatment given for basic elements in [Hun11], which is another possible source for references.

Definition 3.1. A subset $X \subseteq \mathrm{Spec}(R)$ is said to be basic if it is closed under intersections. In other words, for any indexing set Λ and any family $\{\mathfrak{p}_\alpha\}_{\alpha\in\Lambda} \subseteq X$, whenever the intersection $I := \bigcap_{\alpha\in\Lambda} \mathfrak{p}_\alpha$ is a prime ideal, then I is still an element of X.

Examples 3.2. Let R be a commutative Noetherian ring.

(1) $\mathrm{Spec}(R)$ is trivially a basic set.
(2) $\mathrm{Min}(R)$ is a basic set. More generally, for an integer $n \in \mathbb{N}$, the set $X^{(n)} = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathrm{ht}(\mathfrak{p}) \leqslant n\}$ is basic.

(3) The set j-Spec$(R) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p}$ is an intersection of maximal ideals of $R\}$ is basic.

Below is a list of properties about basic sets, which we leave as an exercise to the reader.

Proposition 3.3. *Let R be a commutative Noetherian ring, and X be a basic set.*

(1) If $Y \subseteq X$ is closed, then Y is a basic set.
(2) Every closed set $Y \subseteq X$ is a finite union of irreducible closed sets in X.
(3) If $Y \subseteq X$ is an irreducible closed set, then $Y = V(\mathfrak{p}) \cap X$ for some $\mathfrak{p} \in X$, i.e., Y has a generic point.

Let X be a basic set, and M be a finitely generated R-module. An element $x \in M$ is called X-basic if $x \notin \mathfrak{p}M_{\mathfrak{p}}$, for all $\mathfrak{p} \in X$. Equivalently, x is a minimal generator of $M_{\mathfrak{p}}$, for all $\mathfrak{p} \in X$. The notion of basic element was introduced by Swan in [Swa67], and later used by Eisenbud and Evans for proving Theorem 1.1 and Theorem 1.2. In what follows, given an R-module M and an element $x \in M$, we will denote $\langle x \rangle := (R \cdot x)$. Similarly, given a subset $S \subseteq M$, we will denote $\langle S \rangle := (R \cdot S)$. When P is a projective R-module, an element $x \in P$ is X-free-basic if and only if $\langle x \rangle_{\mathfrak{p}}$ is a non-zero free summand of $P_{\mathfrak{p}}$, for all $\mathfrak{p} \in X$. In particular, when $X = \operatorname{Spec}(R)$, or just $X = j$-Spec(R), and P is projective, $x \in P$ is X-basic if and only if $\langle x \rangle$ is a non-zero free summand of P. When M is not necessarily projective, an X-basic element $x \in M$ may not generate a free summand, in general. We now introduce an invariant that keeps track of the size of the local free summands of a module, rather than its local number of generators as in the theory of basic elements. Before doing so, we impose restrictions on the splitting maps.

Definition 3.4. Let R be a commutative Noetherian ring, M a finitely generated R-module, and $\mathscr{E}$ an R-submodule of $\operatorname{Hom}_R(M, R)$. We say that a free R-submodule $F \cong R^n$ of M is a free $\mathscr{E}$-summand of M if there exists a split surjection $M \to F \cong R^n$ which, when viewed as an element of $\operatorname{Hom}_R(M, R^n) \cong \operatorname{Hom}(M, R)^n$, is a direct sum of elements belonging to $\mathscr{E}$.

Observe that the choice of an isomorphism $F \cong R^n$ does not affect whether or not the projection $M \to F \cong R^n$ is a direct sum of elements of $\mathscr{E}$.

Definition 3.5. Let R be a commutative Noetherian ring, M a finitely generated R-module, $\mathscr{E}$ an R-submodule of $\operatorname{Hom}_R(M, R)$, and S a subset of M. For a prime $\mathfrak{p} \in \operatorname{Spec}(R)$, we denote by $\delta^{\mathscr{E}}_{\mathfrak{p}}(S, M)$ the largest integer n satisfying the following equivalent conditions:

(1) There exists a free $\mathscr{E}_{\mathfrak{p}}$-summand F of $M_{\mathfrak{p}}$ such that the image of S under the natural map $M_{\mathfrak{p}} \to F \to F/\mathfrak{p}F$ generates an $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$-vector subspace of rank n.
(2) There exists a free $\mathscr{E}_{\mathfrak{p}}$-summand G of $M_{\mathfrak{p}}$ of rank n which is contained in $\langle S \rangle_{\mathfrak{p}}$.

Whenever M is clear from the context, we denote $\delta^{\mathscr{E}}_{\mathfrak{p}}(S, M)$ simply by $\delta^{\mathscr{E}}_{\mathfrak{p}}(S)$.

Let $\mathfrak{p} \in \operatorname{Spec}(R)$ be a prime, M a finitely generated R-module, $\mathscr{E}$ an R-submodule of $\operatorname{Hom}_R(M, R)$, and $x \in M$. It follows from our definition that $\delta^{\mathscr{E}}_{\mathfrak{p}}(\{x\}) = 1$ if and only if $\langle x \rangle_{\mathfrak{p}}$ is a non-zero free $\mathscr{E}_{\mathfrak{p}}$-summand of $M_{\mathfrak{p}}$. Also, observe that if $\mathfrak{q} \subseteq \mathfrak{p}$, then $\delta^{\mathscr{E}}_{\mathfrak{q}}(S) \geqslant \delta^{\mathscr{E}}_{\mathfrak{p}}(S)$. In fact, if $G_{\mathfrak{p}} \subseteq \langle S \rangle_{\mathfrak{p}}$ is a free $\mathscr{E}_{\mathfrak{p}}$-summand of $M_{\mathfrak{p}}$ of rank $\delta^{\mathscr{E}}_{\mathfrak{p}}(S)$, then $G_{\mathfrak{q}} \subseteq \langle S\rangle_{\mathfrak{q}}$ is a free $\mathscr{E}_{\mathfrak{q}}$-summand of $M_{\mathfrak{q}}$ of rank $\delta^{\mathscr{E}}_{\mathfrak{p}}(S)$. Since $\delta^{\mathscr{E}}_{\mathfrak{q}}(S)$ is defined as the maximum rank of any such modules, we have $\delta^{\mathscr{E}}_{\mathfrak{q}}(S) \geqslant \delta^{\mathscr{E}}_{\mathfrak{p}}(S)$.

Lemma 3.6. *Let R be a commutative Noetherian ring, M a finitely generated R-module, $\mathscr{E}$ an R-submodule of $\operatorname{Hom}_R(M,R)$, and X a basic set. For any subset S of M and any integer $t \in \mathbb{N}$, the set $Y_t := \{\mathfrak{p} \in X \mid \delta^{\mathscr{E}}_{\mathfrak{p}}(S) \leqslant t\}$ is closed.*

Proof. Observe that $Y_t = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \delta^{\mathscr{E}}_{\mathfrak{p}}(S) \leqslant t\} \cap X$, so it is enough to show that $\{\mathfrak{p} \in \operatorname{Spec}(R) \mid \delta^{\mathscr{E}}_{\mathfrak{p}}(S) \leqslant t\}$ is a closed set, i.e., that $\{\mathfrak{p} \in \operatorname{Spec}(R) \mid \delta^{\mathscr{E}}_{\mathfrak{p}}(S) > t\}$ is open. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ be such that $\delta^{\mathscr{E}}_{\mathfrak{p}}(S) > t$. By definition of $\delta^{\mathscr{E}}_{\mathfrak{p}}(S)$, the identity map $R_{\mathfrak{p}}^{\delta^{\mathscr{E}}_{\mathfrak{p}}(S)} \to R_{\mathfrak{p}}^{\delta^{\mathscr{E}}_{\mathfrak{p}}(S)}$ factors as $R_{\mathfrak{p}}^{\delta^{\mathscr{E}}_{\mathfrak{p}}(S)} \subseteq \langle S\rangle_{\mathfrak{p}} \subseteq M_{\mathfrak{p}} \to R_{\mathfrak{p}}^{\delta^{\mathscr{E}}_{\mathfrak{p}}(S)}$, with $M_{\mathfrak{p}} \to R_{\mathfrak{p}}^{\delta^{\mathscr{E}}_{\mathfrak{p}}(S)}$ a direct sum of elements in $\mathscr{E}_{\mathfrak{p}}$. The inclusion $R_{\mathfrak{p}}^{\delta^{\mathscr{E}}_{\mathfrak{p}}(S)} \subseteq \langle S\rangle_{\mathfrak{p}}$ and the surjection $M_{\mathfrak{p}} \to R_{\mathfrak{p}}^{\delta^{\mathscr{E}}_{\mathfrak{p}}(S)}$ lift to maps $R^{\delta^{\mathscr{E}}_{\mathfrak{p}}(S)} \to \langle S\rangle$ and $M \to R^{\delta^{\mathscr{E}}_{\mathfrak{p}}(S)}$, respectively. In addition, $M \to R^{\delta^{\mathscr{E}}_{\mathfrak{p}}(S)}$ can be chosen such that it is still a direct sum of maps in $\mathscr{E}$. Let K be the kernel and C be the cokernel of the composition $R^{\delta^{\mathscr{E}}_{\mathfrak{p}}(S)} \to \langle S\rangle \subseteq M \to R^{\delta^{\mathscr{E}}_{\mathfrak{p}}(S)}$. As $K_{\mathfrak{p}} = C_{\mathfrak{p}} = 0$ and both these modules are finitely generated, there is an element $s \in R \smallsetminus \mathfrak{p}$ such that $K_{\mathfrak{q}} = C_{\mathfrak{q}}$ for all $\mathfrak{q} \in D(s)$. Thus $D(s)$ is an open neighborhood of $\mathfrak{p}$ such that $\delta^{\mathscr{E}}_{\mathfrak{q}}(S) \geqslant \delta^{\mathscr{E}}_{\mathfrak{p}}(S) > t$ for all $\mathfrak{q} \in D(s)$, which shows that $\{\mathfrak{p} \in \operatorname{Spec}(R) \mid \delta^{\mathscr{E}}_{\mathfrak{p}}(S) > t\}$ is indeed open. $\square$

Lemma 3.7. *Let R be a commutative Noetherian ring, M a finitely generated R-module, $\mathscr{E}$ an R-submodule of $\operatorname{Hom}_R(M,R)$, and X a basic set. For any subset S of M there exists a finite set of primes $\Lambda \subseteq X$ such that if $\mathfrak{p} \in X \smallsetminus \Lambda$, there exists $\mathfrak{q} \in \Lambda$ such that $\mathfrak{q} \subsetneq \mathfrak{p}$, and $\delta^{\mathscr{E}}_{\mathfrak{p}}(S) = \delta^{\mathscr{E}}_{\mathfrak{q}}(S)$.*

Proof. For each $t \in \mathbb{N}$ the sets $Y_t := \{\mathfrak{p} \in X \mid \delta^{\mathscr{E}}_{\mathfrak{p}}(S) \leqslant t\}$ are closed by Lemma 3.6. Also, observe that $Y_t = X$ for all $t \gg 0$. The sets Y_t are finite unions of irreducible closed subsets by Proposition 3.3, and irreducible closed sets of X are of the form $V(\mathfrak{p}) \cap X$ for some $\mathfrak{p} \in X$. Let Λ be the finite collection of generic points of the finitely many irreducible components of the finite collection of closed sets Y_t, as t varies through $\mathbb{N}$. Let $\mathfrak{p} \in X \smallsetminus \Lambda$ and let $t = \delta^{\mathscr{E}}_{\mathfrak{p}}(S)$, so that $\mathfrak{p} \in Y_t$ by definition. Let $\mathfrak{q}$ be the generic point of an irreducible component of Y_t which contains $\mathfrak{p}$. Then $\mathfrak{q} \subsetneq \mathfrak{p}$ because $\mathfrak{q} \in \Lambda$. Also, $t \geqslant \delta^{\mathscr{E}}_{\mathfrak{q}}(S) \geqslant \delta^{\mathscr{E}}_{\mathfrak{p}}(S) = t$, which implies the equality $\delta^{\mathscr{E}}_{\mathfrak{q}}(S) = \delta^{\mathscr{E}}_{\mathfrak{p}}(S)$. $\square$

We now make the key definitions of free-basic set and free-basic element.

Definition 3.8. Let R be a commutative Noetherian ring, M a finitely generated R-module, $\mathscr{E}$ an R-submodule of $\operatorname{Hom}_R(M,R)$, and X a basic set. Given a prime $\mathfrak{p} \in X$ and a finite set $S = \{x_1, \dots, x_n\} \subseteq M$, we say that S is a $(\mathfrak{p}, \mathscr{E})$-free-basic set for M if $\delta^{\mathscr{E}}_{\mathfrak{p}}(S, M) \geqslant \min\{n, 1 + \dim_X(\mathfrak{p})\}$. Here, $\dim_X(\mathfrak{p})$ denotes the supremum over the length of chains $\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ with $\mathfrak{p}_i \in X$ for each i. We say that S is an $(X, \mathscr{E})$-free-basic set for M if it is a $(\mathfrak{p}, \mathscr{E})$-free-basic set for M for all $\mathfrak{p} \in X$. When the module M is clear from the context, we just call S a $(\mathfrak{p}, \mathscr{E})$-free-basic set, or an $(X, \mathscr{E})$-free-basic set. For $x \in M$, if $S = \{x\}$ is a $(\mathfrak{p}, \mathscr{E})$-free-basic set (respectively, an $(X, \mathscr{E})$-free-basic set), we simply say that x is a $(\mathfrak{p}, \mathscr{E})$-free-basic element (respectively, an $(X, \mathscr{E})$-free-basic element).

Let X be a basic set, M a finitely generated R-module, and $\mathscr{E}$ an R-submodule of $\operatorname{Hom}_R(M,R)$. For $\mathfrak{p} \in X$, an element $x \in M$ is $(\mathfrak{p},\mathscr{E})$-free-basic if and only if $\delta_{\mathfrak{p}}^{\mathscr{E}}(\{x\}) = 1$. Therefore, x is $(\mathfrak{p},\mathscr{E})$-free-basic if and only if it generates a non-zero free $\mathscr{E}_{\mathfrak{p}}$-summand of $M_{\mathfrak{p}}$.

Note that, given an R-submodule $\mathscr{E}$ of $\operatorname{Hom}_R(M,R)$, we can view $R \oplus \mathscr{E} \cong \operatorname{Hom}_R(R,R) \oplus \mathscr{E}$ as an R-submodule of $R \oplus \operatorname{Hom}_R(M,R) \cong \operatorname{Hom}_R(R \oplus M, R)$. The following lemma will be crucial in what follows.

Lemma 3.9. *Let R be a commutative Noetherian ring, M a finitely generated R-module, $\mathscr{E}$ an R-submodule of $\operatorname{Hom}_R(M,R)$, and X a basic set. Let $S = \{x_1,\dots,x_n\} \subseteq M$ be an $(X,\mathscr{E})$-free-basic set and $(a,x_1) \in R \oplus M$ be an $(X, R\oplus\mathscr{E})$-free-basic element. Then there exist $a_1,\dots,a_{n-1} \in R$ such that*

$$S' = \{x_1', x_2', \dots, x_{n-1}'\} = \{x_1 + aa_1x_n, x_2 + a_2x_n, \dots, x_{n-1} + a_{n-1}x_n\}$$

is $(X,\mathscr{E})$-free-basic, and $(a,x_1') \in R \oplus M$ is still an $(X, R\oplus\mathscr{E})$-free-basic element.

Proof. We first observe that, since (a,x_1) is $(X,R\oplus\mathscr{E})$-free-basic by assumption, any element of the form $(a, x_1 + az)$, with $z \in M$, is still $(X, R\oplus\mathscr{E})$-free-basic. In fact, let $\mathfrak{p} \in X$. If $a \notin \mathfrak{p}$, then $\langle (a, x_1+az)\rangle_{\mathfrak{p}} \cong R_{\mathfrak{p}}$, and the map $(R \oplus M)_{\mathfrak{p}} \to R_{\mathfrak{p}}$ defined as multiplication by a^{-1} on the first component and as the zero map on the second belongs to $R \oplus \mathscr{E}$. This map provides a splitting to the inclusion $\langle (a,x_1+az)\rangle_{\mathfrak{p}} \subseteq (R\oplus M)_{\mathfrak{p}}$, showing that (a,x_1+az) is $(\mathfrak{p}, R\oplus\mathscr{E})$-free-basic in this case. Assume that $a \in \mathfrak{p}$. Let $\varphi \in (R\oplus\mathscr{E})_{\mathfrak{p}}$ be a map that provides a splitting to the inclusion $\langle (a,x_1)\rangle_{\mathfrak{p}} \subseteq (R\oplus M)_{\mathfrak{p}}$, that is, $\varphi(a,x_1) = 1$. Then $\varphi(a, x_1+az) = 1 + a\varphi(0,z) \in 1 + \mathfrak{p}R_{\mathfrak{p}}$. Setting $\varphi' := (1 + a\varphi(0,z))^{-1} \cdot \varphi$ we see that φ' is still an element of $(R\oplus\mathscr{E})_{\mathfrak{p}}$, and it provides the desired splitting to the inclusion $\langle (a,x_1+az)\rangle_{\mathfrak{p}} \subseteq (R\oplus M)_{\mathfrak{p}}$. Either way, (a,x_1+az) is $(\mathfrak{p}, R\oplus\mathscr{E})$-free-basic for all $\mathfrak{p} \in X$. This shows that the second claim of the lemma follows from the first, since we pick x_1' of the form $x_1 + aa_1x_n$, for some $a_1 \in R$.

We now prove the first claim. Let Λ be as in Lemma 3.7. We claim that for any choice of $a_1, a_2, \dots, a_{n-1} \in R$ and $\mathfrak{p} \in X \smallsetminus \Lambda$ the set $\{x_1 + aa_1x_n, x_2 + a_2x_n, \dots, x_{n-1} + a_{n-1}x_n\}$ is $(\mathfrak{p},\mathscr{E})$-free-basic. Indeed, let $a_1,\dots,a_{n-1} \in R$ and $S' = \{x_1 + aa_1x_n, \dots, x_{n-1} + a_{n-1}x_n\}$. Let F be a free $\mathscr{E}_{\mathfrak{p}}$-summand of $M_{\mathfrak{p}}$ such that the image of the composition $\langle S' \cup \{x_n\}\rangle_{\mathfrak{p}} = \langle S\rangle_{\mathfrak{p}} \subseteq M_{\mathfrak{p}} \to F \to F/\mathfrak{p}F$ generates an $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$-vector space of rank $\delta_{\mathfrak{p}}^{\mathscr{E}}(S)$. Then the image of $\langle S'\rangle_{\mathfrak{p}}$ under this composition is an $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$-vector subspace of dimension at least $\delta_{\mathfrak{p}}^{\mathscr{E}}(S) - 1$. Therefore $\delta_{\mathfrak{p}}^{\mathscr{E}}(S') \geqslant \delta_{\mathfrak{p}}^{\mathscr{E}}(S) - 1$. If $\mathfrak{p} \in X \smallsetminus \Lambda$, then by Lemma 3.7 there is a $\mathfrak{q} \in \Lambda$ such that $\mathfrak{q} \subsetneq \mathfrak{p}$ and $\delta_{\mathfrak{q}}^{\mathscr{E}}(S) = \delta_{\mathfrak{p}}^{\mathscr{E}}(S)$. Hence

$$\delta_{\mathfrak{p}}^{\mathscr{E}}(S') \geqslant \delta_{\mathfrak{p}}^{\mathscr{E}}(S) - 1 = \delta_{\mathfrak{q}}^{\mathscr{E}}(S) - 1 \geqslant \min\{n, 1+\dim_X(\mathfrak{q})\} - 1 \geqslant \min\{n-1, 1+\dim_X(\mathfrak{p})\}.$$

Suppose that $\Lambda = \{\mathfrak{q}_1,\dots,\mathfrak{q}_m\}$, and arrange the primes $\mathfrak{q}_i$ in such a way that, for each $1 \leqslant \ell \leqslant m$, $\mathfrak{q}_\ell$ is a minimal element of $\{\mathfrak{q}_1,\dots,\mathfrak{q}_l\}$. We prove, by induction on $\ell \geqslant 0$, that there exist $a_1, a_2, \dots, a_{n-1}$ such that $\{x_1 + aa_1x_n, x_2 + a_2x_n, \dots, x_{n-1} + a_{n-1}x_n\}$ is $(\mathfrak{q}_i,\mathscr{E})$-free-basic for all $1 \leqslant i \leqslant \ell$. Note that, for $\ell = m$, we get the desired claim for the lemma.

When $\ell = 0$ we are done. By induction on ℓ, there exist $a_1,\dots,a_{n-1} \in R$ such that the set $T := \{x_1 + aa_1x_n, x_2 + a_2x_n, \dots, x_{n-1} + a_{n-1}x_n\}$ is $(\mathfrak{q},\mathscr{E})$-free-basic for each $\mathfrak{q} \in \{\mathfrak{q}_1,\dots,\mathfrak{q}_{\ell-1}\}$. If T happens to be $(\mathfrak{q}_\ell,\mathscr{E})$-free-basic as well, we set $S' = T$, and we are done. Else, $\delta_{\mathfrak{q}_\ell}^{\mathscr{E}}(T) < \min\{n-1, 1+\dim_X(\mathfrak{q}_\ell)\}$. Let $F \subseteq \langle T\rangle_{\mathfrak{q}_\ell}$

be a free $\mathscr{E}_{\mathfrak{q}_\ell}$-summand of $M_{\mathfrak{q}_\ell}$ of rank $\delta^{\mathscr{E}}_{\mathfrak{q}_\ell}(T)$. In this way, the $R_{\mathfrak{q}_\ell}/\mathfrak{q}_\ell R_{\mathfrak{q}_\ell}$-vector space spanned by the image of T under the natural map $M_{\mathfrak{q}_\ell} \to F \to F/\mathfrak{q}_\ell F$ has dimension $\delta^{\mathscr{E}}_{\mathfrak{q}_\ell}(T)$. For each $x \in M$ let $\overline{x}$ denote the image of x in $F/\mathfrak{q}_\ell F$. The condition that $\delta^{\mathscr{E}}_{\mathfrak{q}_\ell}(T) < \min\{n-1, 1+\dim_X(\mathfrak{q}_\ell)\}$ implies that the $R_{\mathfrak{q}_\ell}/\mathfrak{q}_\ell R_{\mathfrak{q}_\ell}$-vector space spanned by $\{\overline{x_1 + aa_1x_n}, \ldots, \overline{x_{n-1} + a_{n-1}x_n}\}$ has dimension strictly smaller than $n-1$. Thus for some $1 \leqslant i \leqslant n-1$, the i-th element in the above set is in the span of the the previous $i-1$ elements. We distinguish two cases.

- Assume $i \neq 1$. Let $r \in (\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{\ell-1}) \smallsetminus \mathfrak{q}_\ell$ and define
$$S' := \{x_1 + aa_1x_n, \ldots, x_i + (a_i + r)x_n, \ldots, x_{n-1} + a_{n-1}x_n\}.$$
Since $r \in R \smallsetminus \mathfrak{q}_\ell$, and because $\overline{x_i + a_ix_n}$ is in the span of the previous $i-1$ elements, $\{\overline{x_1 + aa_1x_n}, \ldots, \overline{x_i + (a_i + r)x_n}, \ldots, \overline{x_{n-1} + a_{n-1}x_n}\}$ spans the same $R_{\mathfrak{q}_\ell}/\mathfrak{q}_\ell R_{\mathfrak{q}_\ell}$-vector space as $\{\overline{x_1 + aa_1x_n}, \ldots, \overline{rx_n}, \ldots, \overline{x_{n-1} + a_{n-1}x_n}\}$ which, in turn, spans the same $R_{\mathfrak{q}_\ell}/\mathfrak{q}_\ell R_{\mathfrak{q}_\ell}$-vector space as $\{\overline{x_1}, \ldots, \overline{x_n}\}$.
- Now assume $i = 1$, which means that $\overline{x_1 + aa_1x_n} = 0$ in $F/\mathfrak{q}_\ell F$. We claim that the element a is not inside $\mathfrak{q}_\ell$. In fact assume, by way of contradiction, that $a \in \mathfrak{q}_\ell$. In such case, as we showed at the beginning of this proof, there is $\varphi \in \mathscr{E}_{\mathfrak{q}_\ell}$ such that $\varphi(a, x_1 + aa_1x_n) = 1$. Then, as $\varphi(0, x_1 + aa_1x_n) = 1-\varphi(a, 0)$ is invertible, we see that $(\varphi(0, x_1+aa_1x_n))^{-1}\cdot\varphi \in \mathscr{E}_{\mathfrak{q}_\ell}$ is a splitting to the inclusion $\langle(0, x_1 + aa_1x_1)\rangle_{\mathfrak{q}_\ell} \subseteq (R\oplus M)_{\mathfrak{q}_\ell}$. Therefore, $(0, x_1+aa_1x_1)$ is $(\mathfrak{q}_\ell, \mathscr{E})$-free-basic. Recall that $F \subseteq \langle T\rangle_{\mathfrak{q}_\ell}$ is a free $\mathscr{E}_{\mathfrak{q}_\ell}$-summand of $M_{\mathfrak{q}_\ell}$ of rank $\delta^{\mathscr{E}}_{\mathfrak{q}_\ell}(T)$. Set $x_1'' := x_1 + aa_1x_n \in T$, and let $\iota : F \to M_{\mathfrak{q}_\ell}$ and $\pi : M_{\mathfrak{q}_\ell} \to F$ denote the natural inclusion and projection, where π is a direct sum of elements of $\mathscr{E}_{\mathfrak{q}_\ell}$. The fact that $\overline{x_1''} = 0$ simply means that $\pi(x_1'') \in \mathfrak{q}_\ell F$. If $x_1'' \in F$, then $x_1'' = \pi\iota(x_1'') \in \mathfrak{q}_\ell F$, so that $(0, x_1'') \in \mathfrak{q}_\ell(R \oplus M)_{\mathfrak{q}_\ell}$, contradicting the fact that $(0, x_1'')$ generates a free summand of $(R\oplus M)_{\mathfrak{q}_\ell}$. Thus, we necessarily have $x_1'' \notin F$. Let $y := x_1'' - \iota\pi(x_1'')$, and note that $y \in \ker(\pi)$. Since $(0, x_1'') - (0, y) = (0, \iota\pi(x_1'')) \in \mathfrak{q}_\ell(R \oplus M)_{\mathfrak{q}_\ell}$, using once again the same argument as above we see that $(0, y)$ is $(\mathfrak{q}_\ell, \mathscr{E})$-free-basic, because $(0, x_1'')$ is. This means that $\langle y\rangle_{\mathfrak{q}_\ell}$ is a free $\mathscr{E}_{\mathfrak{q}_\ell}$-summand of $M_{\mathfrak{q}_\ell}$. Because $M_{\mathfrak{q}_\ell} \cong \ker(\pi)\oplus F$, and because $y \in \ker(\pi)$ generates a free $\mathscr{E}_{\mathfrak{q}_\ell}$-summand of $M_{\mathfrak{q}_\ell}$, we have that $F' = \langle y\rangle_{\mathfrak{q}_\ell} \oplus F$ is a free direct summand of $M_{\mathfrak{q}_\ell}$. In addition, because $x_1'' \in T$ and $\iota\pi(x_1'') \in F \subseteq \langle T\rangle_{\mathfrak{q}_\ell}$, we conclude that $y \in \langle T\rangle_{\mathfrak{q}_\ell}$. Therefore $F' \subseteq \langle T\rangle_{\mathfrak{q}_\ell}$ is a free $\mathscr{E}_{\mathfrak{q}_\ell}$-summand of $M_{\mathfrak{q}_\ell}$ of rank $\delta^{\mathscr{E}}_{\mathfrak{q}_\ell}(T) + 1$, a contradiction. This shows that $a \notin \mathfrak{q}_\ell$.

  Let $r \in (\mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_{\ell-1}) \smallsetminus \mathfrak{q}_\ell$, and define
$$S' := \{x_1 + a(a_1 + r)x_n, x_2 + a_2x_n, \ldots, x_{n-1} + a_{n-1}x_n\}.$$
Since $ar \notin \mathfrak{q}_\ell$ and $\overline{x_1 + aa_1x_n} = 0$, the set
$$\{\overline{x_1 + a(a_1 + r)x_n}, \ldots, \overline{x_i + a_ix_n}, \ldots, \overline{x_{n-1} + a_{n-1}x_n}\}$$
spans the same $R_{\mathfrak{q}_\ell}/\mathfrak{q}_\ell R_{\mathfrak{q}_\ell}$-vector space as the set $\{\overline{x_1}, \ldots, \overline{x_n}\}$ inside $F/\mathfrak{q}_\ell F$.

Either way, we obtain that $\delta^{\mathscr{E}}_{\mathfrak{q}_\ell}(S') \geqslant \delta^{\mathscr{E}}_{\mathfrak{q}_\ell}(S)$. Since S is $(\mathfrak{q}_\ell, \mathscr{E})$-free-basic, we have that $\delta^{\mathscr{E}}_{\mathfrak{q}_\ell}(S) \geqslant \min\{n, 1 + \dim_X(\mathfrak{q}_\ell)\} \geqslant \min\{n-1, 1 + \dim_X(\mathfrak{q}_\ell)\}$. Moreover, $r \in \mathfrak{q}_i$ for each $1 \leqslant i \leqslant \ell - 1$, therefore we also have that $\delta^{\mathscr{E}}_{\mathfrak{q}_i}(S') = \delta^{\mathscr{E}}_{\mathfrak{q}_i}(T) \geqslant \min\{n-1, 1+\dim_X(\mathfrak{q}_i)\}$ for each $1 \leqslant i \leqslant \ell - 1$. This completes the proof of the inductive step. As previously mentioned, the lemma now follows from choosing $\ell = m$. ∎

Theorem 3.10. *Let R be a commutative Noetherian ring, $N \subseteq M$ finitely generated R-modules, $\mathscr{E}$ an R-submodule of $\operatorname{Hom}_R(M,R)$, and X a basic set. Assume that, for each $\mathfrak{p} \in X$, $N_\mathfrak{p}$ contains a free $R_\mathfrak{p}$-module of rank at least $1+\dim_X(\mathfrak{p})$ that is a free $\mathscr{E}_\mathfrak{p}$-summand of $M_\mathfrak{p}$. Then there exists $x \in N$ that is $(X,\mathscr{E})$-free-basic for M. Moreover, if an element $(a,y) \in R \oplus N$ is $(X, R\oplus\mathscr{E})$-free-basic for $R \oplus M$, then x can be chosen to be of the form $x = y + az$ for some $z \in N$.*

Proof. Assume that $(a,y) \in R\oplus N$ is $(X, R\oplus\mathscr{E})$-free-basic for $R\oplus M$. Complete y to a generating set $S = \{y, y_1, \ldots, y_n\}$ for N, which is easily verified to be $(X,\mathscr{E})$-free-basic for M, given our assumptions. Continued use of Lemma 3.9 implies that there exist $a_1,\ldots,a_n \in R$ such that $x = y + a(a_1y_1 + \ldots + a_ny_n)$ is $(X,\mathscr{E})$-free-basic for M, proving the last claim. As for the first claim, notice that $(1,y) \in R\oplus N$ is always $(X, R\oplus\mathscr{E})$-free-basic for $R\oplus M$, for any $y \in N$. Thus existence follows from the last claim. □

Lemma 3.11. *Let $(R,\mathfrak{m})$ be a local ring, M a finitely generated R-module, S a subset of M, and $\mathscr{E}$ an R-submodule of $\operatorname{Hom}_R(M,R)$. Consider $I^{\mathscr{E}}(S,M) := \{y \in \langle S\rangle \mid f(y) \in \mathfrak{m}$ for all $f \in \mathscr{E}\}$, which is an R-submodule of $\langle S\rangle$. Then*

$$\delta^{\mathscr{E}}_{\mathfrak{m}}(S,M) = \lambda_R\left(\frac{\langle S\rangle}{I^{\mathscr{E}}(S,M)}\right).$$

Proof. Let $\delta := \delta^{\mathscr{E}}_{\mathfrak{m}}(S,M)$, and $F \subseteq \langle S\rangle$ be a free $\mathscr{E}$-summand of M such that $F \cong R^\delta$. Let $\varphi : M \to F$ be a split surjection that is a direct sum of elements in $\mathscr{E}$, and let $T = \ker(\varphi)$, so that $M = F \oplus T$. In addition, $T \cap \langle S\rangle$ contains no free $\mathscr{E}$-summands of M. By a slight modification of the argument in [Hun13, Discussion 6.7], we see that $I^{\mathscr{E}}(S,M) = \mathfrak{m}F \oplus (T\cap\langle S\rangle) \cong \mathfrak{m}^\delta \oplus (T \cap \langle S\rangle)$. Since $\langle S\rangle = (F\oplus T)\cap\langle S\rangle = F \oplus (T\cap\langle S\rangle)$, we finally have

$$\delta = \lambda_R\left(\frac{R^\delta \oplus (T\cap\langle S\rangle)}{\mathfrak{m}^\delta \oplus (T\cap\langle S\rangle)}\right) = \lambda_R\left(\frac{F \oplus (T\cap\langle S\rangle)}{\mathfrak{m}F \oplus (T\cap\langle S\rangle)}\right) = \lambda_R\left(\frac{\langle S\rangle}{I^{\mathscr{E}}(S,M)}\right). \qquad \square$$

Lemma 3.12. *Let R be a commutative Noetherian ring, M a finitely generated R-module, $\mathscr{E}$ an R-submodule of $\operatorname{Hom}_R(M,R)$, and S a subset of M. Suppose that $F \subseteq \langle S\rangle$ is a free $\mathscr{E}$-summand of M of rank i, so that we have a split surjection $\varphi : M \to F \cong R^i$ which is a direct sum of elements in $\mathscr{E}$. Define $M' = \ker(\varphi)$, so that we can write $M = F \oplus M'$. Let S' be the projection of S to M' along the internal direct sum, and $\mathscr{E}'$ be the projection of $\mathscr{E}$ to $\operatorname{Hom}_R(M',R)$. For all $\mathfrak{p} \in \operatorname{Spec}(R)$, we have $\delta^{\mathscr{E}'}_{\mathfrak{p}}(S',M') = \delta^{\mathscr{E}}_{\mathfrak{p}}(S,M) - i$.*

Proof. After localizing at $\mathfrak{p}$, we can assume that $(R,\mathfrak{m})$ is a local ring, with $\mathfrak{p} = \mathfrak{m}$. Let $I^{\mathscr{E}}(S,M)$ be as in Lemma 3.11, and similarly define $I^{\mathscr{E}'}(S',M')$. Given that $\langle S\rangle = F \oplus \langle S'\rangle \subseteq F \oplus M' = M$, and that F is a free $\mathscr{E}$-summand, it is straightforward to see that $I^{\mathscr{E}}(S,M) = \mathfrak{m}F \oplus I^{\mathscr{E}'}(S',M')$. This gives rise to the following (actually splitting) short exact sequence:

$$0 \longrightarrow \frac{F}{\mathfrak{m}F} \longrightarrow \frac{\langle S\rangle}{I^{\mathscr{E}}(S,M)} \longrightarrow \frac{\langle S'\rangle}{I^{\mathscr{E}'}(S',M')} \longrightarrow 0.$$

Lemma 3.11 and the short exact sequence above give

$$\delta^{\mathscr{E}'}_{\mathfrak{m}}(S',M') = \lambda_R\left(\frac{\langle S'\rangle}{I^{\mathscr{E}'}(S',M')}\right) = \lambda_R\left(\frac{\langle S\rangle}{I^{\mathscr{E}}(S,M)}\right) - \lambda_R\left(\frac{F}{\mathfrak{m}F}\right) = \delta^{\mathscr{E}}_{\mathfrak{m}}(S,M) - i.$$

□

Theorem C is a consequence of Theorem 3.10 on the existence of free-basic elements for modules that, locally at every prime, have enough free $\mathscr{E}$-summands. With Lemma 3.12, we prove here the most general version of this result, dealing with free $\mathscr{E}$-summands of a given rank that, additionally, are contained in a given submodule.

Theorem 3.13. *Let R be a commutative Noetherian ring, $N \subseteq M$ finitely generated R-modules, $\mathscr{E}$ an R-submodule of $\operatorname{Hom}_R(M,R)$, and $X = j\text{-}\operatorname{Spec}(R)$. Assume that, for each $\mathfrak{p} \in X$, $M_\mathfrak{p}$ contains a free $\mathscr{E}_\mathfrak{p}$-summand $F(\mathfrak{p})$ of rank at least $i + \dim_X(\mathfrak{p})$, for some positive integer i. Then M contains a free $\mathscr{E}$-summand F of rank at least i. Furthermore, if $F(\mathfrak{p}) \subseteq N_\mathfrak{p}$ for all $\mathfrak{p} \in X$, then F can be realized inside N.*

Proof. It suffices to prove the statement involving N, since the first part of the theorem is nothing but the case $N = M$. We proceed by induction on $i \geqslant 1$. If $i = 1$, then our assumptions guarantee that there exists $x \in N$ that is $(X,\mathscr{E})$-free-basic for M, by Theorem 3.10. Thus, the inclusion $R_\mathfrak{m} \cong \langle x \rangle_\mathfrak{m} \subseteq M_\mathfrak{m}$ splits via an element of $\mathscr{E}_\mathfrak{m}$, for all maximal ideals $\mathfrak{m} \in \operatorname{Max}(R)$. In particular, $F := \langle x \rangle \cong R$ is a free R-submodule of N such that the inclusion $F \subseteq M$ splits. We claim that the splitting map $M \to F \cong R$ can be chosen to be inside $\mathscr{E}$. In fact, consider the inclusion $R \cong \langle x \rangle \subseteq M$ and apply the functor $\operatorname{Hom}_R(-,R)$. We get an induced map $h : \mathscr{E} \subseteq \operatorname{Hom}_R(M,R) \to \operatorname{Hom}_R(R,R) \cong R$, and it suffices to show that h is surjective. This follows from the fact that, for each $\mathfrak{m} \in \operatorname{Max}(R)$, the map $h_\mathfrak{m} : \mathscr{E}_\mathfrak{m} \to R_\mathfrak{m}$ is surjective, because we showed that x is $(\mathfrak{m},\mathscr{E})$-free-basic for M. Now assume that $i > 1$. By the base case $i = 1$, there exists an element $x \in N$ with a splitting map $\varphi : M \to \langle x \rangle \cong R$ that belongs to $\mathscr{E}$. Define $M' = \ker(\varphi)$, so that $M = \langle x \rangle \oplus M'$. Let S' be the projection of S to M' along the internal direct sum, and $\mathscr{E}'$ be the projection of $\mathscr{E}$ to $\operatorname{Hom}_R(M',R)$. By Lemma 3.12 we have that $\delta_\mathfrak{p}^{\mathscr{E}'}(S',M') = \delta_\mathfrak{p}^{\mathscr{E}}(S,M) - 1 \geqslant i - 1 + \dim_X(\mathfrak{p})$, and by induction M' has a free $\mathscr{E}'$-summand $F' \subseteq \langle S' \rangle \subseteq N$ of rank at least $i-1$. It follows that $F := \langle x \rangle \oplus F'$ is a free $\mathscr{E}$-summand of M of rank at least i, which is contained in N. $\square$

The case $\mathscr{E} = \operatorname{Hom}_R(M,R)$, $N = M$ and $i = 1$ of Theorem 3.13 gives Theorem A, which generalizes the classical version of Serre's Splitting Theorem 1.1 [Ser58, Theorem 1].

With Theorem 3.10 we are ready to prove Theorem B. That is, a cancellation result for modules that, locally at each prime, have enough free summands.

Theorem 3.14. *Let R be a commutative Noetherian ring, M a finitely generated R-module, and $X = j\text{-}\operatorname{Spec}(R)$. Assume that, for each $\mathfrak{p} \in X$, the module $M_\mathfrak{p}$ contains a free $R_\mathfrak{p}$-summand of rank at least $1 + \dim_X(\mathfrak{p})$. Let Q be a finitely generated projective R-module, and N a finitely generated R-module such that $Q \oplus M \cong Q \oplus N$. Then $M \cong N$.*

Proof. Since Q is projective, we can find another projective module Q' such that $Q \oplus Q' \cong R^a$, for some integer a. We then obtain that $R^a \oplus M \cong R^a \oplus N$ and, by induction on a, we may assume that $R \oplus M \cong R \oplus N$. Let $\alpha : R \oplus N \to R \oplus M$ be an isomorphism, and set $\alpha((1,0)) = (a,x_1)$. We want to show that the composition

$$\begin{tikzcd}
R \oplus N \arrow[r, "\alpha"] & R \oplus M \arrow[r, "\beta"] & R \oplus M \arrow[r, "\gamma"] & R \oplus M \arrow[r, "\eta"] & R \oplus M \\
(1,0) \arrow[r] & (a,x_1) \arrow[rr] & & (1,x) \arrow[r] & (1,0)
\end{tikzcd}$$

is an isomorphism, with β, γ and η to be defined later. Let $\mathscr{E}' = \operatorname{Hom}_R(N, R)$ and $\mathscr{E} = \operatorname{Hom}_R(M, R)$. Note that, since $(1,0)$ is $(X, R \oplus \mathscr{E}')$-free-basic for $R \oplus N$, and α is an isomorphism by assumption, we have that (a, x_1) is $(X, R \oplus \mathscr{E})$-free-basic for $R \oplus M$. Let $\{x_1, \dots, x_n\}$ be a set of generators for M, and recall that $\delta_{\mathfrak{p}}^{\mathscr{E}}(\{x_1, \dots, x_n\}) \geqslant 1 + \dim_X(\mathfrak{p})$ for all $\mathfrak{p} \in X$, by assumption. By Theorem 3.10, there exists an $(X, \mathscr{E})$-free-basic element $x \in M$ of the form $x = x_1 + az$, for some $z \in M$. Define a map $\varphi : R \to M$, by setting $\varphi(1) = z$. Define $\beta : R \oplus M \to R \oplus M$ via the following matrix:

$$\begin{bmatrix} 1_R & 0 \\ \varphi & 1_M \end{bmatrix}.$$

It is easy to check that β is an isomorphism. Note that $\beta((a, x_1)) = (a, \varphi(a) + x_1) = (a, az + x_1) = (a, x)$. Since $x \in M$ is $(X, \mathscr{E})$-free-basic, we have that $M \cong Rx \oplus M'$, for some R-module M'. Therefore, we may define an R-module map $\psi : M \to R$ as $\psi(x) = 1 - a$. Define $\gamma : R \oplus M \to R \oplus M$ via the following matrix:

$$\begin{bmatrix} 1_R & \psi \\ 0 & 1_M \end{bmatrix}.$$

The map γ is an isomorphism, and it is such that $\gamma((a, x)) = (a + \psi(x), x) = (1, x)$. Finally, consider the map $\theta : R \to M$, defined as $\theta(1) = -x$. Define $\eta : R \oplus M \to R \oplus M$ via the matrix:

$$\begin{bmatrix} 1_R & 0 \\ \theta & 1_M \end{bmatrix}.$$

The map η is an isomorphism, and $\eta(1, x) = (1, x + \theta(1)) = (1, 0)$. Consider the composition $\epsilon = \eta \circ \gamma \circ \beta \circ \alpha : R \oplus N \to R \oplus M$, which is an isomorphism such that $\epsilon((1,0)) = (1,0)$. We then have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, equal] & R \oplus N \arrow[r] \arrow[d, "\epsilon", "\cong"'] & N \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & R \arrow[r] & R \oplus M \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

that forces the induced map $N \to M$ to be an isomorphism. $\square$

Bass's Cancellation Theorem for projective modules, Theorem 1.2 [Bas64, Theorem 9.1], now follows immediately from Theorem 3.14.

Acknowledgments

The authors thank Mohan Kumar for many helpful suggestions and ideas regarding the methods used in Section 2. The authors thank Craig Huneke for very useful discussions. The authors also wish to thank the referee for providing helpful feedback.

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