

INVARIANT RADON MEASURES FOR UNIPOTENT FLOWS AND PRODUCTS OF KLEINIAN GROUPS

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ABSTRACT. Let $G = \mathrm{PSL}_2(\mathbb{F})$ where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and consider the space $Z = (\Gamma_1 \times \Gamma_2) \backslash (G \times G)$ where $\Gamma_1 < G$ is a co-compact lattice and $\Gamma_2 < G$ is a geometrically finite discrete Zariski dense subgroup. For a horospherical subgroup N of G , we classify all ergodic, conservative, invariant Radon measures on Z for the diagonal N -action.

1. INTRODUCTION

The celebrated theorem of M. Ratner in 1991 classifies all *finite* invariant measures for unipotent flows on the quotient space of a connected Lie group by its discrete subgroup [16]. The problem of classifying invariant *locally finite* Borel measures (i.e., Radon measures) is far from being understood in general. Most of the known classification results are restricted to the class of horospherical invariant measures on a quotient of a simple Lie group of rank one (e.g., [2, 7, 11, 12, 15, 17, 19, 23]). In this article, we obtain a classification of Radon measures invariant under unipotent flows in one of the most basic examples of the quotient of a higher rank semisimple Lie group by a discrete subgroup of infinite co-volume.

Let $G = \mathrm{PSL}_2(\mathbb{F})$ where the field \mathbb{F} is either \mathbb{R} or \mathbb{C} . Let Γ_1 and Γ_2 be geometrically finite Zariski dense, discrete subgroups of G . Consider the quotient space

$$Z := (\Gamma_1 \times \Gamma_2) \backslash (G \times G) = X_1 \times X_2$$

where $X_i = \Gamma_i \backslash G$ for $i = 1, 2$. For a subset $S \subset G$, $\Delta(S) := \{(s, s) : s \in S\}$ denotes the diagonal embedding of S into $G \times G$.

Theorem 1.1 ([5], Benoist-Quint). *Assume that $\Gamma_1 < G$ is co-compact. Then any ergodic $\Delta(G)$ -invariant Radon measure μ on Z is, up to a constant multiple, one of the following:*

- μ is the product $m^{\mathrm{Haar}} \times m^{\mathrm{Haar}}$ of Haar measures;
- μ is the graph of the Haar measure, in the sense that for some $g_0 \in G$ with $[\Gamma_2 : g_0^{-1} \Gamma_1 g_0 \cap \Gamma_2] < \infty$, $\mu = \phi_* m_{(g_0^{-1} \Gamma_1 g_0 \cap \Gamma_2)}^{\mathrm{Haar}}$, i.e., the push-forward of

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the Haar-measure on $(g_0^{-1}\Gamma_1g_0 \cap \Gamma_2)\backslash G$ to the closed orbit $[(g_0, e)]\Delta(G)$ via the isomorphism ϕ given by $[g] \mapsto [(g_0g, g)]$.

Indeed, it is proved in [5] that any ergodic Γ_2 -invariant Borel probability measure on X_1 is either a Haar measure or supported on a finite orbit of Γ_2 . This result is equivalent to the above theorem, in view of the homeomorphism $\nu \mapsto \tilde{\nu}$ between the space of all Γ_2 -invariant measures on X_1 and the space of all $\Delta(G)$ -invariant measures on Z , given by

$$\tilde{\nu}(f) = \int_{X_2} \int_{X_1} f(\Gamma_1hg, \Gamma_2g) d\nu(h) dm^{\text{Haar}}(g).$$

Since the Haar measure m^{Haar} on X_1 is ergodic for any element of G which generates an unbounded subgroup, it follows that m^{Haar} is Γ_2 -ergodic and hence the product $m^{\text{Haar}} \times m^{\text{Haar}}$ of the Haar measures in $X_1 \times X_2$ is $\Delta(G)$ -ergodic. We also refer to [6] for the topological version of Theorem 1.1 (also see [4]).

In this paper, we consider the action of $\Delta(N)$ on Z where N is a horospherical subgroup of G , i.e., N is conjugate to the subgroup

$$\left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : t \in \mathbb{F} \right\}.$$

A $\Delta(N)$ -invariant Radon measure on Z is said to be *conservative* if for any subset S of positive measure in Z , the measure of $\{n \in N : xn \in S\}$, with respect to the Haar measure of N , is infinite for almost all $x \in S$.

The aim of this paper is to classify all $\Delta(N)$ -invariant ergodic conservative Radon measures on Z , assuming Γ_1 is co-compact. Since Ratner [16] classified all such finite measures, our focus lies on *infinite* Radon measures.

Note that if μ is a $\Delta(N)$ -invariant measure, then the translate $w_*\mu$ is also $\Delta(N)$ -invariant for any w in the centralizer of $\Delta(N)$. The centralizer of $\Delta(N)$ in $G \times G$ is equal to $N \times N$. Hence it suffices to classify $\Delta(N)$ -invariant measures, up to a translation by an element of $N \times N$.

Let $m_{\Gamma_2}^{\text{BR}}$ denote the N -invariant Burger-Roblin measure on X_2 . It is known that, up to a constant multiple, $m_{\Gamma_2}^{\text{BR}}$ is the unique N -invariant ergodic conservative measure on X_2 , which is not supported on a closed N -orbit ([7], [17], [23]). When Γ_2 is of infinite co-volume, $m_{\Gamma_2}^{\text{BR}}$ is an infinite measure.

In the following two theorems, which are main results of this paper, we assume that $\Gamma_1 < G$ is co-compact and Γ_2 is a Zariski dense, geometrically finite subgroup of G with infinite co-volume.

Theorem 1.2. *The product measure $m^{\text{Haar}} \times m_{\Gamma_2}^{\text{BR}}$ on Z is a $\Delta(N)$ -ergodic conservative infinite Radon measure.*

Theorem 1.3. *Any $\Delta(N)$ -invariant, ergodic, conservative, infinite Radon measure μ on Z is one of the following, up to a translation by an element of $N \times N$ and up to a constant multiple:*

- (1) μ is the product measure $m^{\text{Haar}} \times m_{\Gamma_2}^{\text{BR}}$.
- (2) μ is the graph of the BR-measure, in the sense that for some $g_0 \in \text{PSL}_2(\mathbb{F})$ with $[\Gamma_2 : g_0^{-1}\Gamma_1g_0 \cap \Gamma_2] < \infty$,

$$\mu = \phi_*m_{(g_0^{-1}\Gamma_1g_0 \cap \Gamma_2)}^{\text{BR}},$$

i.e., the push-forward of the BR-measure on $(g_0^{-1}\Gamma_1g_0 \cap \Gamma_2)\backslash G$ to the closed orbit $[(g_0, e)]\Delta(G)$ via the isomorphism ϕ given by $[g] \mapsto [(g_0g, g)]$.

- (3) $\mathbb{F} = \mathbb{C}$ and there exists a closed orbit x_2N in X_2 homeomorphic to $\mathbb{R} \times \mathbb{S}^1$ such that μ is supported on $X_1 \times x_2N$. To describe μ more precisely, let $U < N$ denote the one dimensional subgroup containing $\text{Stab}_N(x_2)$ and dn the N -invariant measure on x_2N in X_2 . We then have one of the two possibilities:

(a)

$$\mu = m^{\text{Haar}} \times dn;$$

- (b) there exist a connected subgroup $L \simeq \text{SL}_2(\mathbb{R})$ with $L \cap N = U$, and a compact L orbit Y in X_1 such that

$$\mu = \int_{x_2N} \mu_x dx$$

where $\mu_{x_2n_0}$ is given by $\mu_{x_2n_0}(\psi) = \int_Y \psi(yn_0, x_2n_0) dy$ with dy being the L -invariant probability measure on Y .

We deduce Theorem 1.2 as a consequence of Theorem 1.3 (see subsection 3.2).

Two main ingredients of the proof of Theorem 1.3 are Ratner's classification of probability measures on X_1 which are invariant and ergodic under a one parameter unipotent subgroup of G , and the classification of N -equivariant (set-valued) Borel maps $X_2 \rightarrow X_1$, established in our earlier work [14].

2. RECURRENCE AND ALGEBRAIC ACTIONS ON MEASURE SPACES

In this section, let $G = \text{PSL}_2(\mathbb{C})$ and let $\Gamma < G$ be a Zariski dense geometrically finite discrete subgroup. Set $X = \Gamma \backslash G$. Let N be the horospherical subgroup

$$N = \left\{ n_t := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : t \in \mathbb{C} \right\}$$

and let m^{BR} denote the Burger-Roblin measure on X invariant under N .

Recall that m^{BR} is the unique ergodic N -invariant Radon measure on X which is not supported on a closed N -orbit.

Let $U < N$ be a non-trivial connected subgroup of N . We denote by $\mathcal{P}(U \backslash N)$ the space of probability measures on $U \backslash N$. The natural action N on $U \backslash N$ induces an action of N on $\mathcal{P}(U \backslash N)$.

We will use the following lemma [24, B.5].

Lemma 2.1. *Let H be a locally compact and second countable group. Let X be a standard Borel H -space with quasi-invariant measure and let Y be a standard Borel H -space. Let $f : X \rightarrow Y$ be a Borel function so that for every $h \in H$, $f(hx) = hf(x)$ for almost all $x \in X$. Then there exists an H -invariant co-null Borel subset $X_0 \subset X$ and a Borel H -map $\tilde{f} : X_0 \rightarrow Y$ such that $f = \tilde{f}$ almost surely.*

The aim of this section is to prove the following technical result:

Theorem 2.2. *Let $\Gamma < G$ be a Zariski dense geometrically finite discrete subgroup and $U < N$ be a one dimensional connected subgroup. Then there is no N -equivariant Borel map*

$$f : (X, m^{\text{BR}}) \rightarrow \mathcal{P}(U \backslash N).$$

For the proof, we will first observe that the N -action on $\mathcal{P}(U \setminus N)$ is smooth [24, Def. 2.1.9]. By the fact that m^{BR} is N -ergodic, it then follows that after possibly modifying f on a BR-null set, f is concentrated on a single N -orbit in $\mathcal{P}(U \setminus N)$. We will use a recurrence property of m^{BR} , which is stronger than the conservativity, to prove $U = N$.

We begin with the following lemma. The space $\mathcal{P}(\mathbb{R})$ is equipped with the weak-star topology: i.e., $\nu_n \rightarrow \nu$ if and only if $\nu_n(\psi) \rightarrow \nu(\psi)$ for all $\psi \in C_c(\mathbb{R})$.

Lemma 2.3. *If $\{t_n : n = 1, 2, \dots\}$ is a sequence in \mathbb{R} , so that $t_{n*}\sigma \rightarrow \sigma'$ for some $\sigma, \sigma' \in \mathcal{P}(\mathbb{R})$, then $\{t_n\}$ is bounded.*

Proof. Assume the contrary and after passing to a subsequence suppose $t_n \rightarrow \infty$. Since σ and σ' are probability measures on \mathbb{R} , there is some $M > 1$ such that

$$\sigma([-M, M]) > 0.9 \quad \text{and} \quad \sigma'([-M, M]) > 0.9.$$

Let $\psi \in C_c(\mathbb{R})$ be a continuous function so that $0 \leq \psi \leq 1$, $\psi|_{[-M, M]} = 1$ and $\psi|_{(-\infty, -M-1) \cup (M+1, \infty)} = 0$. Since $t_n \rightarrow \infty$ we have

$$([-M - 1, M + 1] - t_n) \cap [-M - 1, M + 1] = \emptyset \text{ for all large } n.$$

Therefore, $t_{n*}\sigma(\psi) < 0.1$ but $\sigma'(\psi) > 0.9$, which contradicts the assumption that $t_{n*}\sigma \rightarrow \sigma'$. □

As was mentioned above, we will need certain recurrence properties of the action of N on (X, m^{BR}) . This will be deduced from recurrence properties of the Bowen-Margulis-Sullivan measure m^{BMS} on X with respect to the diagonal flow $\text{diag}(e^{t/2}, e^{-t/2})$. We normalize so that m^{BMS} is the probability measure. These two measures m^{BMS} and m^{BR} are quasi-product measures and on weak-stable manifolds (i.e., locally transversal to N -orbits), they are absolutely continuous to each other.

Set $M = \{\text{diag}(z, z^{-1}) : |z| = 1\}$. Then G/M can be identified with the unit tangent bundle of the hyperbolic 3-space \mathbb{H}^3 . Hence for every $g \in G$, we can associate a point g^- in the boundary of \mathbb{H}^3 which is the backward end point of the geodesic determined by the tangent vector gM .

Now the set $X_{\text{rad}} := \{\Gamma g \in X : g^- \text{ is a radial limit point of } \Gamma\}$ has a full BMS-measure as well as a full BR-measure. For $x \in X_{\text{rad}}$, $n \mapsto xn$ is a bijection $N \rightarrow xN$, and μ_x^{PS} denotes the leafwise measure of m^{BMS} , considered as a measure on N (see [14, §2]).

We recall the following result of Rudolph [18, Theorem 17]: for any Borel set B of X and any $\eta > 0$, the set

$$(2.1) \quad \left\{ x \in X_{\text{rad}} : \liminf_T \frac{1}{\mu_x^{\text{PS}}(B_N(T))} \int_{B_N(T)} \chi_B(xn_t) d\mu_x^{\text{PS}}(\mathbf{t}) \geq (1 - \eta)m^{\text{BMS}}(B) \right\}$$

has full BMS-measure m^{BMS} .

Lemma 2.4. *Let U be a one dimensional connected subgroup of N . Then for every subset $B \subset X$ of positive BMS-measure, the set*

$$\{n \in U \setminus N : xn \in B\}$$

is unbounded for m^{BMS} -a.e. $x \in X$.

Proof. We denote by $\text{Nbd}_R(U)$ the R -neighborhood of U , i.e.,

$$\text{Nbd}_R(U) = \{n_t \in N : |t - s| < R \text{ for some } n_s \in U\}.$$

We set $B_N(R) := \text{Nbd}_R(\{e\})$ which is the R -neighborhood of e .

Let $B \subset X$ be any Borel set of positive BMS-measure. Then by (2.1), there is a BMS full measure set X' of X_{rad} with the following property: for all $x \in X'$, there is $T_x > 0$ such that if $T > T_x$, then

$$(2.2) \quad \mu_x^{\text{PS}}\{n_t \in B_N(T) : xn_t \in B\} \geq 0.9 \mu_x^{\text{PS}}(B_N(T))m^{\text{BMS}}(B).$$

Let $x \in X'$. Since x is a radial limit point for Γ , there exists a sequence $T_i \rightarrow \infty$ so that $xa_{-\log T_i}$ converges to some $y \in \text{supp}(m^{\text{BMS}})$. Therefore, we have

$$(2.3) \quad \mu_{xa_{-\log T_i}}^{\text{PS}} \rightarrow \mu_y^{\text{PS}},$$

in the space of regular Borel measures on N endowed with the weak-topology (see [14, Lemma 2.1]).

Moreover, since $\mu_y^{\text{PS}}(U) = 0$, by [14, Lemma 4.3], for every $\epsilon > 0$, there exists $\rho_0 > 0$ such that for every $0 < \rho \leq \rho_0$ we have

$$(2.4) \quad \mu_y^{\text{PS}}(B_N(1) \cap \text{Nbd}_\rho(U)) \leq \epsilon \cdot \mu_y^{\text{PS}}(B_N(1)).$$

Recall that $\mu_x^{\text{PS}}(B_N(T)) = T^\delta \mu_{xa_{-\log T}}^{\text{PS}}(B_N(1))$. Therefore,

$$\frac{\mu_x^{\text{PS}}(B_N(T_i) \cap \text{Nbd}_R(U))}{\mu_x^{\text{PS}}(B_N(T_i))} = \frac{\mu_{xa_{-\log T_i}}^{\text{PS}}(B_N(1) \cap \text{Nbd}_{R/T_i}(U))}{\mu_{xa_{-\log T_i}}^{\text{PS}}(B_N(1))}.$$

Hence, it follows from (2.3) and (2.4) that for every $\epsilon > 0$ and for all sufficiently large i such that $R/T_i < \rho$,

$$(2.5) \quad \mu_x^{\text{PS}}(B_N(T_i) \cap \text{Nbd}_R(U)) \leq \epsilon \cdot \mu_x^{\text{PS}}(B_N(T_i)).$$

Put $\epsilon = 1/10 \cdot m^{\text{BMS}}(B)$. Given any j , there exists $i_j > j$ such that $T_{i_j} > T_x$ and

$$\mu_x^{\text{PS}}(B_N(T_{i_j})) \leq \epsilon \cdot \mu_x^{\text{PS}}(B_N(T_{i_j})).$$

Then for all sufficiently large $i > i_j$, we have

$$\mu_x^{\text{PS}}(B_N(T_j)) + \mu_x^{\text{PS}}(B_N(T_i) \cap \text{Nbd}_R(U)) \leq 2\epsilon \mu_x^{\text{PS}}(B_N(T_i)).$$

Therefore it follows from (2.2) that for any j and for all $i > i_j$,

$$\begin{aligned} \mu_x^{\text{PS}}\{n_t \in B_N(T_i) \setminus (B_N(T_j) \cup \text{Nbd}_R(U)) : xn_t \in B\} \\ \geq 0.5 \mu_x^{\text{PS}}(B_N(T_i))m^{\text{BMS}}(B) > 0. \end{aligned}$$

This implies that the set of x with $xn_t \in B$ cannot be contained in any bounded neighborhood of U , proving the claim. \square

Proof of Theorem 2.2. First by modifying f on a BR-null set, we may assume that for all $x \in X$, and for all $n \in N$,

$$f(xn) = n_* f(x).$$

Fix a compact subset $Q \subset X_{\text{rad}}$ such that f is continuous on Q and $m^{\text{BR}}(Q) > 0$. This is possible by Lusin's theorem. We claim that for some $y \in Q$, the set

$$\{n \in N : yn \in Q\}$$

is unbounded in the quotient space $U \backslash N$.

First note that there exists $\rho_0 > 0$ such that $QB_N(\rho_0)$ has a positive BMS-measure.

By Lemma 2.4 there is a BMS full measure set X' so that for all $x \in X'$,

$$\{n \in N : xn \in QB_N(\rho_0)\} \text{ is unbounded in } U \backslash N.$$

Using the fact that N is abelian, the above implies that

$$(2.6) \quad \{n \in N : yn \in Q\} \text{ is unbounded in } U \backslash N \text{ for all } y \in X'N.$$

The set $X'N$ is a BR-co-null set and $m^{\text{BR}}(Q) > 0$. Therefore, there is some $y \in Q$ which satisfies (2.6), proving the claim. Now, there is a sequence $\{n_{t_i} \in N\}$ such that $n_{t_i} \rightarrow \infty$ in $U \backslash N$ and that $yn_{t_i} \in Q$ and $yn_{t_i} \rightarrow z \in Q$. The function f is continuous on Q . Therefore we get

$$(n_{t_i})_*f(y) \rightarrow f(z).$$

Since $f(y)$ and $f(z)$ are probability measures on $U \backslash N \simeq \mathbb{R}$, and $n_{t_i} \rightarrow \infty$ in $U \backslash N \simeq \mathbb{R}$, this contradicts Lemma 2.3. This yields that $U = N$, yielding a contradiction. \square

3. PROOF OF THEOREMS 1.2 AND 1.3

We continue the notation set up in the introduction. Let $\mathbb{F} = \mathbb{R}$ or \mathbb{C} and $G = \text{PSL}_2(\mathbb{F})$. Let $\Gamma_1 < G$ be a co-compact lattice and $\Gamma_2 < G$ be a geometrically finite and Zariski dense subgroup. Set $X_i = \Gamma_i \backslash G$ for $i = 1, 2$. Let $Z = X_1 \times X_2$. Let $N = \{n_t : t \in \mathbb{F}\} < G$ be a horospherical subgroup.

We denote by $m_{\Gamma_2}^{\text{BR}}$ the N -invariant Burger-Roblin measure on X_2 ; this is unique up to a constant multiple.

Let μ be a $\Delta(N)$ -invariant, ergodic, conservative *infinite* Radon measure on Z . Let

$$\pi : Z \rightarrow X_2$$

be the canonical projection. Since X_1 is compact, the push-forward $\pi_*\mu$ defines an N -invariant ergodic conservative *infinite* Radon measure on X_2 .

Theorem 3.1. *Up to a constant multiple,*

$$\pi_*\mu = m_{\Gamma_2}^{\text{BR}} \quad \text{or} \quad \pi_*\mu = dn$$

for the N -invariant measure dn on a closed orbit x_2N homeomorphic to $\mathbb{R} \times \mathbb{S}^1$. The latter happens only when $\mathbb{F} = \mathbb{C}$ and Γ has a parabolic limit point of rank one.

Proof. Since Γ_2 is assumed to be geometrically finite and Zariski dense, up to a proportionality, the measure $\pi_*\mu$ is either $m_{\Gamma_2}^{\text{BR}}$ or it is the N -invariant measure supported on a closed N -orbit x_2N in X_2 ([17] and [23]). In the latter case, x_2N is homeomorphic to one of the following: $\mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{R} \times \mathbb{R}$, and $\mathbb{R} \times \mathbb{S}^1$. The first possibility cannot happen as that would mean that μ is a finite measure. The second possibility would contradict the assumption that μ is N -conservative. Hence x_2N must be $\mathbb{R} \times \mathbb{S}^1$, up to a homeomorphism. \square

The following is one of the main ingredients of our proof of Theorem 1.3, established in [14].

Theorem 3.2. *One of the following holds, up to a constant multiple:*

- (1) $\pi_*\mu = m_{\Gamma_2}^{\text{BR}}$ and μ is invariant under $U \times \{e\}$ for a non-trivial connected subgroup U of N ;

- (2) $\pi_*\mu = m_{\Gamma_2}^{\text{BR}}$ and the fibers of the map π are finite with the same cardinality almost surely. Moreover, in this case, μ is the graph of the BR-measure in the sense of Theorem 1.3(2);
- (3) $\mathbb{F} = \mathbb{C}$ and $\pi_*\mu = dn$ for the N -invariant measure dn on a closed orbit x_2N homeomorphic to $\mathbb{R} \times \mathbb{S}^1$.

Proof. For the case when $\pi_*\mu = m_{\Gamma_2}^{\text{BR}}$, it follows from [14, Thm. 7.12 and Thm. 7.17] either that the fibers of the map π are finite with the same cardinality almost surely or that μ is invariant under a non-trivial connected subgroup of N , yielding the cases (1) and (2). Indeed [14, Thm. 7.12] states this under the assumption that μ is an N -joining, but all that is used in the proof is the fact that the projection of the measure onto one of the factors is the BR-measure. □

3.1. Proof of Theorem 1.3.

3.1.1. *The case of $G = \text{PSL}_2(\mathbb{R})$.* In this case, $m_{\Gamma_2}^{\text{BR}}$ is the unique infinite conservative N -invariant measure on X_2 . Therefore we may assume, after the normalization of $m_{\Gamma_2}^{\text{BR}}$ if necessary, that $\pi_*\mu = m_{\Gamma_2}^{\text{BR}}$. By the standard disintegration theorem, see [1], we have

$$\mu = \int_{X_2} \mu_x \, dm_{\Gamma_2}^{\text{BR}}(x)$$

where μ_x is a probability measure on X_1 for $m_{\Gamma_2}^{\text{BR}}$ -a.e. x .

Suppose that Theorem 3.2(1) holds, i.e., μ is invariant under $N \times \{e\}$. Then, since every element in the σ -algebra

$$\{X_1 \times B : B \subset X_2 \text{ is a Borel set}\}$$

is invariant under $N \times \{e\}$, we get that μ_x is an N -invariant probability measure on X_1 for $m_{\Gamma_2}^{\text{BR}}$ -a.e. x .

By the unique ergodicity of N on the compact space X_1 [8], we have

$$(3.1) \quad \mu_x = m^{\text{Haar}} \quad \text{for } m_{\Gamma_2}^{\text{BR}}\text{-a.e. } x;$$

hence $\mu = m^{\text{Haar}} \times m_{\Gamma_2}^{\text{BR}}$.

If Theorem 3.2(2) holds, we obtain that μ is the graph of the BR-measure as desired in Theorem 1.3.

3.1.2. *The case of $G = \text{PSL}_2(\mathbb{C})$.* In analyzing the three cases in Theorem 3.2, we use the following special case of Ratner’s measure classification theorem [16]:

Theorem 3.3. *Let $\Gamma_1 < G = \text{PSL}_2(\mathbb{C})$ be a co-compact lattice. Let U be a one parameter unipotent subgroup of G . Let $L \simeq \text{PSL}_2(\mathbb{R})$ be the connected subgroup generated by U and its transpose U^t . Then any ergodic U -invariant probability measure on $\Gamma_1 \backslash G$ is either the Haar measure or a $v^{-1}Lv$ -invariant measure supported on a compact orbit $\Gamma_1 \backslash \Gamma_1 gLv$ for some $g \in G$ and $v \in N$.*

Indeed, since there are no compact U orbits in the compact space $\Gamma_1 \backslash G$, the same conclusion holds for any ergodic u -invariant probability measure on $\Gamma_1 \backslash G$ for any non-trivial element $u \in U$; see [16] and also [22].

Also note that in the second case of Theorem 3.3, the support of the measure is contained in yLN for some compact orbit yL .

We now investigate each case of Theorem 3.2 as follows:

Theorem 3.4. *For $k = 1, 2, 3$, Theorem 3.2(k) implies Theorem 1.3(k).*

Proof. Observe first that the case of $k = 2$ follows directly from Theorem 3.2.

Consider the case $k = 1$: suppose that μ is invariant under a subgroup $U \times \{e\}$ for a non-trivial connected subgroup U of N . We normalize $m_{\Gamma_2}^{\text{BR}}$ so that $\pi_*\mu = m_{\Gamma_2}^{\text{BR}}$. It follows from the standard disintegration theorem, see [1], that

$$(3.2) \quad \mu = \int_{X_2} \mu_x \, dm_{\Gamma_2}^{\text{BR}}(x).$$

Arguing as in section 3.1.1, since μ is invariant under $U \times \{e\}$, we get that μ_x is a U -invariant probability measure on X_1 for $m_{\Gamma_2}^{\text{BR}}$ -a.e. x . We claim that

$$(3.3) \quad \mu_x = m^{\text{Haar}} \quad \text{for } m_{\Gamma_2}^{\text{BR}}\text{-a.e. } x;$$

this implies $\mu = m^{\text{Haar}} \times m_{\Gamma_2}^{\text{BR}}$ and finishes the proof in this case.

We apply Theorem 3.3 to U . Let $L \simeq \text{PSL}_2(\mathbb{R})$ be defined as in Theorem 3.3. Compactness of $\Gamma_1 \backslash \Gamma_1 g L$ implies that $g^{-1} \Gamma_1 g \cap L$ is a co-compact lattice of L . In particular, $g^{-1} \Gamma_1 g \cap L$ is finitely generated and Zariski dense in L . This implies there are only countably many compact L orbits in X_1 .

Let $\{y_i L : i = 0, 1, 2, \dots\}$ be the collection of all compact L -orbits in X_1 . Then for $m_{\Gamma_2}^{\text{BR}}$ -a.e. $x \in X_2$, we have

$$(3.4) \quad \mu_x = c_x m^{\text{Haar}} + \sum_i \mu_{x,i}$$

where $c_x \geq 0$ and $\mu_{x,i}$ is a U -invariant finite measure supported in $y_i L N$.

The set $\{(x_1, x_2) : c_{x_2} > 0\}$ is a $\Delta(N)$ -invariant Borel measurable set. Therefore, (3.3) follows if this set has positive measure.

In view of this, we assume from now that $c_x = 0$ for $m_{\Gamma_2}^{\text{BR}}$ -a.e. x . Then the support of μ is contained in a countable union

$$\bigcup_i (y_i L N \times X_2).$$

Hence for some i ,

$$(3.5) \quad \mu(y_i L N \times X_2) > 0.$$

Without loss of generality, we may assume $i = 0$.

Since $y_0 L N \times X_2$ is $\Delta(N)$ -invariant and μ is $\Delta(N)$ -ergodic, (3.5) implies that $y_0 L N \times X_2$ is μ -co-null. Therefore, μ_x is supported on $y_0 L N$ for $m_{\Gamma_2}^{\text{BR}}$ -a.e. $x \in X_2$.

For each $n \in N$, let η_n be the probability measure supported on $y_0 L n$, invariant under $n^{-1} L n$. Noting that $y_0 L n = y_0 L n'$ if $n \in U n'$, the map $n \mapsto \eta_n$ factors through $U \backslash N$. We also have

$$(3.6) \quad n_{0*} \eta_n = \eta_{n n_0} \quad \text{for any } n, n_0 \in N.$$

By Theorem 3.3, the collection $\{\eta_n : n \in U \backslash N\}$ provides all U -invariant ergodic probability measures on X_1 whose supports are contained in $y_0 L N$.

Hence the U -ergodic decomposition of μ_x gives that for a.e. $x \in X_2$, there is a probability measure σ_x on $U \backslash N$ such that

$$\mu_x = \int_{U \backslash N} \eta_n \, d\sigma_x(n).$$

Since μ is $\Delta(N)$ -invariant, we have

$$(3.7) \quad \mu_{x n_0} = n_{0*} \mu_x \quad \text{for } m_{\Gamma_2}^{\text{BR}}\text{-a.e. } x \in X_2 \text{ and all } n_0 \in N.$$

Observe that

$$(3.8) \quad \mu_{xn_0} = \int_{U \setminus N} \eta_n \, d\sigma_{xn_0}(n),$$

and that

$$n_{0*}\mu_x = \int_{X_1} n_{0*}\eta_n \, d\sigma_x(n) = \int_{X_1} \eta_{nn_0} \, d\sigma_x(n) = \int_{X_1} \eta_n \, d(n_0\sigma_x)(n).$$

Therefore (3.7) implies that for $m_{\Gamma_2}^{\text{BR}}$ -a.e. $x \in X_2$ and for a.e. $n_0 \in N$,

$$(3.9) \quad n_{0*}\sigma_x = \sigma_{xn_0}.$$

It follows that the Borel map $f : (X_2, m_{\Gamma_2}^{\text{BR}}) \rightarrow \mathcal{P}(U \setminus N)$ defined by

$$f(x) := \sigma_x$$

is essentially N -equivariant for the natural action of N on $\mathcal{P}(U \setminus N)$.

As U is one dimensional, this yields a contradiction to Theorem 2.2 and hence completes the proof of case $k = 1$.

We now turn to the proof of the case $k = 3$. The argument is similar to the above case. Let x_2N be a closed orbit as in the statement of Theorem 3.2(3). We disintegrate μ as follows:

$$(3.10) \quad \mu = \int_{x_2N} \mu_x \, dn$$

where μ_x is a probability measure on X_1 for a.e. $x \in x_2N$. As x_2N is homeomorphic to $\mathbb{R} \times \mathbb{S}^1$, the stabilizer of x_2 in N is generated by a unipotent element, say, u . Note that u acts trivially on x_2N and $\Delta(u)$ leaves μ invariant. Hence again we have

$$(3.11) \quad \mu_x \text{ is } u\text{-invariant almost surely.}$$

We apply (3.4) for u -invariant measures μ_x . Let $L \simeq \text{PSL}_2(\mathbb{R})$ denote the connected closed subgroup containing u and u^t and let $\{y_iL : i = 0, 1, \dots\}$ be the collection of all compact L -orbits. Then for almost every $x \in x_2N$ we write

$$\mu_x = c_x m^{\text{Haar}} + \sum_i \mu_{x,i},$$

where $\mu_{x,i}$ is a u -invariant finite measure supported in y_iLN . As before, if $c_x > 0$ on a positive measure subset of x_2N , then $c_x = 1$ almost surely by the $\Delta(N)$ ergodicity of μ . Then $\mu = m^{\text{Haar}} \times dn$; note that this measure is $\Delta(N)$ ergodic since m^{Haar} is u -ergodic. This is the case of Theorem 1.3(3)(a).

Lastly we consider the case when $c_x = 0$ almost surely. As before,

$$\mu(y_iLN \times x_2N) > 0$$

for some i , and hence almost all μ_x is supported on one y_iLN by the ergodicity of μ . We assume $i = 0$ without loss of generality.

Set $U = L \cap N$. Then $\{\eta_n : n \in U \setminus N\}$ (with η_n defined as in the previous case) is the set of all u -ergodic probability measures on X_1 whose supports are contained in y_0LN by Theorem 3.3 and the remark following it. Therefore, we get a probability measure $\sigma_x \in \mathcal{P}(U \setminus N)$ such that

$$\mu_x = \int_{n \in U \setminus N} \eta_n \, d\sigma_x(n).$$

Moreover, $n_*\sigma_x = \sigma_{xn}$ for a.e. x and all $n \in N$.

Put $\sigma := \sigma_x$ for some fixed x . Without loss of generality, we assume $x = x_2$. Then for $\psi \in C_c(Z)$,

$$\mu(\psi) = \int_{n \in U \setminus N} \int_{x_2 n_0 \in x_2 N} \int_Y \psi(y n_0 n, x_2 n_0) dy dn_0 d\sigma(n)$$

where $Y = y_0 L$ and dy is the probability Haar measure on Y .

However for each $n \in U \setminus N$, $\psi \mapsto \int_{x_2 n_0 \in x_2 N} \int_Y \psi(y n_0 n, x_2 n_0) dy dn_0$ defines a $\Delta(N)$ -invariant measure, and hence by the $\Delta(N)$ -ergodicity assumption on μ , σ must be a delta measure at a point, say $n \in U \setminus N$. Therefore we arrive at Theorem 1.3(3)(b). \square

3.2. Proof of Theorem 1.2. Suppose that the product measure

$$\mu := m^{\text{Haar}} \times m_{\Gamma_2}^{\text{BR}}$$

is not ergodic for the action of $\Delta(N)$. Let Ω be the support of μ . We consider the decomposition $\Omega = \Omega_d \cup \Omega_c$ where Ω_d and Ω_c are maximal $\Delta(N)$ -invariant dissipative and conservative subsets respectively. That is, for any positive measure $S \subset \Omega_d$ (resp. $S \subset \Omega_c$), the Haar measure of $\{n \in N : xn \in S\}$ is finite (resp. infinite) for almost all $s \in S$ (see [9]).

Consider the ergodic decomposition of μ . By Theorem 1.3, any ergodic conservative component in the ergodic decomposition, see [1], of μ should be one of the measures as described in Theorem 1.3(2) and 1.3(3).

Now μ gives measure zero to sets of the form

$$(x_1, x_2)\Delta(G)(N \times \{e\})$$

where $(x_1, x_2)\Delta(G)$ is a closed orbit. Moreover, there are only countably many closed $\Delta(G)$ orbits in Z .

Also, any closed N orbit $x_2 N$ gives rise to the family $x_2 N A$ of closed N -orbits where A is the diagonal subgroup. There are only finitely many such AN -orbits in X_2 , as Γ_2 is geometrically finite and hence there are only finitely many Γ orbits of parabolic limit points. Therefore $m_{\Gamma_2}^{\text{BR}}$ gives zero measure to the set of all closed N -orbits in X_2 .

It follows that Ω_c is trivial and hence the product measure $m^{\text{Haar}} \times m_{\Gamma_2}^{\text{BR}}$ is completely dissipative. This is a contradiction since X_1 is compact and $m_{\Gamma_2}^{\text{BR}}$ is N -conservative. This proves Theorem 1.2. \square

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