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EXPANSION BY ORTHOGONAL SYSTEMS WITH RESPECT TO FREUD WEIGHTS RELATED TO HARDY SPACES

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ABSTRACT. For the basic class of Freud weights $w_\alpha(x) = \exp(-|x|^\alpha/2)$, $\alpha > 1$ the coefficients of the expansion of $w_\alpha f \in H_p(R)$ by the Freud orthogonal system $\{w_\alpha p_{n,\alpha}\}_{n=0}^\infty$, where $p_{n,\alpha}$ are polynomials of degree n, are related to the quasi-norm (or norm) of $w_\alpha f$ in $H_p(R)$. Relations are achieved for all $\alpha > 1$ and $\frac{1}{2} < p < 1$, and for some α for a larger range of p. As a result, estimates for $1 < p \le 2$ are also improved.

1. INTRODUCTION

For $f \in L_p(T)$, $f \sim \sum\limits_{n=-\infty}^{\infty} C_n e^{inx}$ the well-known classical inequality

$$(1.1) \qquad \sum_{n=-\infty}^{\infty} |C_n|^p (1+|n|)^{p-2} \le C\|f\|^p_{L_p(T)}, \quad 1 < p \le 2,$$

was proved by Polya. The inequality (1.1) was extended by Hardy and Littlewood to the Hardy space $H_p(T)$ i.e.

$$(1.2) \qquad \sum_{n=-\infty}^{\infty} |C_n|^p (1+|n|)^{p-2} \le C\|f\|^p_{H_p(T)}, \quad f \in H_p(T), \quad 0 < p \le 1.$$

Several analogues of (1.1) and (1.2) for different expansions and spaces were given (see for instance [Ra-Th], [Ka], [Di,13A] [Di,13B] and [Di,16]).

In this paper we deal with expansion with respect to the Freud weights $w_\alpha(x) = \exp(-|x|^\alpha/2)$, $\alpha > 1$ which is the important and typical subset of Freud weights.

The complete orthonormal system $\{w_\alpha p_{n,\alpha}\}_{n=0}^\infty$ is given by

$$(1.3) \qquad \int_{-\infty}^{\infty} p_{n,\alpha}(x) p_{k,\alpha}(x) w_\alpha^2(x)dx = \begin{cases} 1 & n = k, \\ 0 & n \ne k, \end{cases}$$

where $p_{n,\alpha}$ is a polynomial of degree n. We recall that $w_2 p_{n,2}$ is the Hermite function $h_n(x)$ (see [Th, p. 41]).

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For $fw_\alpha \in L_1(R)+L_\infty(R)$ the expansion by $w_\alpha p_{n,\alpha}$ is well defined and given by

$$(1.4)\qquad fw_\alpha \sim \sum_{n=0}^{\infty} C_{n,\alpha} w_\alpha p_{n,\alpha} \quad \text{where} \quad C_{n,\alpha}(fw_\alpha) = C_{n,\alpha} = \int_{-\infty}^{\infty} f p_{n,\alpha} w_\alpha^2 \,.$$

Investigations of $w_\alpha p_{n,\alpha}$, its derivatives, norms and the expansion (1.4) are given in the texts [Di-To], [Fr], [Le-Lu], [Mh] and in numerous articles. When we use a result, we will give the exact location where it is proved (mostly from [Le-Lu] which is the most extensive source). In [Di,13A], we proved

$$(1.5)\qquad \sum_{n=0}^{\infty} (1+n)^{(p-2)(\frac{7}{6}-\frac{1}{2\alpha})} \,|C_{n,\alpha}(fw_\alpha)^p \le C \,\|fw_\alpha\|^p_{L_p(R)}\,, \quad 1<p\le 2,$$

which is a Polya-Hardy-Littlewood type inequality for the Freud expansion. In Section 2, we will prove a Hardy-Littlewood inequality for $fw_\alpha \in H_p(R)$ when $\alpha > 1$ and $\frac{1}{2} < p \le 1$. In Section 3, we will show that when $\alpha = 2k$, the range of p for this inequality is $0 < p \le 1$, and when $\alpha > k$, the range is shown to be $\frac{1}{k+1} < p \le 1$.

In Section 4, we will give an improvement of (1.5) for $L_p(R)$, $1 < p \le 2$ and other comments.

The atomic decomposition of $H_p(R)$ will be used. $a(x)$ is an atom of $H_p(R)$ if for some $r > 0$

$$(1.6)\qquad \operatorname{supp} a(x) \subset \big[x_0 - \frac{r}{2}\,,\, x_0 + \frac{r}{2}\,\big],$$

$$(1.7)\qquad |a(x)| \le r^{-1/p}$$

and

$$(1.8)\qquad \int_{-\infty}^{\infty} a(x)x^\ell dx = 0 \quad \text{for all} \quad \ell \le \big(\frac{1}{p}-1\big),\, \ell = 0,1,\dots\,.$$

Different atoms may have different x_0 and r. The space $H_p(R)$ has atomic decomposition, which means for $fw_\alpha \in H_p(R)$ there exist $H_p(R)$ atoms such that

$$(1.9)\qquad fw_\alpha = \sum \lambda_i a_i(x), \quad C^{-1}\|fw_\alpha\|^p_{H_p(R)} \le \sum |\lambda_i|^p \le C\|fw_\alpha\|^p_{H_p(R)},$$

and the last set of inequalities is denoted by $\sum |\lambda_i|^p \approx \|fw_\alpha\|^p_{H_p(R)}$.

2. The basic result ($\alpha > 1$, $\frac{1}{2} < p \le 1$)

In this section we prove a Hardy-Littlewood type result for expansion with respect to the Freud weight w_α , $\alpha > 1$ in the Hardy space $H_p(R)$, $\frac{1}{2} < p \le 1$.

Theorem 2.1. *For $fw_\alpha \in H_p(R)$ with $\alpha > 1$ and $\frac{1}{2} < p \le 1$ one has*

$$(2.1)\qquad \sum_{n=0}^{\infty} (1+n)^{(\frac{19}{18}-\frac{1}{2\alpha})(p-2)} |C_{n,\alpha}(fw_\alpha)|^p \le C\|fw_\alpha\|^p_{H_p(R)}$$

with C independent of f and p.

Remark 2.2. The coefficients $C_{n,\alpha}(fw_\alpha)$ are well defined by (1.4) as $fw_\alpha \in H_p(R)$ implies $fw_\alpha \in S'$ (the Schwartz space of distribution) and $w_\alpha p_{n,\alpha} \in S$ (the Schwartz space of test functions). The factor $\big(\frac{19}{18}-\frac{1}{2\alpha}\big)$ is smaller than $\big(\frac{7}{6}-\frac{1}{2\alpha}\big)$ and represents an improvement which in Section 4 will imply an inequality stronger than (1.5). For $p = 1$, $\alpha = 2$ (2.1) was proved by Kanjin (see [Ka]).

Proof. Following many articles on the subject (see [Di,13B, p. 37], [Ka, p. 334] and [Ra-Th, p. 3528] for example), we observe that (1.8) implied that it is sufficient to show that for any atom $a(x) \in H_p(R)$ one has

$$I \equiv \sum_{n=0}^{\infty} (1+n)^{(\frac{19}{18}-\frac{1}{2\alpha})(p-2)} |C_{n,\alpha}(a)|^p \le C\|a\|^p_{H_p(R)} = C. \tag{2.2}$$

For $\sigma = \frac{19}{18} - \frac{1}{2\alpha}$ and $\beta = \frac{1}{(2\sigma-1)}$ we write

$$I = \sum_{1+n\le r^{-\beta}} + \sum_{1+n>r^{-\beta}} \equiv S(1) + S(2).$$

To estimate $S(2)$ we use the Hölder inequality and write

$$\begin{aligned} S(2) &\le \Big(\sum_{1+n>r^{-\beta}} (1+n)^{-2\sigma} \Big)^{\frac{2-p}{2}} \Big(\sum |C_{n,\alpha}(a)|^2 \Big)^{p/2} \\ &\le \Big((1+n_0)^{-2\sigma+1} \Big)^{\frac{2-p}{2}} \Big(\frac{1}{r^{2/p}}\, r \Big)^{p/2} \\ &\le C_1 (r)^{1-\frac{p}{2}}\, r^{-1} r^{p/2} \le C_1 \end{aligned}$$

since when $n_0 = \min\{n : 1+n > r^{-\beta}\}$, $1+n_0 \approx r^{-1(2\sigma-1)}$.

We note that in the estimate of $S(2)$ we did not use the fact that $\frac{1}{2} < p$ and hence we can use this estimate in the next section. In addition, we used the fact that $2\sigma > 1$, which we will also use in the next section.

We now estimate $S(1)$. As $w_\alpha p_{n,\alpha}$ for $\alpha > 1$ has at least one continuous derivative, we write the Taylor formula

$$w_\alpha(x) p_{n,\alpha}(x) = w_\alpha(x_0) p_{n,\alpha}(x_0) + (x - x_0) \frac{d}{dx} \big(w_\alpha(x) p_{n,\alpha}(x) \big)\Big|_{x=\xi}. \tag{2.3}$$

We will need the estimate

$$\|(w_\alpha p_{n,\alpha})'\|_{L_\infty(R)} \le C_2 (1+n)^{1-\frac{1}{\alpha}} (1+n)^{(1-\frac{3}{\alpha})/6} = C_2 (1+n)^{\frac{7}{6}-\frac{3}{2\alpha}} \tag{2.4}$$

which we now assume and will prove in the following lemma.

For $\sigma = \frac{19}{18} - \frac{1}{2\alpha}$ and $\beta = 1/(2\sigma - 1)$ we now write

$$\begin{aligned} S(1) &\le \sum_{1+n<r^{-\beta}} (1+n)^{(p-2)\sigma} \Big| \int a(x)(x-x_0)(w_\alpha p_{n,\alpha})'_{x=\xi} dx \Big|^p \\ &\le C_2 \sum_{(1+n)<r^{-\beta}} (1+n)^{p\sigma} (1+n)^{-2\sigma} \Big| \frac{1}{r^{1/p}}\, r^2 (1+n)^{\frac{7}{6}-\frac{3}{2\alpha}} \Big|^p \\ &\le C_3 r^{2p-1} (1+n_0)^{p(\sigma-\frac{7}{6}-\frac{3}{2\alpha})} (1+n_0)^{-2\sigma+1}. \end{aligned}$$

As

$$-\Big(\sigma + \frac{7}{6} - \frac{3}{2\alpha}\Big)/(2\sigma - 1) = -\Big(\frac{40}{18} - \frac{2}{\alpha}\Big) \Big/ \Big(\frac{20}{18} - \frac{1}{\alpha}\Big) = -2$$

and $(1 + n_0) \approx r^{-\beta} = r^{1/(2\sigma-1)}$ when $n_0 = \max\{n : (1+n) < r^{-\beta}\}$, we have $S(1) \le C$.

We observe that $S(1) = 0$ if $r > 1$ but we did not use this fact here. (We will, however, use it in the next section.) We completed the proof of our theorem pending the proof of (2.4). □

Lemma 2.3. *For* $w_\alpha(x) = \exp(-|x|^\alpha/2)$ *and* $p_{n.\alpha}$ *of* (1.1), *the inequality* (2.4) *holds with* C *independent of* n.

Proof. We recall first from [Le-Lu, p. 360, (13.4)] that

$$\|w_\alpha p_{n,\alpha}\|_{L_\infty(R)} \le C(1+n)^{(1-\frac{3}{\alpha})/6} \tag{2.5}$$

since for w_α we have in [Le-Lu] $a_n = a_{-n} \approx \delta_n \approx n^{1/\alpha}$ and $T(a_n) = \alpha$ (see [Le-Lu, pp. 5-10]).

We then use the estimate from [Le-Lu, p. 294, Cor.10.2, (10.3)], which for $p_{n,\alpha}$ (and in fact for any polynomial p_n of degree n), yields

$$\|w_\alpha p'_{n,\alpha}\|_{L_\infty(R)} \le Cn^{1-\frac{3}{2\alpha}}\,\|w_\alpha p_{n,\alpha}\|_{L_\infty(R)} \tag{2.6}$$

where we used a_n, δ_n and $T(a_n)$ as we did for (2.5). Therefore,

$$\|w_\alpha p'_{n,\alpha}\|_{L_\infty(R)} \le C_2(1+n)^{\frac{7}{6}-\frac{2}{\alpha}}\,.$$

For $s \ge \alpha - 1$, we now estimate

$$\|x^{\alpha-1}w_\alpha(x)p_{n,\alpha}(x)\|_{L_\infty(R)} \le \|w_\alpha p_{n,\alpha}\|_{L_\infty(R)}^{(s-\alpha+1)/s}\|x^s w_\alpha p_{n,\alpha}\|_{L_\infty(R)}^{(\alpha-1)/s}\,.$$

We use the result of Mhaskar and Saff (see [Le-Lu, p. 4, (1.12)]) i.e.

$$\|w_\alpha p_{n+s}\|_{L_\infty(R)} = \|w_\alpha p_{n+1}\|_{L_\infty(-a_{n+s},a_{n+s})}, \quad a_n \approx a_{n+1} \approx n^{1/\alpha}\,.$$

We now have

$$\|x^{\alpha-1}w_\alpha p_{n,\alpha}\|_{L_\infty(R)} \le C_3(1+n)^{(1-\frac{1}{\alpha})}(1+n)^{(1+\frac{3}{\alpha})/6} \le C_3(1+n)^{\frac{7}{3}-\frac{3}{2\alpha}}\,. \tag{2.7}$$

Since $1-\frac{1}{\alpha} > 1-\frac{3}{2\alpha}$, we combine (2.6) and (2.7) to obtain (2.4). □

Remark 2.4. We note that for $\alpha = 2$ $\|h_n\|_{L_\infty(R)} \le C(1+n)^{-1/12}$ (see [Th, Lemma 1.5.2(iii), p. 27]) which confirms (2.5) for the special case $\alpha = 2$. To obtain (2.4) for $\alpha = 2$ from the estimate of $\|h_n\|_{L_\infty(R)}$, we may combine

$$\Big(-\frac{d}{dx}-x\Big)h_k(x) = (2k+2)^{1/2}h_{k+1}(x) \quad \text{with} \quad \Big(\frac{d}{dx}+x\Big)h_k(x) = (2k)^{1/2}h_{k-1}(x)$$

(see [Th, pp. 2-5]). To obtain the estimate of $\|x^s w_\alpha p_{n,\alpha}\|$ we may also use repeatedly the formula given in [Mh, (3.1.14), p. 51] since for w_α $\gamma_{n-1}/\gamma_n \approx n^{1/\alpha}$. We used the text [Le-Lu] which contains all the estiamtes we needed.

3. Extending the range of p for some α

In Theorem 2.1 the inequality (2.1) was shown to hold for $\frac{1}{2} < p \le 1$ and all $\alpha > 1$. In the following theorem we show that for a significant subset of α we can extend the range of p for which (2.1) is valid.

Theorem 3.1. *For* $fw_\alpha \in H_p(R)$, *and* $\alpha > L$ *for some integer* L, (2.1) *holds for* $\frac{1}{L+1} < p \le 1$. *When* $\alpha = 2k$, (2.1) *holds for* $0 < p \le 1$. *The constant in* (2.1) *depends on the interval* $I_\ell = \big(\frac{1}{\ell+1}, \frac{1}{\ell}\big]$ *which contains* p.

Proof. When $\alpha > L$ $w_\alpha p_{n,\alpha}$ has (at least) L continuous derivatives and when $\alpha = 2k$, $w_\alpha p_{n,\alpha} \in C^\infty(R)$. Hence we may use the Taylor formula for $\ell \le L$ (in case $\alpha > L$) and all ℓ for $\alpha = 2k$, and write

(3.1)
$$w_\alpha(x)p_{n,\alpha}(x) = w_\alpha(x_0)p_{n,\alpha}(x_0) + \cdots + \frac{(x-x_0)^{\ell-1}}{(\ell-1)!}\big(w_\alpha(x)p_{n,\alpha}(x)\big)^{(\ell-1)}\big|_{x=x_0} + \frac{(x-x_0)^{\ell}}{\ell!}\big(w_\alpha(x)p_{n,\alpha}(x)\big)^{(\ell)}\big|_{x=\xi}.$$

As in the proof of Theorem 2.1, we split the sum in (2.1) into $S(1)$ and $S(2)$, and as commented there, the estimate of $S(2)$ was already proved for all p. We will estimate $S(1)$ for $\frac{1}{\ell+1} < p \le \frac{1}{\ell}$ for which (3.1) holds. Recall that $\sigma = \frac{19}{18} - \frac{1}{2\alpha}$ and $\beta = \frac{1}{(2\sigma-1)}$ and write

$$S(1) = \sum_{(1+n)\le r^{-\beta}} (1+n)^{(p-2)\sigma}|C_{n,\alpha}(a)|^p \tag{3.2}$$

where $a(x)$ satisfies (1.6), (1.7) and (1.8) with x_0 of (3.1) and r of (3.2). We use (3.1), (1.6), (1.7) and (1.8) to estimate $|C_{n,\alpha}(a)|$ and write

$$\begin{aligned} |C_{n,\alpha}(a)| &= \Big|\int a(x)w_\alpha(x)p_{n,\alpha}(x)dx\Big| \\ &\le \frac{1}{\ell!}\int \frac{1}{r}|x-x_0|^\ell \|(w_\alpha p_{n,\alpha})^{(\ell)}\|_{L_\infty(R)} \\ &\le \frac{1}{r} r^{\ell+1}\|(w_\alpha p_{n,\alpha})^{(\ell)}\|_{L_\infty(R)}. \end{aligned} \tag{3.3}$$

We will prove our theorem pending the estimate

$$\begin{aligned} \big\|(w_\alpha p_{n,\alpha})^{(\ell)}\big\|_{L_\infty(R)} &\le C_1(1+n)^{(1-\frac{1}{\alpha})\ell}(1+n)^{(1-\frac{3}{\alpha})/6} \\ &= C_1(1+n)^{\ell+\frac{1}{6}-(\ell+\frac{1}{2})\frac{1}{\alpha}} \end{aligned} \tag{3.4}$$

for $\ell \le L$ in case $\alpha > L$ and for any ℓ in case $\alpha = 2k$. The estimate (3.4) is proved in Lemma 3.2 after the proof of our theorem. For $\frac{1}{\ell+1} < p \le \frac{1}{\ell}$, $\sigma = \frac{19}{18} - \frac{1}{2\alpha}$ and $\beta = 1/(2\sigma-1) = 1/\big(\frac{20}{18} - \frac{1}{\alpha}\big)$, we have

$$\begin{aligned} S(1) &\le C_2\Big(\sum_{(1+n)\le r^{-\beta}} (1+n)^{p\sigma}(1+n)^{[(\ell+\frac{1}{6})-(\ell+\frac{1}{2})\frac{1}{\alpha}]p}(1+n)^{-2\sigma}\Big)r^{-1}r^{(\ell+1)p} \\ &\le C_3\Big[(1+n_0)^{(\frac{19}{18}+\ell+\frac{1}{6})p}(1+n_0)^{-(\ell+1)\frac{1}{\alpha}}(1+n_0)^{-2\sigma+1}\Big]r^{-1}r^{(\ell+1)p} \end{aligned}$$

where $n_0 = \max\{n : (1+n) \le r^{-\beta}\}$ and hence $(1+n_0) \le r^{-\beta}$. We observe that

$$\frac{19}{18} + \ell + \frac{1}{6} - (\ell+1)\frac{1}{\alpha} = (\ell+1)\Big(\frac{20}{18} - \frac{1}{\alpha}\Big) - \frac{(\ell-1)}{9}$$

and hence for $\beta = 1/\big(\frac{20}{18} - \frac{1}{\alpha}\big)$

$$S(1) \le C_3\, r^{(\ell-1)/9\left(\frac{20}{18}-\frac{1}{\alpha}\right)} \le C_3$$

since $S(1) = 0$ whenever $r \ge 1$. We now have $S(1) \le C_3$ pending the proof of (3.4). □

Lemma 3.2. *For $\alpha > L$ (3.4) holds for $\ell \le L$. For $\alpha = 2k$ (3.4) holds for any ℓ. Both hold with constants that depend on ℓ.*

Proof. We write first for any m (not just $m \le \ell$)

$$\begin{aligned}\|w_\alpha p_{n,\alpha}^{(m)}\|_{L_\infty(R)} &\le C(1+n)^{m\left(1-\frac{3}{2\alpha}\right)}\|w_\alpha p_{n,\alpha}\|_{L_\infty(R)} \\ &\le C_1(1+n)^{\left(1-\frac{3}{2\alpha}\right)m+\left(1-\frac{3}{\alpha}\right)/6}\end{aligned} \tag{3.5}$$

by repeating the result of [Le-Lu, p. 294, Cor.10.2, 10.3] m times and using (2.5). We note that the above result of [Le-Lu] is valid for any polynomial of degree n and $p_{n,\alpha}^{(m)}$ is such a polynomial. (The inequality (3.5) applies to $w_\alpha p_{n,\alpha}$ with the special $p_{n,\alpha}$ that is given in (1.3).) We now show for any integer s

$$\|x^s w_\alpha(x) p_{n,\alpha}^{(m)}(x)\|_{L_\infty(R)} \le C(1+n)^{s/\alpha}(1+n)^{\left(1-\frac{3}{2\alpha}\right)m}(1+n)^{\left(1-\frac{3}{\alpha}\right)/6} \tag{3.6}$$

which follows as $x^s p_{n,\alpha}^{(m)}$ is a polynomial of degree $n-m+s$, and using the Mhaskar-Saff result (see [Le-Lu, p. 4,(1.12)]),

$$\|x^s w_\alpha(x) p_{n,\alpha}^{(m)}(x)\|_{L_\infty(R)} = \|x^s w_\alpha(x) p_{n,\alpha}^{(m)}\|_{L_\infty(-a_{n-m+s}\,,a_{n-m+s})}$$

with $|a_{n-m+s}| \approx n^{1/\alpha}$ and hence we have (3.6).

We now obtain for any $\gamma \ge 0$

$$\|x^\gamma w_\alpha(x) p_{n,\alpha}^{(m)}(x)\|_{L_\infty(R)} \le C(1+n)^{\gamma/\alpha}(1+n)^{\left(1-\frac{3}{2\alpha}\right)m}(1+n)^{\left(1-\frac{3}{\alpha}\right)/6} \tag{3.7}$$

by choosing $s \ge \gamma$ ($s = [\gamma]+1$ for instance) and using the interpolation

$$\|x^\gamma w_\alpha(x) p_{n,\alpha}^{(m)}(x)\|_{L_\infty(R)} \le \|w_\alpha p_{n,\alpha}^{(m)}\|_{L_\infty(R)}^{1-\frac{\gamma}{s}}\|x^s w_\alpha(x) p_{n,\alpha}^{(m)}(x)\|_{L_\infty(R)}^{\gamma/s}.$$

To prove (3.4) we note that for $\ell \le L$ in case $\alpha > L$ or any ℓ when $\alpha = 2k$, $(w_\alpha p_{n,\alpha})^{(\ell)}$ is a combination of terms like $(w_\alpha)^{(\ell-m)}(p_{n,\alpha})^{(m)}$ with $m \le \ell$. As in both (3.6) and (3.7) the estimate is monotone increasing in γ or s, and hence the estimate of $(w_\alpha)^{(\ell-m)}(p_{n,\alpha})^{(m)}$ depends only on the highest power of x in $(w_\alpha)^{(\ell-m)}$ which is $x^{(\alpha-1)(\ell-m)}$. Therefore,

$$\begin{aligned}\|w_\alpha^{(\ell-m)} p_{n,\alpha}^{(m)}\|_{L_\infty(R)} &\le C(1+n)^{\left(1-\frac{1}{\alpha}\right)(\ell-m)}(1+n)^{\left(1-\frac{3}{2\alpha}\right)m}(1+n)^{\left(1-\frac{3}{\alpha}\right)/6} \\ &\le C(1+n)^{\left(1-\frac{1}{\alpha}\right)\ell}(1+n)^{\left(1-\frac{3}{\alpha}\right)/6}.\end{aligned} \tag{3.8}$$

Combining (3.8) for $0 \le m \le \ell$, we obtain (3.4). □

4. Extension for $1 < p \le 2$ and other remarks

In [Di,13A] the Hardy-Littlewood type inequality

$$\left\{\sum_{n=0}^{\infty}(1+n)^{(p-2)\left(\frac{7}{6}-\frac{1}{2\alpha}\right)}|C_n|^p\right\}^{1/p} \le C\|fw_\alpha\|_{L_p(R)} \tag{4.1}$$

was proved for $1 < p \le 2$. Using Theorem 2.1, we obtain the following stronger inequality.

Theorem 4.1. *For $w_\alpha = \exp(-|x|^\alpha/2)$, $\alpha > 1$, $C_{n,\alpha} \equiv C_{n,\alpha}(fw_\alpha)$ given by (1.4), $1 < p \le 2$ and $fw_\alpha \in L_p(R)$ one has*

$$\left\{\sum_{n=0}^{\infty}(1+n)^{(p-2)\left(\frac{19}{18}-\frac{1}{2\alpha}\right)}|C_{n,\alpha}|^p\right\}^{1/p} \le C\|fw_\alpha\|_{L_p(R)}. \tag{4.2}$$

Remark 4.2. As $\frac{7}{6} > \frac{19}{18}$ (4.2) is stronger than (4.1). We believe the reason the weak (1.1) result and Marcinkiewicz interpolation used in [Di,13A] did not yield the optimal result is that the maxima of $|w_\alpha(x)p_{n,\alpha}(x)|$ occurs at widely different x_n for different n.

Proof. Clearly, for $p = 2$ (4.2) with equality and $C = 1$ is the Parseval identity. We now use [Ga-Ru, pp. 307-310, Theorem 6.1] with $T(fw_\alpha) \to \left\{C_{n,\alpha}(n+1)^{\left(\frac{19}{18}-\frac{1}{2\alpha}\right)}\right\}_{n=0}^{\infty}$ in the weighted ℓ_p with weights $\left\{(n+1)^{-2\left(\frac{19}{18}-\frac{1}{2\alpha}\right)}\right\}_{n=0}^{\infty}$ to interpolate between T on $fw_\alpha \in H_1(R)$ and T on $fw_\alpha \in L_2(R)$. In fact, the map $T(fw_\alpha)$ from $H_1(R)$ to the corresponding weighted ℓ_1 space is strongly bounded by Theorem 2.1. □

Using duality, we also have the following corollary which is an improvement over [Di,13A, Theorem 3.2].

Corollary 4.3. *If for* q, $2 \le q < \infty$ *there exists a constant* C *and a sequence* $\{C_n\}$ *such that* $\sum_{n=0}^{\infty} |C_n|^q (n+1)^{(q-2)\left(\frac{19}{18}-\frac{1}{2\alpha}\right)} < C$, *then* fw_α *where* $fw_\alpha \sim \sum_{n=0}^{\infty} C_n w_\alpha p_{n,\alpha}$ *satisfies* $fw_\alpha \in L_q(R)$ *and*

$$\|fw_\alpha\|_{L_q(R)} \le C\left\{\sum_{n=0}^{\infty} (n+1)^{(q-2)\left(\frac{19}{18}-\frac{1}{2\alpha}\right)} |C_n|^q\right\}^{1/q}. \tag{4.3}$$

For the Hermite functions (the case $\alpha = 2$) one has

$$\sum_{n=0}^{\infty} (1+n)^{(p-2)\frac{29}{36}} |C_{n,2}(fw_\alpha)|^p \le C\|fw_\alpha\|_{H_p(R)}^p, \quad 0 < p \le 2 \tag{4.4}$$

which was proved by Kanjin (see [Ka]) for important partial range $1 \le p \le 2$.

To answer a question by the referee, we believe (but cannot prove) that (2.1) is optimal when the whole range $\alpha > 1$ and $\frac{1}{2} < p \le 1$ is considered, and that further progress will concentrate on special α and p.

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