PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 4, April 2018, Pages 1585–1598
http://dx.doi.org/10.1090/proc/13808
Article electronically published on December 28, 2017

SOLVING EXISTENCE PROBLEMS VIA F-CONTRACTIONS

DARIUSZ WARDOWSKI

(Communicated by Wenxian Shen)

Abstract. The main results of the paper concern the existence of fixed points of nonlinear F-contraction and the sum of this type of mapping with a compact operator. The results of Krasnosel'skii type are obtained with a usage of the Hausdorff measure of noncompactness and condensing mappings. The presented new tools give the possibility to verify the existence problems of the solutions for some classes of integral equations.

1. Introduction and preliminaries

Krasnosel'skii's fixed point theorem about the sum of contraction and compact mappings is an important result merging two fundamental results in the fixed point theory, Banach contraction principle and Schauder fixed point theorem. The original statement of this theorem can be found in [15] and we may read it as follows: If M is a nonempty closed convex and bounded subset of the Banach space X, $A\colon M \to X$ is a strict contraction, $B\colon M \to X$ is a compact mapping, i.e. continuous and maps a subset of M into a compact subset of X, and $A(M) + B(M) \subset M$, then $A + B$ admits a fixed point. This result can be applied in many interesting contexts, especially in the theory of differential and integral equations. In the literature we can find many important contributions in this direction; see e.g. Burton's results in [2], where Krasnosel'skii's thesis is obtained for large contractions; in [3] Burton and Colleen Kirk combined the contractions with Schaefer's theorem [24]. In [17] Liu and Li investigated the case where the mappings are multivalued and the operator $I - A$ need not be injective. All these improvements have made it easier to apply the obtained more general results. However, a significant breakthrough was achieved when it started to involve the measure of noncompactness. Then it was possible to replace the difficulty of verifying the condition $A(M) + B(M) \subset M$ onto the weaker

$$(A + B)(M) \subset M. \tag{1.1}$$

We recall the notion of the Hausdorff measure of noncompactness; i.e. for any bounded subset $C \subset X$, there is assigned a nonnegative number $\beta(C)$ by the formula

$$\beta(C) := \inf\{r > 0\colon C \subset \bigcup_{i=1}^{N} B(x_i, r),\ x_i \in X,\ i = 1, \dots, N\},$$

Received by the editors February 2, 2017 and, in revised form, May 4, 2017.
2010 *Mathematics Subject Classification.* Primary 47H10, 47H09, 47H08, 47N20.
Key words and phrases. Fixed point, nonlinear F-contraction, compact operator, condensing mapping, integral equation.

where $B(x_i, r)$ denotes the closed ball centred at x_i with a radius r. Some of the basic properties of the Hausdorff measure of noncompactness are the following: $\beta(C) = 0$ if and only if C is relatively compact (i.e. cl$C$ is compact), $\beta(C) = \beta(\mathrm{cl}C)$; if we assume that $D \subset X$ is bounded, then we also have $C \subset D$ implies $\beta(C) \leq \beta(D)$ and $\beta(C + D) \leq \beta(C) + \beta(D)$. More information about the measure of noncompactness and its properties can be found e.g. in [1]. The classical fixed point results where measure of noncompactness is applied are due to Darbo [5] and Sadovskii [23]. If M is a nonempty bounded closed convex subset of a Banach space X, $T\colon M \to M$ is a continuous mapping such that one of the following conditions holds:

(a) There exists $k \in [0, 1)$ such that for any set $C \subset M$,

$$\text{(Darbo)} \quad \beta(T(C)) \leq k\beta(C).$$

(b) For any set $C \subset M$ with positive measure of noncompactness

$$\text{(Sadovskii)} \quad \beta(T(C)) < \beta(C).$$

Then T has a fixed point.

The mappings satisfying the contraction condition in Darbo's and Sadovskii's results are called *k-set contraction* and *β-condensing* respectively. Using these results we can obtain the existence of a fixed point of $A + B$, provided that $A + B$ is β-condensing or a *k-set contraction*. In this way Przeradzki in his work [21] showed that the sum of a generalized contraction (sometimes called Krasnosel'skii's contraction) and compact mapping satisfying (1.1) has a fixed point. In the literature, one can find also other significant contributions inspired by Krasnosel'skii's result. For example Kryszewski and Mederski in their work [16] for the set-valued operators of Krasnosel'skii type (the sum $A + B$ was replaced by a general nonlinear operator) obtained fixed point results on the complete absolute neighbourhood retracts. In [6] Garcia-Falset et al. proved a Krasnosel'skii-Schaefer type theorem, where the investigated operators need not be weakly continuous.

As was mentioned, some authors made an effort to improve Krasnosel'skii's theorem by extending the family of mappings satisfying the appropriate new, more general contraction condition. There are many generalizations of Banach's contraction. One can mention here e.g. the recent paper by Włodarczyk [36], where the author proves fixed point results in a very general setting in the so-called quasi-triangular spaces. In [36] there is also included a comprehensive list of references of the articles where other known contraction conditions have been investigated. In Kirk's handbook [13] we can also find out about the recent developments in metric fixed point theory. However many new defined contractions are special cases of the others. The information on this topic can be found for example in Rhoades' article [22] or in Jachymski's papers [11] and [12]. The inspiration for our investigations is the aforementioned Przeradzki article, where there were taken into consideration generalized contractions. If X is a nonempty set and (X, d) is a metric space with a metric function $\mathrm{d}\colon X \times X \to \mathbb{R}$, then a mapping $T\colon X \to X$ is called *a generalized contraction* if there exists a function $\Gamma\colon X \times X \to [0, \infty)$ such that:

(Γ1) $\sup_{a \leq \mathrm{d}(x,y) \leq b} \Gamma(x, y) < 1$ for all $0 < a \leq b$,
(Γ2) $\mathrm{d}(Tx, Ty) \leq \Gamma(x, y)\mathrm{d}(x, y)$ for all $x, y \in X$.

Many known contractions are equivalent to generalized contractions; for details see [12]. Przeradzki showed that generalized contractions are a proper generalization of the so-called *large contractions* (introduced by Burton in [2]), i.e. the mappings $T\colon X \to X$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$,

$$\mathrm{d}(x,y) \geq \varepsilon \text{ implies } \mathrm{d}(Tx,Ty) < \varepsilon.$$

Przeradzki in [21] also showed that generalized contractions are β-condensing, which together with Sadovskii's result and condition (1.1) derived a significant improvement of Krasnosel'skii's result. On the other hand, in [35] there was introduced a new type of contraction condition, called *F-contraction*, i.e. a mapping $T\colon X \to X$ satisfying

$$\tau + F(\mathrm{d}(Tx,Ty)) \leq F(\mathrm{d}(x,y)) \text{ for all } x,y \in X \text{ with } Tx \neq Ty,$$

where $\tau > 0$ and $F\colon (0,\infty) \to \mathbb{R}$ satisfies the following conditions:

(F1) for all $t_1, t_2 > 0$, $t_1 > t_2$ implies $F(t_1) > F(t_2)$;
(F2) for any sequence $(t_n) \subset (0,\infty)$, $t_n \to 0$ if and only if $F(t_n) \to -\infty$;
(F3) there exists $k \in (0,1)$ satisfying $\lim_{t\to 0^+} t^k F(t) = 0$.

It was also proved that every F-contraction defined on a complete metric space has a unique fixed point. Using concrete forms of F it is possible to obtain other known types of contractions; e.g. for $F(t) = \ln(t)$, $t > 0$, we get a Banach contraction (for details, see [35]). Many articles concerning F-contractions and their extensions have appeared so far; see e.g. [7–10, 18–20, 25–34]. Turinici in [32] observed that the condition (F2) can be relaxed to the form

(F2') $\lim_{t\to 0^+} F(t) = -\infty$.

Then the implication

(F2")] $F(t_n) \to -\infty \Rightarrow t_n \to 0$

can be derived from (F1). In the present paper, we are going to reconsider this type of mapping in a broader setting, i.e. with τ taken as a function. A mapping $T\colon X \to X$ is said to be a *(φ, F)-contraction* (or *nonlinear F-contraction*) if there exist the functions $F\colon (0,\infty) \to \mathbb{R}$ and $\varphi\colon (0,\infty) \to (0,\infty)$ satisfying

(H1) F satisfies (F1) and (F2');
(H2) $\liminf_{s\to t^+} \varphi(s) > 0$ for all $t \geq 0$;
(H3) $\varphi(\mathrm{d}(x,y)) + F(\mathrm{d}(Tx,Ty)) \leq F(\mathrm{d}(x,y))$ for all $x,y \in X$ such that $Tx \neq Ty$.

Nonlinear F-contractions have been studied e.g. in [14], however observe that in the present investigations we omit the condition (F3). The proposed type of the contraction condition is not a special case of a generalized contraction. In the following, there is presented the example of a (φ, F)-contraction which is not a generalized contraction.

Example 1.1. Let (x_n) be a sequence given by the formula

$$x_n := n - 2 + 2^{-n+1},\ n \geq 1.$$

The set $X = \{x_n : n \in \mathbb{N}\}$ together with the metric $\mathrm{d}(x,y) = |x-y|$, $x,y \in X$, is a complete metric space. Consider the mapping $T\colon X \to X$ defined by the formula

$$Tx_n := \begin{cases} x_{n-1}, & \text{for} \quad n \geq 2, \\ x_1, & \text{for} \quad n = 1. \end{cases}$$

First, observe that T is not a generalized contraction. Indeed, taking any $n \in \mathbb{N}$ we have

$$|x_{n+1} - x_n| = 1 - 2^{-n}$$

and hence

$$\frac{1}{2} \leq |x_{n+1} - x_n| < 1 \text{ for all } n \in \mathbb{N}.$$

If there existed a function Γ satisfying (Γ2), then for all $n \geq 2$ we would get

$$\Gamma(x_{n+1}, x_n) \geq \frac{|Tx_{n+1} - Tx_n|}{|x_{n+1} - x_n|} = \frac{|x_n - x_{n-1}|}{|x_{n+1} - x_n|} = \frac{2^n - 2}{2^n - 1}.$$

In consequence, by (Γ1), we would obtain

$$1 > \sup_{\frac{1}{2} \leq \mathrm{d}(x,y) \leq 1} \Gamma(x, y) \geq \Gamma(x_{n+1}, x_n) \geq \frac{2^n - 2}{2^n - 1}.$$

Tending with $n \to \infty$ we get a contradiction. In order to show that T is a nonlinear F-contraction let us consider the mapping $\varphi\colon (0, \infty) \to (0, \infty)$ by the formula

$$\varphi(t) := \begin{cases} -t + 1, & \text{for} \quad 0 < t < 1, \\ -t + n, & \text{for} \quad n - 1 \leq t < n, \ n \geq 2. \end{cases}$$

Obviously $\liminf_{s \to t^+} \varphi(s) > 0$ for any $t \geq 0$, and for any $m, n \in \mathbb{N}$, $2 \leq m < n$ we have

$$|x_m - x_n| = n - m + 2^{-n+1} - 2^{-m+1} \text{ and } n - m - 1 < |x_m - x_n| < n - m.$$

Hence we get

$$\begin{aligned} \frac{|Tx_m - Tx_n|}{|x_m - x_n|} \mathrm{e}^{|Tx_m - Tx_n| - |x_m - x_n|} &< \mathrm{e}^{x_{n-1} - x_{m-1} - (x_n - x_m)} \\ &= \mathrm{e}^{2^{-n+1} - 2^{-m+1}} = \mathrm{e}^{-\varphi(|x_m - x_n|)}. \end{aligned}$$

Next, for any $m \geq 3$ we obtain

$$|x_1 - x_m| = m - 2 + 2^{-m+1} \text{ and } m - 2 < |x_1 - x_m| < m - 1,$$

which gives

$$\begin{aligned} \frac{|Tx_1 - Tx_m|}{|x_1 - x_m|} \mathrm{e}^{|Tx_1 - Tx_m| - |x_1 - x_m|} &< \mathrm{e}^{x_{m-1} - x_m} \\ &= \mathrm{e}^{2^{-m+1} - 1} = \mathrm{e}^{-\varphi(|x_1 - x_m|)}. \end{aligned}$$

Now, after some simple calculations we can observe that T satisfies (H3) for $F(t) = \ln t + t$, $t > 0$.

In the present paper first we prove a fixed point theorem for nonlinear F-contractions; next we show that some of these contractions are β-condensing. Finally, using Sadovskii's result, we get that the sum of compact mapping with a (φ, F)-contraction has a fixed point. In the last section we apply our results to certain classes of integral equations.

Within the article we will use the following notation: $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{Q}$ the set of all rational numbers, $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and all nonnegative real numbers respectively.

2. The results

First we enunciate a fixed point result concerning nonlinear F-contractions. Observe that in the proof of this fact we will not use the condition (F3).

Theorem 2.1. *Let (X, d) be a complete metric space and let $T\colon X \to X$ be a (φ, F)-contraction. Then T has a unique fixed point.*

Proof. T has at most one fixed point, which is an immediate consequence of (H3) and the fact that $\varphi > 0$.

Take any $x_0 \in X$ and define the sequence $x_n = T^n x_0$, $n = 1, 2, \ldots$. Denote the sequence $\gamma_n = \mathrm{d}(x_{n-1}, x_n)$, $n \in \mathbb{N}$. Without loss of generality we can assume that $\gamma_n > 0$ for all $n \in \mathbb{N}$. From (H3) we have

$$F(\gamma_{n+1}) \leq F(\gamma_n) - \varphi(\gamma_n) < F(\gamma_n) \text{ for all } n \in \mathbb{N}.$$

From the above and from (F1) we get that (γ_n) is decreasing, and hence, $\gamma_n \searrow t$, $t \geq 0$. From (H2) there exists $c > 0$ and $n_0 \in \mathbb{N}$ such that $\varphi(\gamma_n) > c$ for all $n \geq n_0$. In consequence, we have

$$\begin{aligned} F(\gamma_n) &\leq F(\gamma_{n-1}) - \varphi(\gamma_{n-1}) \leq \cdots \leq F(\gamma_1) - \sum_{i=1}^{n-1} \varphi(\gamma_i) \\ &= F(\gamma_1) - \sum_{i=1}^{n_0-1} \varphi(\gamma_i) - \sum_{i=n_0}^{n-1} \varphi(\gamma_i) < F(\gamma_1) - (n - n_0)c, \ n > n_0. \end{aligned}$$

Tending with $n \to \infty$ we get $F(\gamma_n) \to -\infty$ and, by (F2"), $\gamma_n \to 0$.

To show that (x_n) is the Cauchy sequence, we will use the technique due to Turinici [32]. Suppose on the contrary that (x_n) is not Cauchy. From (F1) the set Δ of all discontinuity points of F is at most countable. There exists $\eta > 0$, $\eta \notin \Delta$ such that for every $k \geq 0$ one can find $m_k, n_k \in \mathbb{N}$ satisfying

$$k \leq m_k < n_k \text{ and } \mathrm{d}(x_{m_k}, x_{n_k}) > \eta. \tag{2.2}$$

Denote by $\overline{m}_k$ the least of m_k satisfying (2.2) and by $\overline{n}_k$ the least of n_k such that $\overline{m}_k < n_k$ and $\mathrm{d}(x_{\overline{m}_k}, x_{n_k}) > \eta$. Naturally

$$\mathrm{d}(x_{\overline{m}_k}, x_{\overline{n}_k}) > \eta \text{ for all } k \geq 0. \tag{2.3}$$

Observe that taking $k_0 \in \mathbb{N}$ such that $\gamma_k < \eta$ for all $k \geq k_0$, we have

$$\eta < \mathrm{d}(x_{\overline{m}_k}, x_{\overline{n}_k}) \leq \mathrm{d}(x_{\overline{m}_k}, x_{\overline{n}_k - 1}) + \mathrm{d}(x_{\overline{n}_k - 1}, x_{\overline{n}_k}) \leq \eta + \gamma_{\overline{n}_k}, \text{ for all } k \geq k_0.$$

Tending with $k \to \infty$ we obtain

$$\mathrm{d}(x_{\overline{m}_k}, x_{\overline{n}_k}) \to \eta. \tag{2.4}$$

Also observe that for all $k \geq 0$ we have

$$\mathrm{d}(x_{\overline{m}_k}, x_{\overline{n}_k}) - \gamma_{\overline{m}_k+1} - \gamma_{\overline{n}_k+1} \leq \mathrm{d}(x_{\overline{m}_k+1}, x_{\overline{n}_k+1}) \leq \gamma_{\overline{m}_k+1} + \mathrm{d}(x_{\overline{m}_k}, x_{\overline{n}_k}) + \gamma_{\overline{n}_k+1}.$$

Again, tending with $k \to \infty$ we have

$$\mathrm{d}(x_{\overline{m}_k+1}, x_{\overline{n}_k+1}) \to \eta. \tag{2.5}$$

Finally observe that from (H3) we get

$$\varphi(\mathrm{d}(x_{\overline{m}_k}, x_{\overline{n}_k})) \leq F(\mathrm{d}(x_{\overline{m}_k}, x_{\overline{n}_k})) - F(\mathrm{d}(x_{\overline{m}_k+1}, x_{\overline{n}_k+1})), \ k \geq 0.$$

Now, from the above inequality, using (2.3)-(2.5) and the fact that F is continuous at η one gets

$$\begin{aligned}\liminf_{s\to\eta^+}\varphi(s) \le \liminf_{k\to\infty}\varphi(\mathrm{d}(x_{\overline{m}_k},x_{\overline{n}_k}))&\le \lim_{k\to\infty}\left(F(\mathrm{d}(x_{\overline{m}_k},x_{\overline{n}_k}))-F(\mathrm{d}(x_{\overline{m}_k+1},x_{\overline{n}_k+1}))\right)\\ &= 0,\end{aligned}$$

which contradicts (H2). Therefore (x_n) is Cauchy.

The completeness of X and the continuity of T end the proof. □

Before we prove that some class of nonlinear F-contractions consists of β-condensing mappings we introduce a notion which will enable us to measure how much a given function is discontinuous. It will help us to prove that (φ, F)-contractions can be condensing even for discontinuous F.

Let U be a nonempty subset of $\mathbb{R}$, $x_0 \in U$, $f\colon U \to \mathbb{R}$ a function and $\eta \ge 0$.

Definition 2.1. We say that f is continuous at x_0 with accuracy η if for any $\varepsilon > \eta$ there exists $\delta > 0$ such that for all $x \in U$, $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. The set of all $\eta \ge 0$ such that f is continuous at x_0 with accuracy η will be denoted by $\mathrm{disc}(f, x_0)$.

Remark 2.1. Note that the set $\mathrm{disc}(f, x_0)$ is closed. Indeed, if we assume that $\mathrm{disc}(f, x_0)$ is nonempty and consider a sequence $(\eta_n) \subset \mathrm{disc}(f, x_0), \eta_n \to \eta \in \mathbb{R}$, then taking any $\varepsilon > \eta$ one can find $N \in \mathbb{N}$ such that $\varepsilon > \eta_N$. Since f is continuous with accuracy η_N, there exists $\delta > 0$ such that $|x-x_0| < \delta$ implies $|f(x)-(x_0)| < \varepsilon$. In consequence f is continuous at x_0 with accuracy η, i.e. $\eta \in \mathrm{disc}(f, x_0)$.

In light of the above remark we can establish the following definition.

Definition 2.2. If $\mathrm{disc}(f, x_0) \ne \varnothing$, then the measure of discontinuity of f at x_0, denoted by $\sigma(f, x_0)$, will be called the nonnegative number given by the formula

$$\sigma(f, x_0) := \min \mathrm{disc}(f, x_0). \tag{2.6}$$

In case $\mathrm{disc}(f, x_0) = \varnothing$ we put $\sigma(f, x_0) := \infty$.

Lemma 2.1. If $\sigma(f, x_0) < \infty$, then for any sequence $(x_n) \subset U$ such that $x_n \to x_0$, the following inequalities hold:

$$\limsup_{n\to\infty} f(x_n) - \sigma(f, x_0) \le f(x_0) \le \liminf_{n\to\infty} f(x_n) + \sigma(f, x_0). \tag{2.7}$$

Proof. Suppose that $\limsup_{n\to\infty} f(x_n) - \sigma(f, x_0) > f(x_0)$. There exists $c > f(x_0)$ and a subsequence (x_{n_k}) of (x_n) satisfying

$$f(x_{n_k}) - \sigma(f, x_0) \ge c \text{ for all } k \in \mathbb{N}.$$

Letting $\varepsilon = \sigma(f, x_0) + c - f(x_0)$ and taking any $\delta > 0$ one can find $N \in \mathbb{N}$ such that $|x_{n_N} - x_0| < \delta$ and $|f(x_{n_N}) - f(x_0)| \ge f(x_{n_N}) - f(x_0) \ge c + \sigma(f, x_0) - f(x_0) = \varepsilon$, which gives that f is not continuous at x_0 with accuracy $\sigma(f, x_0)$, which is impossible. By an analogous argument we prove the second inequality. □

Observe that from the definition of the measure of discontinuity, the equality $\sigma(f, x_0) = 0$ easily implies the continuity of the function f at x_0, and then inequality (2.7) takes the form $\lim_{n\to\infty} f(x_n) = f(x_0)$.

A straightforward consequence of Lemma 2.1 is the following.

Remark 2.2. If $\sigma(f,x_0)<\infty$, then for any sequence $(x_n)\subset U$ such that $x_n\to x_0$ the following holds:

$$\limsup_{n\to\infty} f(x_n)-\liminf_{n\to\infty} f(x_n)\le 2\sigma\left(f,x_0\right). \tag{2.8}$$

Example 2.1.

1) For the Dirichlet function $I_{\mathbb{Q}}\colon\mathbb{R}\to\{0,1\}$, where $I_{\mathbb{Q}}(x)=1$ for $x\in\mathbb{Q}$ and $I_{\mathbb{Q}}(x)=0$ otherwise, $\sigma\left(I_{\mathbb{Q}},x_0\right)=1$ for every $x_0\in\mathbb{R}$.
2) Consider the function $f\colon(-\infty,0)\to(-\infty,0)$ of the form
$$f(t)=\begin{cases} t-1, & \text{if } t<-1;\\ t-\frac{1}{n+1} & \text{if } t\in\left[-\frac{1}{n},-\frac{1}{n+1}\right),\ n\in\mathbb{N}.\end{cases}$$
Then $\sigma\left(f,-\frac{1}{n}\right)=\frac{1}{n(n+1)}$ and $\sigma\left(f,x\right)=0$ for $x\neq-\frac{1}{n}$, $n\in\mathbb{N}$.
3) If $f(t)=1/t, t>0$ and $f(0)=0$, then $\sigma(f,0)=\infty$.

Now, we are in a position to state our main results.

Theorem 2.2. *Let (X,d) be a metric space and let $T\colon X\to X$ be a (φ,F)-contraction. If*

$$\liminf_{s\to t}\varphi(s)>2\sigma(F,t) \tag{2.9}$$

for all $t>0$, then T is β-condensing.

Proof. First, observe that by (F1) we have $\sigma(F,t)<\infty$ and thus inequality (2.9) is valid. Take any $C\subset X$ with positive Hausdorff measure of noncompactness. Since $2\sigma(F,\beta(C))<\liminf_{s\to\beta(C)}\varphi(s)$, there exists $\varepsilon>0$ such that

$$2\sigma(F,\beta(C))<\varphi(s) \text{ for all } s\in[\beta(C)-\varepsilon,\beta(C)+\varepsilon]. \tag{2.10}$$

Applying to F Remark 2.2 and using (F1) we have

$$\lim_{t\to\beta(C)^+}F(t)-\lim_{t\to\beta(C)^-}F(t)\le 2\sigma(F,\beta(C))<\varphi(s) \tag{2.11}$$

for all $s\in[\beta(C)-\varepsilon,\beta(C)+\varepsilon]$. Taking in (2.10) appropriately small enough ε, we can deduce from (2.11) the inequality

$$F(r)-F(t)<\varphi(s) \tag{2.12}$$

for all $r\in(\beta(C),\beta(C)+\varepsilon]$, $t\in[\beta(C)-\varepsilon,\beta(C))$, $s\in[\beta(C)-\varepsilon,\beta(C)+\varepsilon]$. Consider $R=\beta(C)+\varepsilon$ and take a finite R-net of C, i.e.

$$C\subset\bigcup_{i=1}^{k}B(x_i,R),\ x_1,\dots,x_k\in X. \tag{2.13}$$

Denote $R'=\beta(C)-\varepsilon$. We will show that the open balls $B(Tx_i,R')$, $1\le i\le k$, cover $T(C)$. Let $y\in T(C)$ and let $x\in C$ be such that $Tx=y$. From (2.13) there exists $i\in\{1,\dots,k\}$ such that $\mathrm{d}(x,x_i)<R$. If $Tx=Tx_i$, then obviously $\mathrm{d}(y,Tx_i)<R'$. Suppose that $Tx\neq Tx_i$ and consider two cases. If $0<\mathrm{d}(x,x_i)<R'$, then, since T is a (φ,F)-contraction, we have

$$F(\mathrm{d}(Tx,Tx_i))\le F(\mathrm{d}(x,x_i))-\varphi(\mathrm{d}(x,x_i))\le F(R')-\varphi(\mathrm{d}(x,x_i))<F(R').$$

If $R'\le\mathrm{d}(x,x_i)<R$, then due to (2.12) we obtain the inequalities

$$F(\mathrm{d}(Tx,Tx_i))\le F(\mathrm{d}(x,x_i))-\varphi(\mathrm{d}(x,x_i))<F(R)-\varphi(\mathrm{d}(x,x_i))<F(R').$$

In the above both cases (F1) implies $d(Tx, Tx_i) < R'$, which gives

$$\beta(T(C)) \leq R' < \beta(C).$$

□

The immediate consequence of Theorem 2.2 is the following result.

Corollary 2.1. *Every (φ, F)-contraction with continuous F satisfying $\liminf_{s\to t} \varphi(s) > 0$ for all $t > 0$ is β-condensing.*

The consequence of Sadovskii's result is the following theorem.

Theorem 2.3. *Let C be a closed bounded and convex subset of a Banach space X. If $A\colon C \to X$ is a (φ, F)-contraction with $\liminf_{s\to t} \varphi(s) > 2\rho(F, t)$ for all $t > 0$, $B\colon C \to X$ is a compact operator and $(A + B)(C) \subset C$, then $A + B$ has a fixed point.*

Proof. Take any $K \subset C$ which is not relatively compact. Using the properties of the Hausdorff measure of noncompactness, Theorem 2.2 and the fact that $\beta(B(C)) = 0$ we obtain

$$\beta((A + B)(C)) \leq \beta(A(C) + B(C)) \leq \beta(A(C)) + \beta(B(C)) < \beta(C),$$

which gives that $A+B$ is a β-condensing mapping; thus by Sadovskii's result, $A+B$ has a fixed point. □

Corollary 2.1 and Theorem 2.3 simply yield the following result, which will be applied in the next section.

Corollary 2.2. *Let X be a Banach space, C a closed bounded and convex subset of X. If $A\colon C \to X$ is a (φ, F)-contraction with continuous F satisfying $\liminf_{s\to t} \varphi(s) > 0$ for all $t > 0$, $B\colon C \to X$ a compact operator and $(A+B)(C) \subset C$, then $A + B$ has a fixed point.*

3. The applications of nonlinear F-contractions

In this section we show the applicability of the obtained results.

3.1. Integral equation of Volterra type.

First we present the application of the existence of fixed point for (φ, F)-contractions to the following equation of Volterra type:

$$x(t) = \int_0^t K(t, s, x(s))ds + h(t),\ t \in I, \tag{3.1}$$

where $T > 0$, $I = [0, T]$, $K\colon I \times I \times \mathbb{R} \to \mathbb{R}$, $h\colon I \to \mathbb{R}$.

In order to obtain our claims, we will need the following assumptions:

(C1) The functions h, K are continuous.

(C2) There exists a strictly increasing sequence $(\alpha_n)_{n\in\mathbb{N}\cup\{0\}}$ satisfying $\alpha_0 = 0$, $\alpha_n \geq 1$, $\alpha_n - \alpha_{n-1} \leq 1$ for all $n \in \mathbb{N}$, $\alpha_n \to \infty$ such that for any $n \in \mathbb{N}$,

$$|K(t, s, u) - K(t, s, v)| \leq \frac{\alpha_n}{1 + \alpha_n(\alpha_n - \alpha_{n-1})} e^{-t\alpha_n} |u - v| \tag{3.2}$$

for all $s, t \in I$ and $u, v \in \mathbb{R}$ such that $|u - v| < \alpha_n e^T$.

Consider the Banach space $C(I)$ of all continuous functions $x\colon I \to \mathbb{R}$ equipped with Bielecki's norm:

$$\|x\| = \sup_{t\in I} \mathrm{e}^{-t}|x(t)|.$$

Now, we are ready to enunciate our first existence result.

Theorem 3.1. *If* (C1) *and* (C2) *are satisfied, then the nonlinear problem* (3.1) *has a unique solution in* $C(I)$.

Proof. Consider the operator $\mathcal{L}\colon C(I) \to C(I)$ as follows:

$$(\mathcal{L}x)(t) = \int_0^t K(t,s,x(s))ds + h(t),\ x \in C(I).$$

A fixed point of the operator $\mathcal{L}$ will be a solution of the equation (3.1). In order to fulfil all the assumptions of Theorem 2.1 let us consider a function $F(t) = -1/t$, $t > 0$, and $\varphi\colon (0,\infty) \to (0,\infty)$ of the form

$$\varphi(t) = \begin{cases} -t + \alpha_1, & 0 < t < \alpha_1, \\ -t + \alpha_n, & \alpha_{n-1} \le t < \alpha_n,\ n \ge 2. \end{cases}$$

In this case one can calculate that the contraction condition (H3) takes the following form:

$$\|Tx - Ty\| \le \frac{\|x-y\|}{1 + \|x-y\|\big[\alpha_n - \|x-y\|\big]}, \tag{3.3}$$

for all $x, y \in C(I)$ satisfying $\alpha_{n-1} \le \|x-y\| < \alpha_n$ when $n \ge 2$ and $0 < \|x-y\| < \alpha_1$ for $n = 1$.

We will show that $\mathcal{L}$ satisfies (3.3). Fix $n \ge 2$ and take any $x, y \in C(I)$ such that $\alpha_{n-1} \le \|x-y\| < \alpha_n$. Observe that for each $s \in I$ we have

$$|x(s) - y(s)| \le \mathrm{e}^s \sup_{s\in I} \mathrm{e}^{-s}|x(s) - y(s)| < \mathrm{e}^s\alpha_n \le \mathrm{e}^T\alpha_n.$$

Therefore, due to (C2), we obtain

$$\begin{aligned} |(\mathcal{L}x)(t) - (\mathcal{L}y)(t)| &\le \int_0^t |K(t,s,x(s)) - K(t,s,y(s))|ds \\ &\le \frac{\alpha_n}{1 + \alpha_n(\alpha_n - \alpha_{n-1})}\mathrm{e}^{-t\alpha_n}\int_0^t |x(s) - y(s)|ds,\ t \in I. \end{aligned}$$

Next, we see that

$$1 + \|x-y\|\big(\alpha_n - \|x-y\|\big) < 1 + \alpha_n(\alpha_n - \alpha_{n-1}),$$

and, since $\alpha_{n+1} > 1$, $-s\alpha_{n+1} \le -s$ for all $s \in I$. In consequence, the following holds:

$$\begin{aligned}
|(\mathcal{L}x)(t) - (\mathcal{L}y)(t)| &\le \frac{\alpha_n}{1 + \|x-y\|\big(\alpha_n - \|x-y\|\big)} \mathrm{e}^{-t\alpha_n} \int_0^t |x(s) - y(s)| ds \\
&= \frac{\alpha_n}{1 + \|x-y\|\big(\alpha_n - \|x-y\|\big)} \mathrm{e}^{-t\alpha_n} \int_0^t |x(s) - y(s)| \mathrm{e}^{-s\alpha_{n+1}} \mathrm{e}^{s\alpha_{n+1}} ds \\
&\le \frac{\alpha_n \|x-y\|}{1 + \|x-y\|\big(\alpha_n - \|x-y\|\big)} \mathrm{e}^{-t\alpha_n} \int_0^t \mathrm{e}^{s\alpha_{n+1}} ds \\
&< \frac{\alpha_n \|x-y\|}{1 + \|x-y\|\big(\alpha_n - \|x-y\|\big)} \mathrm{e}^{-t\alpha_n} \frac{1}{\alpha_{n+1}} \mathrm{e}^{t\alpha_{n+1}} \\
&= \frac{\alpha_n \|x-y\|}{\alpha_{n+1}\Big(1 + \|x-y\|\big(\alpha_n - \|x-y\|\big)\Big)} \mathrm{e}^{t(\alpha_{n+1} - \alpha_n)}, \ t \in I.
\end{aligned}$$

Using again the properties of the sequence (α_n) we obtain

$$\mathrm{e}^{-t} |(\mathcal{L}x)(t) - (\mathcal{L}y)(t)| \le \frac{\|x-y\|}{1 + \|x-y\|\big(\alpha_n - \|x-y\|\big)}, \ t \in I.$$

Taking the supremum with respect to t in the above inequality we obtain (3.3). The analogous calculations are for $n = 1$. Theorem 2.1 ends the proof. □

3.2. **Implicit integral equations.** In this section we apply our result of Krasnosel'skii type to the general class of integral equations

$$V(t, x(t)) = S\left(t, \int_0^t H(t, s, x(s)) ds\right), \tag{3.4}$$

where $V, S \colon [-\alpha, \alpha] \times [-\alpha, \alpha] \to \mathbb{R}$ and $H \colon [-\alpha, \alpha] \times [-\alpha, \alpha] \times [-\alpha, \alpha] \to \mathbb{R}$ are continuous, $\alpha > 0$. The equations of this type were studied e.g. in [2] and [4]. We will look for the solution of (3.4) in a subset C of the Banach space X of continuous functions $\zeta \colon [-\beta, \beta] \to \mathbb{R}$, $0 < \beta < \alpha$, equipped with the supremum norm of the form

$$C := \{\zeta \in X \colon \zeta(0) = 0, \ \|\zeta\| \le \alpha\}.$$

Theorem 3.2. *If $S(0,0) = V(t,0) = 0$ for all $t \in [-\alpha, \alpha]$ and the operator $(A\zeta)(t) = \zeta(t) - V(t, \zeta(t))$ is a (φ, F)-contraction on C with continuous F and nonincreasing φ, then* (3.4) *has a solution in C.*

Proof. Denote $\theta(t) = 0$ for all $t \in [-\alpha, \alpha]$. We have $(A\theta)(t) = \theta(t) - V(t, \theta(t)) = 0$, $t \in [-\alpha, \alpha]$. Taking any $\zeta \in C$ with $A\zeta \ne \theta$, by (H3), we have

$$F(\|A\zeta\|) = F(\|A\zeta - A\theta\|) \le F(\|\zeta\|) - \varphi(\|\zeta\|) < F(\|\zeta\|),$$

which, by (F1), gives

$$\|A\zeta\| < \|\zeta\|. \tag{3.5}$$

We show that for any $\zeta \in C$,

$$\|A\zeta\| \le \gamma < \alpha \text{ for some } \gamma > 0. \tag{3.6}$$

In another case there exists a sequence $(\zeta_n) \subset C$ such that $(\|A\zeta_n\|)$ is increasing and $\lim_{n\to\infty} \|A\zeta_n\| = \alpha$. Since $\zeta_n \in C$, (3.5) implies $\|A\zeta_n\| \le \|\zeta_n\| \le \alpha$ and thus $\lim_{n\to\infty} \|\zeta_n\| = \alpha$. In consequence, we obtain

$$\varphi(\alpha) \le \varphi(\|\zeta_n\|) \le F(\|\zeta_n\|) - F(\|A\zeta_n\|).$$

Tending with $n \to \infty$ and using a continuity of F we get $\varphi(\alpha) \le 0$, which is impossible.

The rest of the proof goes as in Burton's result, Theorem 3 in [2]. Defining a mapping $B\colon C \to C$ by $(B\zeta)(t) = S(t, \int_0^t H(t,s,\zeta(s))ds)$ and using a continuity of H on its compact domain, continuity of S and the fact that $S(0,0) = 0$ we show that there exists $0 < \beta < \alpha$ such that $\zeta \in C$ and $t < \beta$ imply $|(B\zeta)(t)| \le \alpha - \gamma$, which together with (3.6) gives $\|A\zeta + B\zeta\| \le \alpha$. A simple fact $(A\zeta)(0) + (B\zeta)(0) = 0$ finally gives $(A+B)(\zeta) \in C$. The compactness of B is received by showing that BC is an equicontinuous set (for details, see [2]). Finally, note that $\liminf_{s\to t^+} \varphi(s) > 0$ for all $t \ge 0$, since φ is decreasing. Thus, all the assumptions of Corollary 2.2 are satisfied. □

In many practical situations a common difficulty which may occur is when we want to verify the contractivity of the operator A. However, when we reduce our investigations to nonnegative solutions, we may obtain more comfortable and applicable tools. Applying the analogous methods as in the proof of the above result we can prove the following theorem.

Theorem 3.3. *If $S \ge 0$, $S(0,0) = V(t,0) = 0$ for all $t \in [-\alpha,\alpha]$ and the operator $(A\zeta)(t) = \zeta(t) - V(t,\zeta(t))$ is a (φ,F)-contraction on C^+ of the form*

$$C^+ := \{\zeta \in C\colon \zeta \ge 0\}$$

with continuous F, nonincreasing φ and $0 \le V(t,\zeta(t)) \le \zeta(t)$ for every $\zeta \in C^+$, $t \in [-\alpha,\alpha]$, then (3.4) has a solution in C^+.

Proof. In light of the proof of Theorem 3.2 it is enough to observe that $A\zeta \in C^+$ and $(A+B)(\zeta) \ge 0$ for each $\zeta \in C^+$. □

We show the applicability of the above result by the following example.

Example 3.1. Consider the differential equation of the form

$$2xx'(x+1) = \big(2t + G(t,x)\big)(2x+1)^2, \tag{3.7}$$

where $G\colon [-\alpha,\alpha]\times[-\alpha,\alpha] \to \mathbb{R}^+$ is continuous, $\alpha > 0$. We will verify the existence of a solution in C^+. The equation (3.7) can be transformed into the form

$$\frac{2xx'(x+1)}{(2x+1)^2} - 2t = G(t,x),$$

where x is evaluated at each t. Next, we obtain

$$\begin{aligned} t^2 + \int_0^t G(s,x(s))ds &= \int_0^t \frac{2x(s)x'(s)(x(s)+1)}{(2x(s)+1)^2}ds = \int_0^{x(t)} \frac{2u(u+1)}{(2u+1)^2}du \\ &= \int_0^{x(t)} \left[\frac{1}{2} - \frac{1}{2(2u+1)^2}\right]du = \frac{x(t)}{2} - \frac{2x(t)}{4(2x(t)+1)} \\ &= \frac{x^2(t)}{2x(t)+1}. \end{aligned}$$

Now, we can see that the equation (3.7) can be presented in the form (3.4). In order to fulfil the assumptions of Theorem 3.2 we show that the operator

$$Ax := x - \frac{x^2}{2x+1} = \frac{x^2+x}{2x+1}$$

is a (φ, F)-contraction on C^+. Indeed, taking any $x, y \in C^+$ we have (in the following x, y are evaluated in each $t \in [-\alpha, \alpha]$)

$$|Ax - Ay| = \frac{|x-y|(x+y+2xy+1)}{2x+2y+4xy+1}.$$

Observe that

$$|x-y| \leq x+y+2xy,$$

which together with the fact that a function $t \mapsto \frac{1+t}{1+2t}, t \geq 0$, is decreasing gives us

$$|Ax - Ay| \leq \frac{|x-y|(1+|x-y|)}{1+2|x-y|}.$$

Next, using the increasing function $t \mapsto \frac{t(1+t)}{1+2t}, t \geq 0$, we have

$$\|Ax-Ay\| \leq \frac{\sup_{t\in[0,\alpha]}|x(t)-y(t)|\big(1+\sup_{t\in[0,\alpha]}|x(t)-y(t)|\big)}{1+2\sup_{t\in[0,\alpha]}|x(t)-y(t)|} = \frac{1+\|x-y\|}{1+2\|x-y\|}\|x-y\|.$$

The last inequality, if $Ax \neq Ay$, can be presented in the form

$$\frac{1}{\|x-y\|+1} - \frac{1}{\|Ax-Ay\|} \leq -\frac{1}{\|x-y\|}.$$

Now one can observe that the above contraction condition is a (φ, F)-contraction for $\varphi(s) = 1/(s+1)$, $s > 0$, and $F(s) = -1/s$, $s > 0$.

References

[1] J. Banaś and K. Goebel, *Measures of noncompactness in Banach spaces*, Lecture Notes in Pure and Applied Mathematics, vol. 60, Marcel Dekker, Inc., New York, 1980. MR591679

[2] T. A. Burton, *Integral equations, implicit functions, and fixed points*, Proc. Amer. Math. Soc. **124** (1996), no. 8, 2383–2390, DOI 10.1090/S0002-9939-96-03533-2. MR1346965

[3] T. A. Burton and C. Kirk, *A fixed point theorem of Krasnoselskii-Schaefer type*, Math. Nachr. **189** (1998), 23–31, DOI 10.1002/mana.19981890103. MR1492921

[4] C. Corduneanu, *Integral equations and applications*, Cambridge University Press, Cambridge, 1991. MR1109491

[5] G. Darbo, *Punti uniti in trasformazioni a codominio non compatto* (Italian), Rend. Sem. Mat. Univ. Padova **24** (1955), 84–92. MR0070164

[6] J. Garcia-Falset, K. Latrach, E. Moreno-Gálvez, and M.-A. Taoudi, *Schaefer-Krasnoselskii fixed point theorems using a usual measure of weak noncompactness*, J. Differential Equations **252** (2012), no. 5, 3436–3452, DOI 10.1016/j.jde.2011.11.012. MR2876659

[7] N. Hussain, A. Latif, and I. Iqbal, *Fixed point results for generalized F-contractions in modular metric and fuzzy metric spaces*, Fixed Point Theory Appl., 2015:158, 20 pp., DOI 10.1186/s13663-015-0407-1. MR3392555

[8] N. Hussain and P. Salimi, *Suzuki-Wardowski type fixed point theorems for α-GF-contractions*, Taiwanese J. Math. **18** (2014), no. 6, 1879–1895, DOI 10.11650/tjm.18.2014.4462. MR3284036

[9] I. Iqbal and N. Hussain, *Fixed point theorems for generalized multivalued nonlinear $\mathbb{F}$-contractions*, J. Nonlinear Sci. Appl. **9** (2016), no. 11, 5870–5893. MR3584046

[10] I. Iqbal, N. Hussain, and N. Sultana, *Fixed points of multivalued non-linear F-contractions with application to solution of matrix equations*, Filomat **31** (2017), no. 11, 3319-3333.

[11] J. R. Jachymski, *Equivalence of some contractivity properties over metrical structures*, Proc. Amer. Math. Soc. **125** (1997), no. 8, 2327–2335, DOI 10.1090/S0002-9939-97-03853-7. MR1389524

[12] J. Jachymski, *Equivalent conditions for generalized contractions on (ordered) metric spaces*, Nonlinear Anal. **74** (2011), no. 3, 768–774, DOI 10.1016/j.na.2010.09.025. MR2738628

[13] W. A. Kirk, *Contraction mappings and extensions*, Handbook of metric fixed point theory, Kluwer Acad. Publ., Dordrecht, 2001, pp. 1–34. MR1904272

[14] D. Klim and D. Wardowski, *Fixed points of dynamic processes of set-valued F-contractions and application to functional equations*, Fixed Point Theory Appl., 2015:22, 9 pp., DOI 10.1186/s13663-015-0272-y. MR3311515

[15] M. A. Krasnosel′skiĭ, *Some problems of nonlinear analysis*, American Mathematical Society Translations, Ser. 2, Vol. 10, American Mathematical Society, Providence, R.I., 1958, pp. 345–409. MR0094731

[16] W. Kryszewski and J. Mederski, *Fixed point index for Krasnosel′skii-type set-valued maps on complete ANRs*, Topol. Methods Nonlinear Anal. **28** (2006), no. 2, 335–384. MR2289691

[17] Y. Liu and Z. Li, *Krasnoselskii type fixed point theorems and applications*, Proc. Amer. Math. Soc. **136** (2008), no. 4, 1213–1220, DOI 10.1090/S0002-9939-07-09190-3. MR2367095

[18] G. Mınak, A. Helvacı, and I. Altun, *Ćirić type generalized F-contractions on complete metric spaces and fixed point results*, Filomat **28** (2014), no. 6, 1143–1151, DOI 10.2298/FIL1406143M. MR3360088

[19] D. Paesano and C. Vetro, *Multi-valued F-contractions in 0-complete partial metric spaces with application to Volterra type integral equation*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **108** (2014), no. 2, 1005–1020, DOI 10.1007/s13398-013-0157-z. MR3249990

[20] H. Piri and P. Kumam, *Some fixed point theorems concerning F-contraction in complete metric spaces*, Fixed Point Theory Appl., 2014:210, 11 pp., DOI 10.1186/1687-1812-2014-210. MR3357360

[21] B. Przeradzki, *A generalization of Krasnosel'skii fixed point theorem for sums of compact and contractible maps with application*, Cent. Eur. J. Math. **10** (2012), no. 6, 2012–2018, DOI 10.2478/s11533-012-0102-y. MR2983143

[22] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **226** (1977), 257–290, DOI 10.2307/1997954. MR0433430

[23] B. N. Sadovskiĭ, *Limit-compact and condensing operators* (Russian), Uspehi Mat. Nauk **27** (1972), no. 1(163), 81–146. MR0428132

[24] H. Schaefer, *Über die Methode der a priori-Schranken* (German), Math. Ann. **129** (1955), 415–416, DOI 10.1007/BF01362380. MR0071723

[25] N. A. Secelean, *Weak F-contractions and some fixed point results*, Bull. Iranian Math. Soc. **42** (2016), no. 3, 779–798. MR3518218

[26] N.-A. Secelean, *Generalized F-iterated function systems on product of metric spaces*, J. Fixed Point Theory Appl. **17** (2015), no. 3, 575–595, DOI 10.1007/s11784-015-0235-2. MR3411806

[27] N.-A. Secelean, *Iterated function systems consisting of F-contractions*, Fixed Point Theory Appl., 2013:277, 13 pp., DOI 10.1186/1687-1812-2013-277. MR3213104

[28] N.A. Secelean and D. Wardowski, *New fixed point tools in non-metrizable spaces*, Results. Math. (2017), DOI: 10.1007/s00025-017-0688-2.

[29] N.-A. Secelean and D. Wardowski, *ψF-contractions: not necessarily nonexpansive Picard operators*, Results Math. **70** (2016), no. 3-4, 415–431, DOI 10.1007/s00025-016-0570-7. MR3544869

[30] M. Sgroi and C. Vetro, *Multi-valued F-contractions and the solution of certain functional and integral equations*, Filomat **27** (2013), no. 7, 1259–1268, DOI 10.2298/FIL1307259S. MR3243997

[31] T. Suzuki, *Discussion of several contractions by Jachymski's approach*, Fixed Point Theory Appl., 2016:91, 11 pp., DOI 10.1186/s13663-016-0581-9. MR3548683

[32] M. Turinici, Wardowski implicit contractions in metric spaces, (2013) arXiv:1212.3164v2 [Math.GN].

[33] X. Udo-utun, *On inclusion of F-contractions in (δ, k)-weak contractions*, Fixed Point Theory Appl., posted on 2013, 2014:65, 6 pp., DOI 10.1186/1687-1812-2014-65. MR3345452

[34] D. Wardowski and N. Van Dung, *Fixed points of F-weak contractions on complete metric spaces*, Demonstr. Math. **47** (2014), no. 1, 146–155, DOI 10.2478/dema-2014-0012. MR3200192

[35] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., 2012:94, 6 pp., DOI 10.1186/1687-1812-2012-94. MR2949666

[36] K. Włodarczyk, *Periodic and fixed points of the Leader-type contractions in quasi-triangular spaces*, Fixed Point Theory Appl., 2016:85, 28 pp., DOI 10.1186/s13663-016-0575-7. MR3538658

Department of Nonlinear Analysis, Faculty of Mathematics and Computer Science, University of Łódź, Banacha 22, 90-238 Łódź, Poland

E-mail address: `wardd@math.uni.lodz.pl`