PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 4, April 2018, Pages 1431–1437
http://dx.doi.org/10.1090/proc/13835
Article electronically published on November 7, 2017

WHEN IS $R \ltimes I$ AN ALMOST GORENSTEIN LOCAL RING?

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(Communicated by Irena Peeva)

Abstract. Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension $d > 0$ and let I be an ideal of R such that $(0) \neq I \subsetneq R$ and R/I is a Cohen-Macaulay ring of dimension d. There is given a complete answer to the question of when the idealization $A = R \ltimes I$ of I over R is an almost Gorenstein local ring.

1. Introduction

Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension $d > 0$ with infinite residue class field. Assume that R is a homomorphic image of a regular local ring. With this notation the purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let I be a non-zero ideal of R and suppose that R/I is a Cohen-Macaulay ring of dimension d. Let $A = R \ltimes I$ denote the idealization of I over R. Then the following conditions are equivalent:*

(1) *$A = R \ltimes I$ is an almost Gorenstein local ring.*
(2) *R has the presentation $R = S/[(X) \cap (Y)]$ where S is a regular local ring of dimension $d+1$ and X, Y is a part of a regular system of parameters of S such that $I = XR$.*

The notion of an almost Gorenstein local ring (*AGL ring* for short) is one of the generalizations of Gorenstein rings, which originated in the paper [1] of V. Barucci and R. Fröberg in 1997. They introduced the notion for one-dimensional analytically unramified local rings and developed a beautiful theory, investigating the semigroup rings of numerical semigroups. In 2013 the first author, N. Matsuoka, and T. T. Phuong [5] extended the notion to arbitrary Cohen-Macaulay local rings but still of dimension one. The research of [5] has been succeeded by two works [11] and [3] in 2015 and 2017, respectively. In [3] one can find the notion of a 2-almost Gorenstein local ring (2-*AGL ring* for short) of dimension one, which is a generalization of AGL rings. Using the Sally modules of canonical ideals, the authors show that 2-AGL rings behave well as if they were twins of AGL rings. The purpose of the research [11] of the first author, R. Takahashi, and N, Taniguchi started in a different direction. They have extended the notion of an AGL ring to higher dimensional Cohen-Macaulay local/graded rings, using the notion of Ulrich modules ([2]). Here let us briefly recall their definition for the local case.

Received by the editors March 17, 2017 and, in revised form, May 11, 2017.
2010 *Mathematics Subject Classification.* Primary 13H10; Secondary 13H05, 13H15.
The first author was partially supported by JSPS Grant-in-Aid for Scientific Research (C) 25400051. Both authors are partially supported by JSPS Bilateral Programs (Joint Research) and International Research Supporting Program of Meiji University.

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Definition 1.2. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, possessing the canonical module K_R. Then we say that R is an AGL ring, if there exists an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that either $C = (0)$ or $C \neq (0)$ and $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$, where $\mu_R(C)$ denotes the number of elements in a minimal system of generators of C and

$$\mathrm{e}^0_{\mathfrak{m}}(C) = \lim_{n\to\infty} (d-1)! \cdot \frac{\ell_R(C/\mathfrak{m}^{n+1}C)}{n^{d-1}}$$

denotes the multiplicity of C with respect to the maximal ideal $\mathfrak{m}$ (here $\ell_R(*)$ stands for the length).

We explain a little about Definition 1.2. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and assume that R possesses the canonical module K_R. The condition of Definition 1.2 requires that R is embedded into K_R and even though $R \neq \mathrm{K}_R$, the difference $C = \mathrm{K}_R/R$ between K_R and R is an Ulrich R-module ([2]) and behaves well. In particular, the condition is equivalent to saying that $\mathfrak{m}C = (0)$, when $\dim R = 1$ ([11, Proposition 3.4]). In general, if R is an AGL ring of dimension $d > 0$, then $R_{\mathfrak{p}}$ is a Gorenstein ring for every $\mathfrak{p} \in \operatorname{Ass} R$, because $\dim_R C \leq d-1$ ([11, Lemma 3.1]).

The research on almost Gorenstein local/graded rings is still in progress, exploring, e.g., the problem of when the Rees algebras of ideals/modules are almost Gorenstein rings (see [6–10, 15]) and the reader can consult [11] for several basic results on almost Gorenstein local/graded rings. For instance, non-Gorenstein AGL rings are G-regular in the sense of [14] and all the known Cohen-Macaulay local rings of finite Cohen-Macaulay representation type are AGL rings. Besides, the authors explored the question of when the idealization $A = R \ltimes M$ is an AGL ring, where $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring and M is a maximal Cohen-Macaulay R-module. Because $A = R \ltimes M$ is a Gorenstein ring if and only if $M \cong \mathrm{K}_R$ as an R-module ([13]), this question seems quite natural and in [11, Section 6] the authors actually gave a complete answer to the question in the case where M is a faithful R-module, that is, the case $(0) :_R M = (0)$. However, the case where M is not faithful has been left open, which our Theorem 1.1 settles in the special case where R is a Gorenstein local ring and $M = I$ is an ideal of R such that R/I is a Cohen-Macaulay ring with $\dim R/I = \dim R$. For the case where $\dim R/I = d$ but $\operatorname{depth} R/I = d - 1$ the question remains open (see Remark 2.6).

2. Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. To begin with, let us fix our notation. Unless otherwise specified, throughout this paper let $(R, \mathfrak{m})$ be a Gorenstein local ring with $d = \dim R > 0$. Let I be a non-zero ideal of R such that R/I is a Cohen-Macaulay ring with $\dim R/I = d$. Let $A = R \ltimes I$ be the idealization of I over R. Therefore, $A = R \oplus I$ as an R-module and the multiplication in A is given by

$$(a, x)(b, y) = (ab, bx + ay)$$

where $a, b \in R$ and $x, y \in I$. Hence A is a Cohen-Macaulay local ring with $\dim A = d$, because I is a maximal Cohen-Macaulay R-module.

For each R-module N let $N^\vee = \operatorname{Hom}_R(N, R)$. We set $L = I^\vee \oplus R$ and consider L to be an A-module under the following action of A:

$$(a, x) \circ (f, y) = (af, f(x) + ay),$$

where $(a, x) \in A$ and $(f, y) \in L$. Then it is standard to check that the map

$$A^\vee \to L, \ \alpha \mapsto (\alpha \circ j, \alpha(1))$$

is an isomorphism of A-modules, where $j : I \to A$, $x \mapsto (0, x)$ and $1 = (1, 0)$ denotes the identity of the ring A. Hence by [12, Satz 5.12] we get the following.

Fact 2.1. $\mathrm{K}_A = L$, where K_A denotes the canonical module of A.

We set $J = (0) :_R I$. Let $\iota : I \to R$ denote the embedding. Then taking the R-dual of the exact sequence

$$0 \to I \xrightarrow{\iota} R \to R/I \to 0,$$

we get the exact sequence

$$0 \to (R/I)^\vee \to R^\vee \xrightarrow{\iota^\vee} I^\vee \to 0 = \operatorname{Ext}^1_R(R/I, R) \to \cdots$$

of R-modules, which shows $I^\vee = R{\cdot}\iota$. Hence $J = (0) :_R I^\vee$ because $I = I^{\vee\vee}$ ([12, Korollar 6.8]), so that $I^\vee = R{\cdot}\iota \cong R/J$ as an R-module. Hence $I \cong (R/J)^\vee = \mathrm{K}_{R/J}$ ([12, Satz 5.12]). Therefore, taking again the R-dual of the exact sequence

$$0 \to J \to R^\vee \xrightarrow{\iota^\vee} I^\vee \to 0,$$

we get the exact sequence $0 \to I \xrightarrow{\iota} R \to J^\vee \to 0$ of R-modules, whence $J^\vee \cong R/I$, so that $J \cong (R/I)^\vee = \mathrm{K}_{R/I}$. Summarizing the arguments, we get the following.

Fact 2.2. $I \cong (R/J)^\vee = \mathrm{K}_{R/J}$ and $J \cong (R/I)^\vee = \mathrm{K}_{R/I}$.

Notice that $\mathrm{r}(A) = 2$ by [12, Satz 6.10] where $\mathrm{r}(A)$ denotes the Cohen-Macaulay type of A, because A is not a Gorenstein ring (as $I \not\cong R$; see [13]) but K_A is generated by two elements; $\mathrm{K}_A = R{\cdot}(\iota, 0) + R{\cdot}(0, 1)$.

We denote by $\mathfrak{M} = \mathfrak{m} \times I$ the maximal ideal of A. Let us begin with the following.

Lemma 2.3. *Let $d = 1$. Then the following conditions are equivalent:*

(1) *A is an* AGL *ring.*

(2) $I + J = \mathfrak{m}$.

When this is the case, $I \cap J = (0)$.

Proof. (2) $\Rightarrow$ (1) We set $f = (\iota, 1) \in \mathrm{K}_A$ and $C = \mathrm{K}_A/Af$. Let $\alpha \in \mathfrak{m}$ and $\beta \in I$. Let us write $\alpha = a + b$ with $a \in I$ and $b \in J$. Then because

$$(\alpha, 0)(0, 1) = (0, \alpha) = (b\iota, a + b) = (b, a)(\iota, 1), \ (0, \beta)(0, 1) = (0, 0),$$

we get $\mathfrak{M}C = (0)$, whence A is an AGL ring.

(1) $\Rightarrow$ (2) We have $I \cap J = (0)$. In fact, let $\mathfrak{p} \in \operatorname{Ass} R$ and set $P = \mathfrak{p} \times I$. Hence $P \in \operatorname{Min} A$. Assume that $IR_\mathfrak{p} \neq (0)$. Then since $A_P = R_\mathfrak{p} \ltimes IR_\mathfrak{p}$ and A_P is a Gorenstein local ring, $IR_\mathfrak{p} \cong R_\mathfrak{p}$ ([13]), so that $JR_\mathfrak{p} = (0)$. Therefore, $(I \cap J)R_\mathfrak{p} = (0)$ for every $\mathfrak{p} \in \operatorname{Ass} R$, whence $I \cap J = (0)$.

Now consider the exact sequence

$$0 \to A \xrightarrow{\varphi} \mathrm{K}_A \to C \to 0$$

of A-modules such that $\mathfrak{M}C = (0)$. We set $f = \varphi(1)$. Then $f \notin \mathfrak{M}\mathrm{K}_A$ by [11, Corollary 3.10], because A is not a discrete valuation ring (DVR for short).

We identify $\mathrm{K}_A = I^\vee \times R$ (Fact 2.1) and write $f = (a\iota, b)$ with $a, b \in R$. Then $a \notin \mathfrak{m}$ or $b \notin \mathfrak{m}$, since $f = (a, 0)(\iota, 0) + (b, 0)(0, 1) \notin \mathfrak{M}\mathrm{K}_A$.

First, assume that $a \notin \mathfrak{m}$. Without loss of generality, we may assume $a = 1$, whence $f = (\iota, b)$. Let $\alpha \in \mathfrak{m}$. Then since $(\alpha, 0)(0, 1) \in Af$, we can write $(\alpha, 0)(0, 1) = (r, x)(\iota, b)$ with some $r \in R$ and $x \in I$. Because

$$(0, \alpha) = (\alpha, 0)(0, 1) = (r, x)(\iota, b) = (r\iota, x + rb),$$

we get

$$r \in (0) :_R \iota = J, \quad \alpha = x + rb \in I + J.$$

Therefore, $\mathfrak{m} = I + J$.

Now assume that $a \in \mathfrak{m}$. Then since $b \notin \mathfrak{m}$, we may assume $b = 1$, whence $f = (a\iota, 1)$. Let $\alpha \in \mathfrak{m}$ and write $(\alpha, 0)(\iota, 0) = (r, x)(a\iota, 1)$ with $r \in R$ and $x \in I$. Then since $(\alpha\iota, 0) = ((ra)\iota, ax + r)$, we get

$$\alpha - ra \in J, \quad r = -xa \in (a),$$

so that $\alpha \in J + (a^2) \subseteq J + \mathfrak{m}^2$, whence $\mathfrak{m} = J$. Because $I \cap J = (0)$, this implies $I = (0)$, which is absurd. Therefore, $a \notin \mathfrak{m}$, whence $I + J = \mathfrak{m}$. ☐

Corollary 2.4. *Let $d = 1$. Assume that $A = R \ltimes I$ is an* AGL *ring. Then both R/I and R/J are discrete valuation rings and $\mu_R(I) = \mu_R(J) = 1$. Consequently, if R is a homomorphic image of a regular local ring, then R has the presentation*

$$R = S/[(X) \cap (Y)]$$

for some two-dimensional regular local ring $(S, \mathfrak{n})$ with $\mathfrak{n} = (X, Y)$, so that $I = (x)$ and $J = (y)$, where x, y respectively denote the images of X, Y in R.

Proof. Since $I + J = \mathfrak{m}$ and $I \cap J = (0)$, $\mathrm{K}_{R/I} \cong J \cong \mathfrak{m}/I$ by Fact 2.2. Hence R/I is a DVR by Burch's Theorem (see, e.g., [4, Theorem 1.1 (1)]), because $\mathrm{id}_{R/I}\, \mathfrak{m}/I = \mathrm{id}_{R/I}\, \mathrm{K}_{R/I} = 1 < \infty$, where $\mathrm{id}_{R/I}(*)$ denotes the injective dimension. We similarly get that R/J is a DVR, since $\mathrm{K}_{R/J} \cong I \cong \mathfrak{m}/J$. Consequently, $\mu_R(I) = \mu_R(J) = 1$. We write $I = (x)$ and $J = (y)$. Hence $\mathfrak{m} = I + J = (x, y)$. Since $xy = 0$, we have $\mathfrak{m}^2 = (x^2, y^2) = (x + y)\mathfrak{m}$. Therefore, $\mathrm{v}(R) = \mathrm{e}(R) = 2$ because R is not a DVR, where $\mathrm{v}(R)$ (resp. $\mathrm{e}(R)$) denotes the embedding dimension of R (resp. the multiplicity $\mathrm{e}^0_{\mathfrak{m}}(R)$ of R with respect to $\mathfrak{m}$). Suppose now that R is a homomorphic image of a regular local ring. Let us write $R = S/\mathfrak{a}$ where $\mathfrak{a}$ is an ideal in a two-dimensional regular local ring $(S, \mathfrak{n})$ and choose $X, Y \in \mathfrak{n}$ so that x, y are the images of X, Y in R, respectively. Then $\mathfrak{n} = (X, Y)$, since $\mathfrak{a} \subseteq \mathfrak{n}^2$. We consider the canonical epimorphism

$$\varphi : S/[(X) \cap (Y)] \to R$$

and get that φ is an isomorphism, because

$$\ell_S\left(S/(XY, X + Y)\right) = 2 = \ell_R\left(R/(x + y)R\right).$$

Thus $\mathfrak{a} = (X) \cap (Y)$ and $R = S/[(X) \cap (Y)]$. ☐

We note the following.

Proposition 2.5. *Let S be a regular local ring of dimension $d + 1$ $(d > 0)$ and let X, Y be a part of a regular system of parameters of S. We set $R = S/[(X) \cap (Y)]$ and $I = (x)$, where x denotes the image of X in R. Then $I \neq (0)$, R/I is a Cohen-Macaulay ring with* $\dim R/I = d$, *and the idealization $A = R \ltimes I$ is an* AGL *ring.*

Proof. Let y be the image of Y in R. Then $(y) = (0) :_R x$ and we have the presentation

$$0 \to (y) \to R \to (x) \to 0$$

of the R-module $I = (x)$, whence $A = R[T]/(yT, T^2)$, where T is an indeterminate. Therefore

$$A = S[T]/(XY, YT, T^2).$$

Notice that (XY, YT, T^2) is equal to the ideal generated by the 2×2 minors of the matrix $\mathbb{M} = \left(\begin{smallmatrix} X & Y & T \\ T & Y & 0 \end{smallmatrix}\right)$ and we readily get by [11, Theorem 7.8] that $A = R \ltimes I$ is an AGL ring, because X, Y, T is a part of a regular system of parameters of the regular local ring $S[T]_{\mathfrak{P}}$, where $\mathfrak{P} = \mathfrak{n}S[T] + (T)$. $\square$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2.5 we have only to show the implication (1) $\Rightarrow$ (2). Consider the exact sequence

$$0 \to A \to \mathrm{K}_A \to C \to 0$$

of A-modules such that C is an Ulrich A-module. Let $\mathfrak{M} = \mathfrak{m} \times I$ stand for the maximal ideal of A. Then since $\mathfrak{m}A \subseteq \mathfrak{M} \subseteq \overline{\mathfrak{m}A}$ (here $\overline{\mathfrak{m}A}$ denotes the integral closure of $\mathfrak{m}A$) and the field $R/\mathfrak{m}$ is infinite, we can choose a superficial sequence $f_1, f_2, \ldots, f_{d-1} \in \mathfrak{m}$ for C with respect to $\mathfrak{M}$ so that $f_1, f_2, \ldots, f_{d-1}$ is also a part of a system of parameters for both R and R/I. We set $\mathfrak{q} = (f_1, f_2, \ldots, f_{d-1})$ and $\overline{R} = R/\mathfrak{q}$. Let $\overline{I} = (I + \mathfrak{q})/\mathfrak{q}$ and $\overline{J} = (J + \mathfrak{q})/\mathfrak{q}$. Then since $f_1, f_2, \ldots, f_{d-1}$ is a regular sequence for R/I, by the exact sequence

$$0 \to I \to R \to R/I \to 0$$

we get the exact sequence

$$0 \to I/\mathfrak{q}I \to \overline{R} \to R/(I + \mathfrak{q}) \to 0,$$

so that $I/\mathfrak{q}I \cong \overline{I}$ as an $\overline{R}$-module. Hence

$$A/\mathfrak{q}A = \overline{R} \ltimes (I/\mathfrak{q}I) \cong \overline{R} \ltimes \overline{I}.$$

Remember that $A/\mathfrak{q}A$ is an AGL ring by [11, Theorem 3.7], because $f_1, f_2, \ldots, f_{d-1}$ is a superficial sequence of C with respect to $\mathfrak{M}$ and $f_1, f_2, \ldots, f_{d-1}$ is an A-regular sequence. Consequently, thanks to Corollary 2.4, $\overline{R}/\overline{I}$ is a DVR and $\mu_{\overline{R}}(\overline{I}) = 1$. Hence R/I is a regular local ring and $\mu_R(I) = 1$, because $I/\mathfrak{q}I \cong \overline{I}$. Let $I = (x)$. Then $R/J \cong I = (x)$, since $J = (0) :_R I$. Because $f_1, f_2, \ldots, f_{d-1}$ is a regular sequence for the R-module I, $f_1, f_2, \ldots, f_{d-1}$ is a regular sequence for R/J, so that we get the exact sequence

$$0 \to J/\mathfrak{q}J \to \overline{R} \to R/(J + \mathfrak{q}) \to 0.$$

Therefore, $\overline{J} \cong J/\mathfrak{q}J$ and since $R/(J + \mathfrak{q}) \cong I/\mathfrak{q}I \cong \overline{I}$, we have $\overline{J} = (0) :_{\overline{R}} \overline{I}$. Hence R/J is a regular local ring and $\mu_R(J) = 1$, because $\overline{R}/\overline{J}$ is a DVR and $\mu_{\overline{R}}(\overline{J}) = 1$ by Corollary 2.4.

Let $J = (y)$ and let $\overline{\mathfrak{m}} = \mathfrak{m}/\mathfrak{q}$. Then by Lemma 2.3 we have $\overline{\mathfrak{m}} = \overline{I} + \overline{J}$, whence $\mathfrak{m} = (x, y, f_1, f_2, \ldots, f_{d-1})$. Therefore $\mu_R(\mathfrak{m}) = d + 1$, since R is not a regular local ring. On the other hand, since both R/I and R/J are regular local rings, considering the canonical exact sequence

$$0 \to R \to R/I \oplus R/J \to R/(I + J) \to 0$$

(notice that $I \cap J = (0)$ for the same reason as in the proof of Lemma 2.3), we readily get $\mathrm{e}(R) = 2$. We now choose a regular local ring $(S, \mathfrak{n})$ of dimension $d+1$ and an ideal $\mathfrak{a}$ of S so that $R = S/\mathfrak{a}$. Let $X, Y, \mathbb{Z}_1, Z_2, \ldots, Z_{d-1}$ be the elements of $\mathfrak{n}$ whose images in R are equal to $x, y, f_1, f_2, \ldots, f_{d-1}$, respectively. Then $\mathfrak{n} = (X, Y, Z_1, Z_2, \ldots, Z_{d-1})$, since $\mathfrak{a} \subseteq \mathfrak{n}^2$. Because $(X) \cap (Y) \subseteq \mathfrak{a}$ as $xy = 0$, we get a surjective homomorphism

$$S/[(X) \cap (Y)] \to R$$

of rings, which has to be an isomorphism, because both the Cohen-Macaulay local rings $S/[(X) \cap (Y)]$ and R have the same multiplicity 2. This completes the proof of Theorem 1.1. $\square$

Remark 2.6. Let $(S, \mathfrak{n})$ be a two-dimensional regular local ring and let X, Y be a regular system of parameters of S. We set $R = S/[(X) \cap (Y)]$. Let x, y denote the images of X, Y in R, respectively. Let $n \geq 2$ be an integer. Then $\dim R/(x^n) = 1$ but $\operatorname{depth} R/(x^n) = 0$. We have $x^n = x^{n-1}(x+y)$, whence $(x^n) \cong (x)$ as an R-module because $x+y$ is a non-zerodivisor of R, so that $R \ltimes (x^n)$ is an AGL ring (Proposition 2.5). This example shows that there are certain ideals I in Gorenstein local rings R of dimension $d > 0$ such that $\dim R/I = d$ and $\operatorname{depth} R/I = d-1$, for which the idealizations $R \ltimes I$ are AGL rings. However, we have no idea how to control them.

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