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CONFIRMING A q-TRIGONOMETRIC CONJECTURE OF GOSPER

MOHAMED EL BACHRAOUI

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Abstract. We shall confirm a conjecture of Gosper on the q-analogue of the function $\cos(2z)$ and we shall give a short proof for his other related identity on the q-analogue of $\sin(2z)$ which was recently proved by Mező.

1. Introduction

Throughout the paper let $q = e^{\pi i \tau}$ with $\mathrm{Im}(\tau) > 0$, let $\tau' = -\frac{1}{\tau}$, and let $p = e^{\pi i \tau'}$. Note that the assumption $\mathrm{Im}(\tau) > 0$ guarantees that $|q| < 1$ and $|p| < 1$. For a complex variable a, the q-shifted factorials are given by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i), \quad (a;q)_\infty = \lim_{n\to\infty}(a;q)_n,$$

and for brevity let

$$(a_1,\dots,a_k;q)_n = (a_1;q)_n \cdots (a_k;q)_n, \quad (a_1,\dots,a_k;q)_\infty = (a_1;q)_\infty \cdots (a_k;q)_\infty.$$

The four Jacobi's theta functions (with *nome* q) are defined as follows:

$$\theta_1(z,q) = \theta_1(z \mid \tau) = 2\sum_{n=0}^{\infty}(-1)^n q^{(2n+1)^2/4}\sin(2n+1)z,$$

$$\theta_2(z,q) = \theta_2(z \mid \tau) = 2\sum_{n=0}^{\infty} q^{(2n+1)^2/4}\cos(2n+1)z,$$

$$\theta_3(z,q) = \theta_3(z \mid \tau) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz,$$

$$\theta_4(z,q) = \theta_4(z \mid \tau) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nz.$$

A standard reference for information about theta functions is the book by Whittaker and Watson [11]. By Jacobi's triple product identity (see [11, p. 469] and [3, p. 15])

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} z^n = (zq, z^{-1}q, q^2; q^2)_\infty,$$

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it can be seen that each of the Jacobi's theta functions have infinite product representations. In particular, we have

$$\theta_1(z \mid \tau) = iq^{\frac{1}{4}} e^{-iz}(q^2 e^{-2iz}, e^{2iz}, q^2; q^2)_\infty,$$

and

$$\theta_2(z \mid \tau) = q^{\frac{1}{4}} e^{-iz}(-q^2 e^{-2iz}, -e^{2iz}, q^2; q^2)_\infty.$$

It is clear that the function θ_1 is odd and the function θ_2 is even. For the purpose of this work we will need the following basic properties of θ_1 and θ_2 which can be derived straightforwardly by the definitions:

$$\begin{aligned} \theta_1(k\pi) &= 0 \quad (k \in \mathbb{Z}), \\ \theta_1(z+\pi \mid \tau) &= -\theta_1(z \mid \tau), \\ \theta_1(z+\pi\tau \mid \tau) &= -q^{-1} e^{-2iz} \theta_1(z \mid \tau), \\ \theta_1\left(z+\pi\tau \mid \frac{\tau}{2}\right) &= q^{-2} e^{-4iz} \theta_1\left(z \mid \frac{\tau}{2}\right), \end{aligned} \tag{1}$$

$$\begin{aligned} \theta_2(k\frac{\pi}{2}) &= 0 \quad (k \in \mathbb{Z}), \\ \theta_2(z \mid \tau) &= \theta_1(z+\frac{\pi}{2} \mid \tau), \\ \theta_2(z+\pi\tau \mid \tau) &= q^{-1} e^{-2iz} \theta_2(z \mid \tau). \end{aligned} \tag{2}$$

Jacobi's imaginary transformation for the function θ_1 states that

$$\theta_1(z \mid \tau) = (-i\tau)^{-\frac{1}{2}}(-i) e^{\frac{i\tau' z^2}{\pi}} \theta_1(z\tau' \mid \tau'). \tag{3}$$

See [11, p. 475]. Gosper [4] introduced q-analogues of $\sin(z)$ and $\cos(z)$ as follows:

$$\sin_q(\pi z) = q^{(z-1/2)^2} \prod_{n=1}^{\infty} \frac{(1-q^{2n-2z})(1-q^{2n+2z-2})}{(1-q^{2n-1})^2} = q^{(z-\frac{1}{2})^2} \frac{(q^{2z}, q^{2-2z}; q^2)_\infty}{(q;q^2)_\infty^2},$$

$$\cos_q(\pi z) = q^{z^2} \prod_{n=1}^{\infty} \frac{(1-q^{2n-2z-1})(1-q^{2n+2z-1})}{(1-q^{2n-1})^2} = q^{z^2} \frac{(q^{1+2z}, q^{1-2z}; q^2)_\infty}{(q;q^2)_\infty^2}.$$

It is easy to see that $\cos_q(z) = \sin_q(\pi/2 - z)$. Gosper proved a variety of identities involving these two functions. In particular, he showed that both $\sin_q(z)$ and $\cos_q(z)$ in fact are ratios of Jacobi's theta functions with nome p. More specifically, he showed that

$$\sin_q(z) = \frac{\theta_1(z,p)}{\theta_1\left(\frac{\pi}{2}, p\right)} \quad \text{where } (\ln p)(\ln q) = \pi^2,$$

which is readily seen to be equivalent to

$$\sin_q(z) = \frac{\theta_1(z \mid \tau')}{\theta_1\left(\frac{\pi}{2} \mid \tau'\right)}. \tag{4}$$

As to $\cos_q(z)$, clearly the formula (4) combined with the identities $\cos_q(z) = \sin_q(\pi/2 - z)$ and $\theta_1(z+\pi) = -\theta_1(z)$ yield

$$\cos_q(z) = \frac{\theta_1\left(z+\frac{\pi}{2}, p\right)}{\theta_1\left(\frac{\pi}{2}, p\right)} = \frac{\theta_1\left(z+\frac{\pi}{2} \mid \tau'\right)}{\theta_1\left(\frac{\pi}{2} \mid \tau'\right)}. \tag{5}$$

See Gosper [4, p. 98]. The author after introducing the function $\cos_q z$ proved that

$$\sin_q(2z) = q^{-\frac{1}{4}} \frac{(q^2;q^4)_\infty^4}{(q;q^2)_\infty^2} \cdot \sin_{q^2}(z) \cos_{q^2}(z) \tag{6}$$

which can be seen to be a q-analogue for the famous trigonometric identity $\sin 2z = 2\sin z\cos z$; refer to [4, p. 92]. Mező [8] gave another proof for (6). Besides, in an attempt to give a q-analogue for the related identity $\cos 2z = \cos^2 z - \sin^2 z$, Gosper conjectured that

$$\cos_q(2z) = (\cos_{q^2}(z))^2 - (\sin_{q^2}(z))^2, \tag{7}$$

and noted that he found "empirical confirmation"; see Gosper [4, p. 93]. Note that taking into account the relations (4) and (5), formula (7) can be written as

$$\frac{\theta_1\left(2z+\frac{\pi}{2}\mid\tau'\right)}{\theta_1\left(\frac{\pi}{2}\mid\tau'\right)} = \left(\frac{\theta_1\left(z+\frac{\pi}{2}\mid\frac{\tau'}{2}\right)}{\theta_1\left(\frac{\pi}{2}\mid\frac{\tau'}{2}\right)}\right)^2 - \left(\frac{\theta_1\left(z\mid\frac{\tau'}{2}\right)}{\theta_1\left(\frac{\pi}{2}\mid\frac{\tau'}{2}\right)}\right)^2,$$

which after rearrangement becomes

$$\begin{aligned}\theta_1\left(2z+\frac{\pi}{2}\mid\tau'\right)\theta_1^2\left(\frac{\pi}{2}\mid\frac{\tau'}{2}\right)\\ = \theta_1\left(\frac{\pi}{2}\mid\tau'\right)\theta_1^2\left(z+\frac{\pi}{2}\mid\frac{\tau'}{2}\right) - \theta_1\left(\frac{\pi}{2}\mid\tau'\right)\theta_1^2\left(z\mid\frac{\tau'}{2}\right).\end{aligned}$$

Furthermore, again by virtue of (4) and (5) note that formula (6) means

$$\frac{\theta_1(2z\mid\tau')}{\theta_1\left(\frac{\pi}{2}\mid\tau'\right)} = C(q)\frac{\theta_1\left(z\mid\frac{\tau'}{2}\right)}{\theta_1\left(\frac{\pi}{2}\mid\frac{\tau'}{2}\right)}\cdot\frac{\theta_1\left(z+\frac{\pi}{2}\mid\frac{\tau'}{2}\right)}{\theta_1\left(\frac{\pi}{2}\mid\frac{\tau'}{2}\right)},$$

or equivalently,

$$\theta_1(2z\mid\tau')\theta_1^2\left(\frac{\pi}{2}\mid\frac{\tau'}{2}\right) = C(q)\theta_1\left(\frac{\pi}{2}\mid\tau'\right)\theta_1\left(z\mid\frac{\tau'}{2}\right)\theta_1\left(z+\frac{\pi}{2}\mid\frac{\tau'}{2}\right).$$

Therefore, Gosper's identities (6) and (7) both can be seen as three-term addition formulas involving theta functions. The theory of elliptic functions proved to be a powerful tool to study this type of addition formulas. For recent papers dealing with addition formulas using elliptic functions, we refer to Liu [6, 7]. See also Whittaker and Watson [11], Lawden [5], and Shen [9, 10] for more additive formulas involving theta functions and applications. In this paper we will confirm conjecture (7) and we will reproduce a short proof for formula (6) by employing the theory of elliptic functions. We shall prove the following results.

Theorem 1. *For all complex number z we have*

$$\begin{aligned}\theta_1\left(2z+\frac{\pi}{2}\mid\tau'\right)\theta_1^2\left(\frac{\pi}{2}\mid\frac{\tau'}{2}\right) &= \theta_1\left(\frac{\pi}{2}\mid\tau'\right)\theta_1^2\left(z+\frac{\pi}{2}\mid\frac{\tau'}{2}\right)\\ &\quad - \theta_1\left(\frac{\pi}{2}\mid\tau'\right)\theta_1^2\left(z\mid\frac{\tau'}{2}\right).\end{aligned}$$

Theorem 2. *For all complex number z we have*

$$\theta_1(2z\mid\tau')\theta_1^2\left(\frac{\pi}{2}\mid\frac{\tau'}{2}\right) = C(q)\theta_1\left(\frac{\pi}{2}\mid\tau'\right)\theta_1\left(z\mid\frac{\tau'}{2}\right)\theta_1\left(z+\frac{\pi}{2}\mid\frac{\tau'}{2}\right),$$

where

$$C(q) = \frac{\theta_1^2\left(\frac{\pi}{2}\mid\frac{\tau'}{2}\right)}{\theta_1^2\left(\frac{\pi}{4}\mid\frac{\tau'}{2}\right)} = q^{-\frac{1}{4}}\frac{(q^2;q^4)_\infty^4}{(q;q^2)_\infty^2}.$$

It turns out that Theorem 1 and Theorem 2 are direct consequences of the following result.

Theorem 3. *For all complex number $x, y,$ and z we have*

$$\begin{aligned}&\theta_1(z-x-y \mid \tau)\theta_1\left(x-y \mid \frac{\tau}{2}\right)\theta_1\left(z-\frac{\pi}{2} \mid \frac{\tau}{2}\right)\\ &=\theta_1(y-x-z \mid \tau)\theta_1\left(x-z \mid \frac{\tau}{2}\right)\theta_1\left(y-\frac{\pi}{2} \mid \frac{\tau}{2}\right)\\ &-\theta_1(x-y-z \mid \tau)\theta_1\left(y-z \mid \frac{\tau}{2}\right)\theta_1\left(x-\frac{\pi}{2} \mid \frac{\tau}{2}\right).\end{aligned}$$

To prove Theorem 3, we shall need the following more general result.

Theorem 4. *Let $f(u)$ be an entire function such that*

$$f(u+\pi)=-f(u) \quad \textit{and } f\left(u+\frac{\pi\tau}{2}\right)=q^{-\frac{1}{2}}e^{-2iu}f(u).$$

Then for all complex numbers $x, y,$ and z we have

$$\begin{aligned}\frac{\theta_1(z-x-y \mid \tau)f(z)}{\theta_1\left(x-z \mid \frac{\tau}{2}\right)\theta_1\left(y-z \mid \frac{\tau}{2}\right)} &= \frac{\theta_1(y-x-z \mid \tau)f(y)}{\theta_1\left(x-y \mid \frac{\tau}{2}\right)\theta_1\left(y-z \mid \frac{\tau}{2}\right)}\\ &-\frac{\theta_1(x-y-z \mid \tau)f(x)}{\theta_1\left(x-y \mid \frac{\tau}{2}\right)\theta_1\left(x-z \mid \frac{\tau}{2}\right)}.\end{aligned}$$

2. Proof of Theorem 4

Let

$$g(u)=\frac{\theta_1(2u-x-y-z \mid \tau)f(u)}{\theta_1(u-x \mid \frac{\tau}{2})\theta_1(u-y \mid \frac{\tau}{2})\theta_1(u-z \mid \frac{\tau}{2})},$$

where $x, y,$ and z are different from the zeros of $\theta_1(2u-x-y-z \mid \tau)f(u)$. Suppose for the moment that $0 < x, y, z < \pi$. Then by the properties of the function θ_1 and the assumptions on the function $f(u)$ we can easily check that

$$g(u+\pi)=g(u) \quad \text{and } g\left(u+\frac{\pi\tau}{2}\right)=g(u),$$

showing that $g(u)$ is an elliptic function with periods π and $\frac{\pi\tau}{2}$. Clearly, the function $g(u)$ has simple poles at $x, y,$ and z in the fundamental parallelogram $0, \pi, \frac{\pi\tau}{2}, \pi+\frac{\pi\tau}{2}$. We have

$$\begin{aligned}\operatorname{Res}(g;x) &= \lim_{u\to x}\frac{u-x}{\theta_1\left(u-x \mid \frac{\tau}{2}\right)}\cdot\frac{\theta_1(x-y-z \mid \tau)f(x)}{\theta_1\left(x-y \mid \frac{\tau}{2}\right)\theta_1\left(x-z \mid \frac{\tau}{2}\right)}\\ &=\frac{\theta_1(x-y-z \mid \tau)f(x)}{\theta'\left(0 \mid \frac{\tau}{2}\right)\theta_1\left(x-y \mid \frac{\tau}{2}\right)\theta_1\left(x-z \mid \frac{\tau}{2}\right)},\end{aligned} \tag{8}$$

and similarly,

$$\begin{aligned}\operatorname{Res}(g;y) &= \frac{\theta_1(y-x-z \mid \tau)f(y)}{\theta'\left(0 \mid \frac{\tau}{2}\right)\theta_1\left(y-x \mid \frac{\tau}{2}\right)\theta_1\left(y-z \mid \frac{\tau}{2}\right)},\\ \operatorname{Res}(g;z) &= \frac{\theta_1(z-x-y \mid \tau)f(z)}{\theta'\left(0 \mid \frac{\tau}{2}\right)\theta_1\left(z-x \mid \frac{\tau}{2}\right)\theta_1\left(z-y \mid \frac{\tau}{2}\right)}.\end{aligned} \tag{9}$$

Hence by the residue theorem for elliptic functions and the formulas in (8) and (9) we obtain the desired identity which holds for all complex $x, y,$ and z by analytic continuation.

3. Proof of Theorem 3

Let $f(u) = \theta_2(u \mid \frac{\tau}{2})$. Then it is easily verified by the properties (2) that the function $f(u)$ satisfies the two conditions of Theorem 4 and so,

$$\frac{\theta_1(z-x-y \mid \tau)\theta_2\left(z \mid \frac{\tau}{2}\right)}{\theta_1\left(x-z \mid \frac{\tau}{2}\right)\theta_1\left(y-z \mid \frac{\tau}{2}\right)} = \frac{\theta_1(y-x-z \mid \tau)\theta_2\left(y \mid \frac{\tau}{2}\right)}{\theta_1\left(x-y \mid \frac{\tau}{2}\right)\theta_1\left(y-z \mid \frac{\tau}{2}\right)} - \frac{\theta_1(x-y-z \mid \tau)\theta_2\left(x \mid \frac{\tau}{2}\right)}{\theta_1\left(x-y \mid \frac{\tau}{2}\right)\theta_1\left(x-z \mid \frac{\tau}{2}\right)}.$$

Now rearranging and using the basic fact $\theta_2(z \mid \tau) = \theta_1(z - \pi/2 \mid \tau)$, the previous formula yields

$$\begin{aligned}
&\theta_1(z-x-y \mid \tau)\theta_1\left(x-y \mid \frac{\tau}{2}\right)\theta_1\left(z-\frac{\pi}{2} \mid \frac{\tau}{2}\right)\\
&= \theta_1(y-x-z \mid \tau)\theta_1\left(x-z \mid \frac{\tau}{2}\right)\theta_1\left(y-\frac{\pi}{2} \mid \frac{\tau}{2}\right)\\
&-\theta_1(x-y-z \mid \tau)\theta_1\left(y-z \mid \frac{\tau}{2}\right)\theta_1\left(x-\frac{\pi}{2} \mid \frac{\tau}{2}\right),
\end{aligned}$$

as desired.

4. Proof of Theorem 1

Letting in Theorem 3, $x - z = y - \pi/2$, $y - z = x - 3\pi/2$, and so $z = \pi$, gives

$$\begin{aligned}
\theta_1^2\left(\frac{\pi}{2} \mid \frac{\tau}{2}\right)\theta_1\left(\pi - 2y - \frac{\pi}{2} \mid \tau\right) &= \theta_1\left(-\frac{3\pi}{2} \mid \tau\right)\theta_1^2\left(y-\frac{\pi}{2} \mid \frac{\tau}{2}\right)\\
&-\theta_1\left(-\frac{\pi}{2} \mid \tau\right)\theta_1\left(y-\pi \mid \frac{\tau}{2}\right)\theta_1\left(y \mid \frac{\tau}{2}\right)
\end{aligned}$$

which by the basic properties (1) is equivalent to

$$-\theta_1^2\left(\frac{\pi}{2} \mid \frac{\tau}{2}\right)\theta_1\left(2y-\frac{\pi}{2} \mid \tau\right) = \theta_1(\frac{\pi}{2} \mid \tau)\theta_1^2\left(y-\frac{\pi}{2} \mid \frac{\tau}{2}\right) - \theta_1\left(\frac{\pi}{2} \mid \tau\right)\theta_1^2\left(y \mid \frac{\tau}{2}\right).$$

Then using the substitution $z := y - \pi/2$ in the previous identity gives the desired formula.

5. Proof of Theorem 2

Let $z = y - x$ in Theorem 3 and use the basic properties in (1) to get

$$\begin{aligned}
&\theta_1(2x \mid \tau)\theta_1\left(y-x \mid \frac{\tau}{2}\right)\theta_1\left(y-x-\frac{\pi}{2} \mid \frac{\tau}{2}\right)\\
&= \theta_1(2y-2x \mid \tau)\theta_1\left(x \mid \frac{\tau}{2}\right)\theta_1\left(x-\frac{\pi}{2} \mid \frac{\tau}{2}\right).
\end{aligned}$$

Then the substitution $z := x - \pi/2$ in the previous identity implies

$$\begin{aligned}
&\theta_1(2z+\pi \mid \tau)\theta_1\left(y-z-\frac{\pi}{2} \mid \frac{\tau}{2}\right)\theta_1\left(y-z-\pi \mid \frac{\tau}{2}\right)\\
&= \theta_1(2y-2z-\pi \mid \tau)\theta_1\left(z+\frac{\pi}{2} \mid \frac{\tau}{2}\right)\theta_1\left(z \mid \frac{\tau}{2}\right),
\end{aligned}$$

or, equivalently

$$\begin{aligned}
&-\theta_1(2z \mid \tau)\theta_1\left(y-z-\frac{\pi}{2} \mid \frac{\tau}{2}\right)\theta_1\left(y-z \mid \frac{\tau}{2}\right)\\
&= \theta_1(2y-2z \mid \tau)\theta_1\left(z+\frac{\pi}{2} \mid \frac{\tau}{2}\right)\theta_1\left(z \mid \frac{\tau}{2}\right).
\end{aligned}$$

Finally, let in the previous identity $y - z = \pi/4$ to get

$$\theta_1(2z \mid \tau)\theta_1^2\left(\frac{\pi}{4} \mid \frac{\tau}{2}\right) = \theta_1\left(\frac{\pi}{2} \mid \tau\right)\theta_1\left(z + \frac{\pi}{2} \mid \frac{\tau}{2}\right)\theta_1\left(z \mid \frac{\tau}{2}\right),$$

or, equivalently

$$\theta_1(2z \mid \tau)\theta_1^2\left(\frac{\pi}{2} \mid \frac{\tau}{2}\right) = \frac{\theta_1^2\left(\frac{\pi}{2} \mid \frac{\tau}{2}\right)}{\theta_1^2\left(\frac{\pi}{4} \mid \frac{\tau}{2}\right)}\theta_1\left(\frac{\pi}{2} \mid \tau\right)\theta_1\left(z + \frac{\pi}{2} \mid \frac{\tau}{2}\right)\theta_1\left(z \mid \frac{\tau}{2}\right).$$

It remains to prove that if we replace τ by τ' in the previous identity, then

$$\frac{\theta_1^2\left(\frac{\pi}{2} \mid \frac{\tau'}{2}\right)}{\theta_1^2\left(\frac{\pi}{4} \mid \frac{\tau'}{2}\right)} = q^{-\frac{1}{4}}\frac{(q^2;q^4)_\infty^4}{(q;q^2)_\infty^2}. \tag{10}$$

Indeed, by virtue of Jacobi's imaginary transformation (3) we have

$$\begin{aligned}\theta_1\left(\frac{\pi}{2} \mid \frac{\tau'}{2}\right) &= \left(-i\frac{\tau'}{2}\right)^{-\frac{1}{2}}(-i)e^{i(2\tau)\frac{\pi}{4}}\theta_1\left(\frac{\pi}{2}(2\tau) \mid 2\tau\right)\\ &= \left(-i\frac{\tau'}{2}\right)^{-\frac{1}{2}}(-i)q^{\frac{1}{2}}iq^{-\frac{1}{2}}(q^2,q^2,q^4;q^4)_\infty\\ &= \left(-i\frac{\tau'}{2}\right)^{-\frac{1}{2}}(q^2;q^4)_\infty^2(q^4;q^4)_\infty,\end{aligned} \tag{11}$$

and similarly,

$$\begin{aligned}\theta_1\left(\frac{\pi}{4} \mid \frac{\tau'}{2}\right) &= \left(-i\frac{\tau'}{2}\right)^{-\frac{1}{2}}(-i)e^{i(2\tau)\frac{\pi}{16}}iq^{\frac{1}{2}}e^{-i\frac{\pi\tau}{2}}(q^3,q,q^4;q^4)_\infty\\ &= \left(-i\frac{\tau}{2}\right)^{-\frac{1}{2}}q^{\frac{1}{8}}(q;q^2)_\infty(q^4;q^4)_\infty.\end{aligned} \tag{12}$$

Finally take the squares in the relations (11) and (12) and divide to establish identity (10). This completes the proof.

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Department of Mathematical Sciences, United Arab Emirates University, P.O. Box 15551, Al-Ain, United Arab Emirates

E-mail address: melbachraoui@uaeu.ac.ae