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BRIESKORN SPHERES BOUNDING RATIONAL BALLS

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ABSTRACT. Fintushel and Stern showed that the Brieskorn sphere $\Sigma(2,3,7)$ bounds a rational homology ball, while its non-trivial Rokhlin invariant obstructs it from bounding an integral homology ball. It is known that their argument can be modified to show that the figure-eight knot is rationally slice, and we use this fact to provide the first additional examples of Brieskorn spheres that bound rational homology balls but not integral homology balls: the families $\Sigma(2,4n+1,12n+5)$ and $\Sigma(3,3n+1,12n+5)$ for n odd. We also provide handlebody diagrams for a rational homology ball containing a rationally slice disk for the figure-eight knot, as well as for a rational homology ball bounded by $\Sigma(2,3,7)$. These handle diagrams necessarily contain 3-handles.

1. INTRODUCTION

A classic question in low-dimensional topology asks which 3-dimensional integral homology spheres smoothly bound integral homology balls. Due to their nice properties, a reasonable starting point to address this question is to consider the Brieskorn homology spheres $\Sigma(p,q,r) = \{x^p + y^q + z^r = 0\} \cap S^5 \subset \mathbb{C}^3$, with p,q,r positive and relatively prime. A large number of Brieskorn spheres are known to bound integral homology balls (or even contractible 4-manifolds), for example see [AK79], [CH81], [Ste78], [FS81], and [Fic84]. One can weaken the above question to ask which integral homology spheres bound *rational* homology balls; however, it turns out that this has not helped much in producing more examples. Indeed it appears to be a difficult problem to find Brieskorn spheres that bound rational homology balls but not integral homology balls. Fintushel and Stern [FS84] provided the first example by constructing a rational homology ball bounded by the Brieskorn sphere $\Sigma(2,3,7)$. Since $\Sigma(2,3,7)$ has Rokhlin invariant $\mu = 1$, it cannot bound an integral homology ball. In this note we give the first new examples. Using the fact that the figure-eight knot is rationally slice (see Section 3), a simple observation shows that the $\mu = 1$ Brieskorn sphere $\Sigma(2,3,19)$ bounds a rational homology ball (Proposition 3). More interesting are two infinite families of Brieskorn spheres.

Theorem 1. *The Brieskorn spheres $\Sigma(2,4n+1,12n+5)$ and $\Sigma(3,3n+1,12n+5)$ bound rational homology balls, and when n is odd they have $\mu = 1$ and so do not bound integral homology balls.*

The difficulty of this problem is related to handle decompositions of 4-manifolds. A simple homological argument shows that if an integral homology 3-sphere bounds

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a rational homology ball X that is not an integral homology ball, then any handle decomposition of X must contain 3-handles. Therefore the difficulty of working with handle decompositions with 3-handles points to the challenge and interest of finding these examples.

One of the main motivations for our work relates to the group of integral homology spheres $\Theta^3_{\mathbb{Z}}$ and the group of rational homology spheres $\Theta^3_{\mathbb{Q}}$. There is a canonical homomorphism $\psi\colon \Theta^3_{\mathbb{Z}} \to \Theta^3_{\mathbb{Q}}$ induced by inclusion (see [AL16] for more discussion of this homomorphism), and the work of [FS84] can be interpreted as showing that the kernel of ψ is non-trivial. Since $\Sigma(2,3,7)$ has infinite order in $\Theta^3_{\mathbb{Z}}$ we get that the kernel contains a subgroup isomorphic to $\mathbb{Z}$. Beyond this nothing is known of its structure, but it seems likely that the kernel is in fact much larger. Theorem 1 gives a large collection of additional Brieskorn spheres that represent non-trivial elements in the kernel of ψ, but it is unknown if they are linearly independent in $\Theta^3_{\mathbb{Z}}$. Since Brieskorn spheres are often amenable to the computation of gauge and Floer theoretic invariants, it is possible that the following is a tractable question.

Question. Is some subset of the Brieskorn spheres $\Sigma(2,3,7)$, $\Sigma(2,3,19)$, $\Sigma(2,4n+1,12n+5)$, and $\Sigma(3,3n+1,12n+5)$ linearly independent in $\Theta^3_{\mathbb{Z}}$?

Now we outline the proof of Theorem 1. Recall that $\Sigma(2,3,7)$ can be obtained by $+1$-surgery on the figure-eight knot, and in fact the construction in [FS84] can be modified to show that the figure-eight knot is rationally slice, that is, bounds a smooth disk in a rational homology ball bounded by S^3 (see [Cha07]). This fact was apparently also known by Kawauchi[1] [Kaw80], whose argument moreover generalizes to show that all strongly negative-amphicheiral knots are rationally slice (see [Kaw09], [KW16]). In Section 3 we give a handle proof that the figure-eight knot is rationally slice. Our proof uses the fact that the linking circle (meridian) of the 1-handle of $-W^+(0,2)$ is slice in $-W^+(0,2)$ (see p. 23 of [Akb16] for the notation).

If Y denotes the 3-manifold obtained by 0-surgery on the figure-eight knot, it then follows that Y bounds a 4-manifold with the rational homology of $S^1 \times D^3$. Hence any homology sphere obtained by integral surgery on Y will bound a rational homology ball (the surgery corresponds to attaching a 2-handle to the rational homology $S^1 \times D^3$ which kills the non-torsion part of the homology). Theorem 1 is then proved by showing that each $\Sigma(2,4n+1,12n+5)$ and $\Sigma(3,3n+1,12n+5)$ can be obtained by an integral surgery on Y. We do this in Section 2.

In Section 3 we give handle diagrams for some of the relevant rational homology balls. In particular we give handle diagrams for a rational homology ball bounded by $\Sigma(2,3,7)$, and a rational homology ball bounded by S^3 showing a rationally slice disk for the figure-eight knot. While the arguments in [FS84] and [Cha07] are constructive, they do not give explicit handle diagrams for the rational homology balls they construct.

[1]Kawauchi adds an extra algebraic condition in his definition of rationally slice and shows that the (2,1)-cable of the figure-eight knot satisfies this stronger condition. However, it is implicit in his argument that the figure-eight knot satisfies the weaker definition we use.

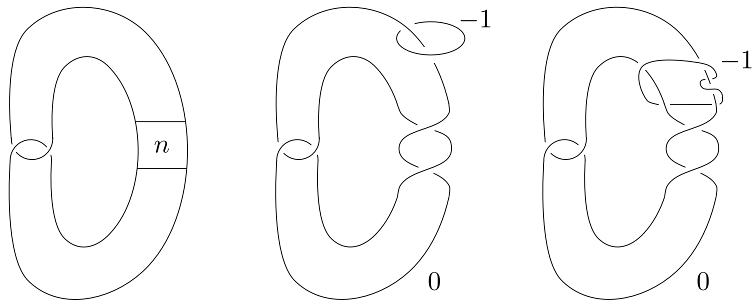


FIGURE 1. On the left we have the knot K_n, where the box denotes n full right-handed twists. In the middle is $\Sigma(2,3,7)$, and on the right is $\Sigma(2,3,19)$.

2. Proof of Theorem 1

We start with the following lemma whose proof was outlined in Section 1. As before, let Y denote 0-surgery on the figure-eight knot.

Lemma 2. *Any integral homology sphere obtained by integral surgery on Y bounds a rational homology ball.*

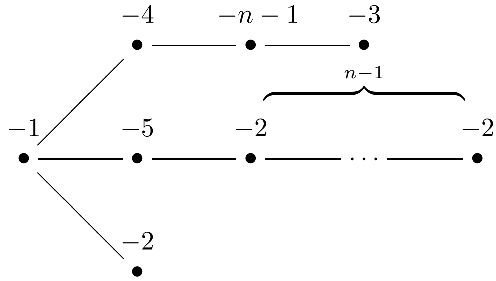
Proof. Let X be a rational homology ball with boundary S^3 such that the figure-eight knot bounds a smooth properly embedded disk D in X (for example, see Section 3). Then the Mayer-Vietoris long exact sequence shows that $C := X \setminus \nu D$ has the rational homology of $S^1 \times D^3$. The manifold C has boundary Y (the induced framing on ∂D is the 0-framing on the figure-eight knot because gluing D to a Seifert surface for the figure-eight knot gives a closed surface in X, a 4-manifold with trivial intersection form). An integral surgery on Y corresponds to attaching a 2-handle to C. If the resulting 3-manifold is an integral homology sphere, then the 4-manifold W obtained by attaching the corresponding 2-handle to C must be a rational homology ball, as can be seen from the Mayer-Vietoris long exact sequence and the long exact sequence of the pair $(W, \partial W)$. $\square$

We remark that in the previous lemma we can use 0-surgery on any rationally slice knot, and not just the figure-eight knot.

Proposition 3. $\Sigma(2,3,19)$ *bounds a rational homology ball.*

Proof. Using Lemma 2 it suffices to show that $\Sigma(2,3,19)$ can be obtained by an integral surgery on Y. The Brieskorn sphere $\Sigma(2,3,6n+1)$ admits a surgery description as $+1$-surgery on the twist knot K_n defined as in Figure 1 (for this and plumbing descriptions of Brieskorn spheres see [Sav02]). Note that K_1 is the figure-eight knot. Blowing down the -1-framed components in the middle and right pictures of Figure 1 results in $+1$-surgery on K_1 and K_3, respectively, showing that $\Sigma(2,3,7)$ and $\Sigma(2,3,19)$ bound rational homology balls. $\square$

Proof of Theorem 1. For the families $\Sigma(2, 4n+1, 12n+5)$ and $\Sigma(3, 3n+1, 12n+5)$ we use the dual approach, giving integral surgeries from their canonical negative definite plumbings to Y. For $\Sigma(2, 4n+1, 12n+5)$ these plumbings take the form



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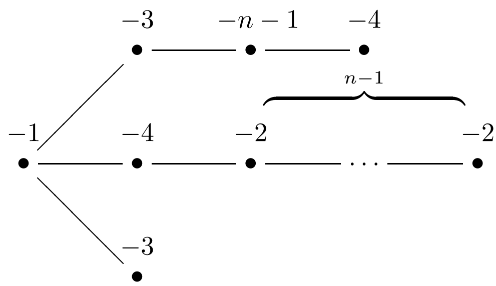
and for $\Sigma(3, 3n+1, 12n+5)$ we have



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When $n = 1$ we get $\Sigma(2, 5, 17)$ and $\Sigma(3, 4, 17)$. Surgery diagrams for their canonical negative definite plumbings appear as the gray components in Figure 2 and Figure 3, and the black component gives the necessary surgery to Y. This can be seen from a straightforward sequence of blowdowns which we leave to the reader.

Now we describe an iterative procedure to obtain the whole families. Starting with the plumbing for either $\Sigma(2, 5, 17)$ or $\Sigma(3, 4, 17)$, in Figure 2 or Figure 3 we can blow up to unlink the black -1-framed component from the -2-framed component. This is demonstrated in Figure 4. The result is to lower the framing of the -2-framed component to -3, and the previous surgery curve becomes a -2-framed component in the bottom chain. If we forget about the newly introduced -1-framed unknot we see a plumbing for the $n = 2$ case, and the -1-framed unknot again provides the required surgery to Y since blowing up does not change the boundary 3-manifold. It is clear that we can keep blowing up in this fashion, each time adding a -2-framed component to the bottom chain and decreasing the framing on the appropriate component in the upper-right chain by 1. Hence if we start with $\Sigma(2, 5, 17)$, blowing up in this way will generate the family $\Sigma(2, 4n+1, 12n+5)$, and starting with $\Sigma(3, 4, 17)$ generates $\Sigma(3, 3n+1, 12n+5)$. In each case the -1-framed unknot coming from the blow up provides the surgery to Y, and so by Lemma 2 these Brieskorn spheres bound rational homology balls.

It is not hard to compute the Rokhlin invariant for our examples from the diagrams of their canonical negative definite plumbings, as described for example in [NR78]. In fact it is just as easy to compute the more powerful Neumann-Siebenmann $\bar{\mu}$ invariant [Neu80]. The plumbings have signature $-5 - n$, and when n is odd the square of their spherical Wu class is $-13 - n$. Hence for odd n these Brieskorn spheres have $\bar{\mu} = (-5 - n) - (-13 - n) = 8$. Since $\mu = \frac{\bar{\mu}}{8} \mod (2)$, we see that these examples have non-trivial Rokhlin invariant. $\square$

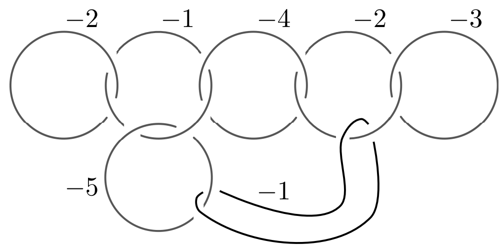

FIGURE 2. An integral surgery from $\Sigma(2,5,17)$ to Y.

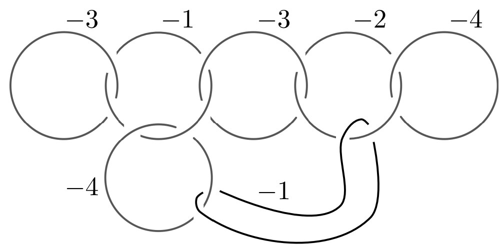

FIGURE 3. An integral surgery from $\Sigma(3,4,17)$ to Y.

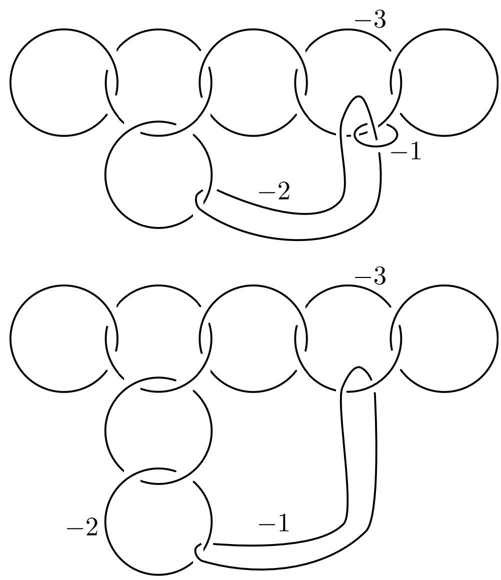

FIGURE 4. Blowing up in the diagram.

3. Handle diagrams

Here we will first construct a rational ball W_0 with boundary S^3 where the figure-eight knot K is slice. Then by blowing down this slice disk we will get a specific handlebody of a rational ball W which $\Sigma(2,3,7)$ bounds. We start with the figure-eight knot in $S^3 = \partial B^4$, drawn as the yellow curve in Figure 6. Then we attach a canceling 2/3 handle pair to B^4. After this, we apply the obvious boundary diffeomorphisms to get the last picture of Figure 6. Then we go from Figure 6 to

Figure 7 by applying the local diffeomorphism described in Figure 5. This brings us to the first picture of Figure 7. Then an isotopy and handle slide (indicated by the dotted arrow) and turning a zero framed 2-handle to a dotted circle gives us the last picture of Figure 7, which is a rational ball and the figure-eight knot K (drawn in yellow color) is obviously slice there. Then by blowing down this slice K, in Figure 8 with $+1$-framing, gives a rational ball W which $\Sigma(2,3,7)$ bounds. The notation in Figure 8 means that everything going through K is twisted by a -1-twist. From the picture we see that W has one 1-handle, two 2-handles, and one 3-handle.

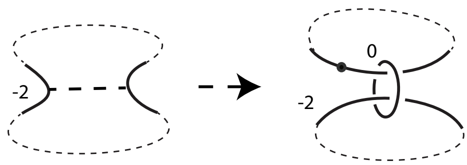

Figure 5

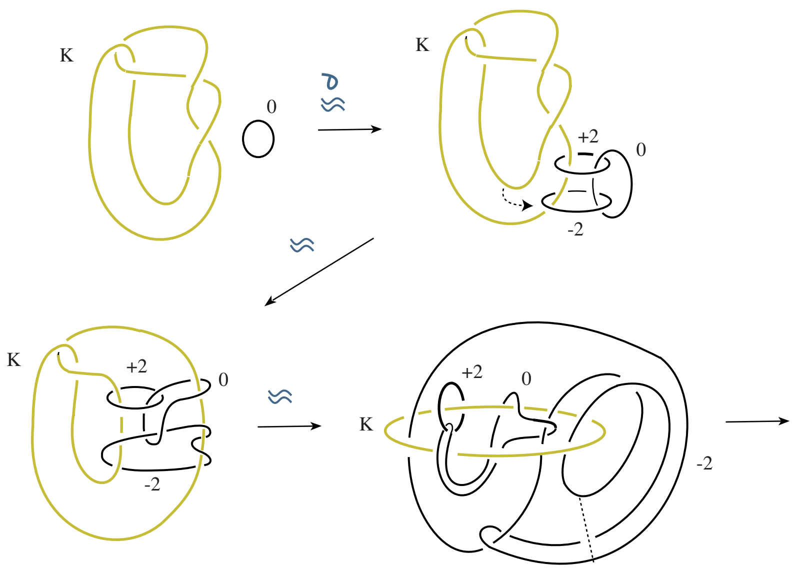

Figure 6

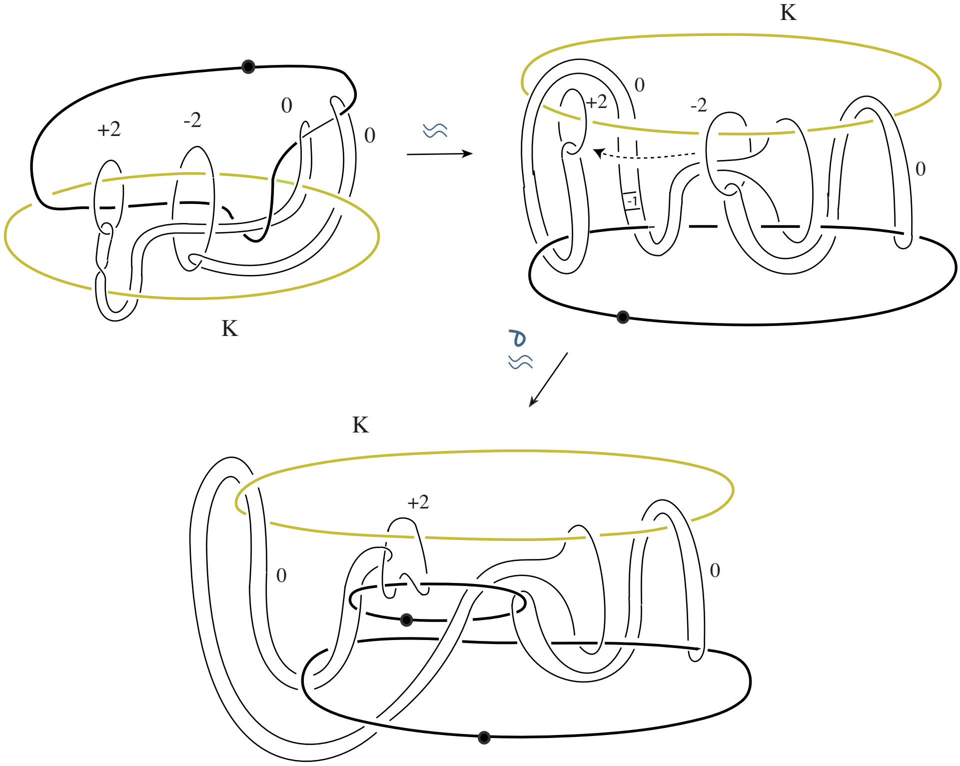

FIGURE 7. W_0

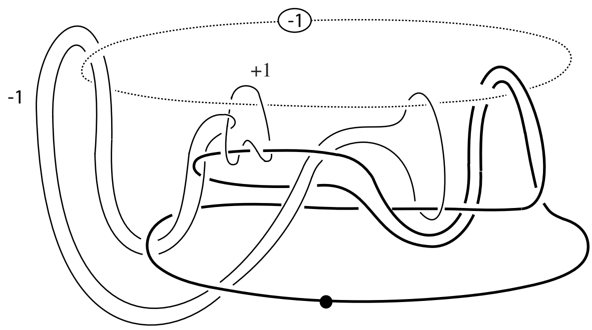

FIGURE 8. W

Acknowledgements

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