PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 4, April 2018, Pages 1825–1832
http://dx.doi.org/10.1090/proc/13843
Article electronically published on November 10, 2017

BANACH SPACES WITH WEAK*-SEQUENTIAL DUAL BALL

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(Communicated by Thomas Schlumprecht)

ABSTRACT. A topological space is said to be sequential if every subspace closed under taking limits of convergent sequences is closed. We consider Banach spaces with weak*-sequential dual ball. In particular, we show that if X is a Banach space with weak*-sequentially compact dual ball and $Y \subset X$ is a subspace such that Y and X/Y have weak*-sequential dual ball, then X has weak*-sequential dual ball. As an application we obtain that the Johnson-Lindenstrauss space JL_2 and $C(K)$ for K a scattered compact space of countable height are examples of Banach spaces with weak*-sequential dual ball. These results provide a negative solution to a question of A. Plichko, who asked whether the dual ball of a Banach space is weak*-angelic whenever it is weak*-sequential.

1. Introduction

All topological spaces considered in this paper are Hausdorff. The symbol w^* denotes the weak* topology of the corresponding Banach space. A topological space T is said to be *sequentially compact* if every sequence in T contains a convergent subsequence. Moreover, T is said to be *Fréchet-Urysohn* (FU for short) if for every subspace F of T, every point in the closure of F is the limit of a sequence in F. Every FU compact space is sequentially compact. A Banach space with weak*-FU dual ball is said to have *weak*-angelic dual*. Some examples of Banach spaces with weak*-angelic dual are WCG Banach spaces (i.e., Banach spaces generated by a weakly compact set) and, in general, WLD Banach spaces (i.e., Banach spaces whose dual ball with the weak*-topology is Corson). On the other hand, every weak Asplund Banach space and every Banach space without copies of ℓ_1 in the dual have weak*-sequentially compact dual ball [7, Chapter XIII].

In this paper we are going to focus on sequential spaces, which is a generalization of the FU property. If T is a topological space and F is a subspace of T, the *sequential closure* of F is the set of all limits of sequences in F. F is said to be *sequentially closed* if it coincides with its sequential closure. A topological space is said to be *sequential* if any sequentially closed subspace is closed. Thus, every FU

Received by the editors December 19, 2016 and, in revised form, May 17, 2017.

2010 *Mathematics Subject Classification.* Primary 57N17, 54D55, 46A50; Secondary 46B20, 46B50.

Key words and phrases. Sequential, sequentially compact, countable tightness, Fréchet-Urysohn, angelic, Efremov property, Banach space.

The author was partially supported by the research project 19275/PI/14 funded by Fundación Séneca - Agencia de Ciencia y Tecnología de la Región de Murcia within the framework of PCTIRM 2011-2014 and by Ministerio de Economía y Competitividad and FEDER (project MTM2014-54182-P).

space is sequential. Another natural generalization of the FU property is countable tightness. A topological space T is said to have *countable tightness* if for every subspace F of T, every point in the closure of F is in the closure of a countable subspace of F. It can be proved that every sequential space has countable tightness. However, whether the converse implication in the class of compact spaces is true is known as the Moore-Mrowka Problem and it is undecidable in ZFC (i.e., in the usual axioms of set-theory)[2]. Therefore, for a compact space K, we have the following implications:

$$\begin{array}{ccccc} K \text{ is FU} & \Longrightarrow & K \text{ is sequential} & \Longrightarrow & K \text{ is sequentially compact} \\ & & \Downarrow & & \\ & & K \text{ has countable tightness} & & \end{array}$$

In [20, Question 10] A. Plichko asked whether every Banach space with weak*-sequential dual ball has weak*-angelic dual. In the next section we prove the following theorem, which is applied to prove that the Johnson-Lindenstrauss space JL_2 provides a negative answer to Plichko's question:

Theorem 1.1. *Let X be a Banach space with weak*-sequentially compact dual ball. Let $Y \subset X$ be a subspace with weak*-sequential dual ball with sequential order $\leq \gamma_1$ and such that X/Y has weak*-sequential dual ball with sequential order $\leq \gamma_2$. Then X has weak*-sequential dual ball with sequential order $\leq \gamma_1 + \gamma_2$.*

One of the properties studied by Plichko in [20] is property $\mathcal{E}$ of Efremov. A Banach space X is said to have *property* $\mathcal{E}$ if every point in the weak*-closure of any *convex* subset $C \subset B_{X^*}$ is the weak*-limit of a sequence in C. We say that X has *property* $\mathcal{E}'$ if every weak*-sequentially closed *convex* set in the dual ball is weak*-closed. Thus, if X has weak*-angelic dual, then it has property $\mathcal{E}$; and if X has weak*-sequential dual ball, then X has property $\mathcal{E}'$. We also provide a convex version of Theorem 1.1 (see Theorem 2.3).

Other related Banach space properties are the Mazur property and property (C). A Banach space X has *Mazur property* if every $x^{**} \in X^{**}$ which is weak*-sequentially continuous on X^* is weak*-continuous and, therefore, $x^{**} \in X$. Notice that if a topological space T is sequential, then any sequentially continuous function $f : T \to \mathbb{R}$ is continuous. Thus, it follows from the Banach-Dieudonné Theorem that every Banach space with weak*-sequential dual ball has the Mazur property. Moreover, property $\mathcal{E}'$ also implies the Mazur property.

A Banach space X has *property (C)* of Corson if and only if every point in the closure of C is in the weak*-closure of a countable subset of C for every convex set C in B_{X^*} (this characterization of property (C) is due to R. Pol [19]).

Thus, we have the following implications among these Banach space properties:

$$\begin{array}{ccccc} \text{weak*-angelic dual} & \Rightarrow & \text{weak*-sequential dual ball} & \Rightarrow & \text{weak*-seq. compact dual ball} \\ \Downarrow & & \Downarrow & & \\ \text{property } \mathcal{E} & \Longrightarrow & \text{property } \mathcal{E}' & \Longrightarrow & \text{property } (C) \\ & & \Downarrow & & \\ & & \text{Mazur property} & & \end{array}$$

Notice that $\mathcal{C}([0,\omega_1])$ has weak*-sequentially compact dual ball, but it is not weak*-sequential. Moreover, $\ell_1(\omega_1)$ has the Mazur property [9, Section 5], but it does not have property (C).

In [21, p. 352] it is asked whether property (C) implies property $\mathcal{E}$. J.T. Moore in an unpublished paper and C. Brech in her PhD thesis [6] provided a negative answer under some additional consistent axioms, but the question is still open in ZFC. Notice that the convex version of Plichko's question is whether property $\mathcal{E}'$ implies property $\mathcal{E}$. A negative answer to this question would provide an example of a Banach space with property (C) not having property $\mathcal{E}$.

In [10, Lemma 2.5] it is proved that the dual ball of $\mathcal{C}(K)$ does not contain a copy of $\omega_1 + 1 = [0,\omega_1]$ when K is a scattered compact space of finite height satisfying some properties. It is also proved in [17] that $\mathcal{C}(K)$ has the Mazur property whenever K is a scattered compact space of countable height. We generalize these results by proving that $\mathcal{C}(K)$ has weak*-sequential dual ball whenever K is a scattered compact space of countable height (Theorem 3.2).

2. Banach spaces with weak*-sequential dual ball

Definition 2.1. *Let T be a topological space and F a subspace of T. For any $\alpha \leq \omega_1$ we define $S_\alpha(F)$ as the αth sequential closure of F by induction on α: $S_0(F) = F$, $S_{\alpha+1}(F)$ is the sequential closure of $S_\alpha(F)$ for every $\alpha < \omega_1$ and $S_\alpha(F) = \bigcup_{\beta<\alpha} S_\beta(F)$ if α is a limit ordinal.*

Notice that $S_{\omega_1}(F)$ is sequentially closed for every subspace F. Thus, a topological space T is sequential if and only if $S_{\omega_1}(F) = \overline{F}$ for every subspace F of T. We say that T has *sequential order* α if $S_\alpha(F) = \overline{F}$ for every subspace F of T and for every $\beta < \alpha$ there exists F with $S_\beta(F) \neq \overline{F}$. Therefore, a topological space T is sequential with sequential order ≤ 1 if and only if it is FU. We will use the following lemma in the proof of Theorem 1.1:

Lemma 2.2. *Let $f : K \to L$ be a continuous function, where K, L are topological spaces and K is sequentially compact. Then, $f(S_\alpha(F)) = S_\alpha(f(F))$ for every $F \subset K$ and every ordinal α.*

Proof. The inclusion $f(S_\alpha(F)) \subset S_\alpha(f(F))$ follows from the continuity of f.

We prove the other inclusion by induction on α. The case $\alpha = 0$ is immediate. Suppose $\alpha = 1$. Take $s \in S_1(f(F))$. Then, there exists a sequence t_n in F such that $f(t_n)$ converges to s. Since K is sequentially compact, without loss of generality we may suppose t_n is converging to some point t. Then, it follows from the continuity of f that $f(t) = s$. Thus, $s \in f(S_1(F))$.

Now suppose the result is true for every $\beta < \alpha$ and $\alpha \geq 2$. If α is a limit ordinal, then

$$f(S_\alpha(F)) = f(\bigcup_{\beta<\alpha} S_\beta(F)) = \bigcup_{\beta<\alpha} f(S_\beta(F)) = \bigcup_{\beta<\alpha} S_\beta(f(F)) = S_\alpha(f(F)).$$

If $\alpha = \beta + 1$ is a successor ordinal, then

$$f(S_\alpha(F)) = f(S_1(S_\beta(F))) = S_1(f(S_\beta(F))) = S_1(S_\beta(f(F))) = S_\alpha(f(F)).$$

□

Proof of Theorem 1.1. It is enough to prove that if $F \subset B_{X^*}$ and $0 \in \overline{F}^{w^*}$, then $0 \in S_{\gamma_1+\gamma_2}(F)$. Let $R : X^* \to Y^*$ be the restriction operator. For each finite set $A \subset X$ and each $\varepsilon > 0$, define

$$F_{A,\varepsilon} = \{x^* \in F : |x^*(x)| \leq \varepsilon \text{ for all } x \in A\}.$$

Since R is weak*-weak* continuous and $0 \in \overline{F_{A,\varepsilon}}^{w^*}$, we have that

$$0 \in \overline{R(F_{A,\varepsilon})}^{w^*} = S_{\gamma_1}(R(F_{A,\varepsilon})) = R(S_{\gamma_1}(F_{A,\varepsilon})),$$

where the last equality follows from Lemma 2.2.

Thus, for every finite set $A \subset X$ and every $\varepsilon > 0$ we can take $x^*_{A,\varepsilon} \in S_{\gamma_1}(F_{A,\varepsilon})$ such that $R(x^*_{A,\varepsilon}) = 0$.

Therefore, $0 \in \overline{G}^{w^*}$, where

$$G := \{x^*_{A,\varepsilon} : A \subset X \text{ finite}, \varepsilon > 0\} \subset Y^\perp \cap B_{X^*}.$$

Note that $(Y^\perp \cap B_{X^*}, w^*)$ is homeomorphic to the dual ball of $(X/Y)^*$ with the weak* topology. Hence

$$0 \in S_{\gamma_2}(G) \subset S_{\gamma_2}(S_{\gamma_1}(F)) = S_{\gamma_1+\gamma_2}(F).$$

□

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in a Banach space, we say that $(y_k)_{k\in\mathbb{N}}$ is a *convex block subsequence* of $(x_n)_{n\in\mathbb{N}}$ if there is a sequence $(I_k)_{k\in\mathbb{N}}$ of subsets of $\mathbb{N}$ with $\max(I_k) < \min(I_{k+1})$ and a sequence $a_n \in [0,1]$ with $\sum_{n\in I_k} a_n = 1$ for every $k \in \mathbb{N}$ such that $y_k = \sum_{n\in I_k} a_n x_n$. A Banach space X is said to have *weak*-convex block compact dual ball* if every bounded sequence in X^* has a weak*-convergent convex block subsequence. Every Banach space containing no isomorphic copies of ℓ_1 has weak*-convex block compact dual ball [5]. Therefore, every WPG Banach space (i.e. every Banach space with a linearly dense weakly precompact set) also has weak*-convex block compact dual ball.

For any ordinal $\gamma \leq \omega_1$, we say that X has *property* $\mathcal{E}(\alpha)$ if $S_\alpha(C) = C$ for every convex subset C in (B_{X^*}, w^*). Thus, property $\mathcal{E}$ is property $\mathcal{E}(1)$ and property $\mathcal{E}'$ is property $\mathcal{E}(\omega_1)$. The proof of the following theorem is an immediate adaptation of the proof of Lemma 2.2 and Theorem 1.1.

Theorem 2.3. *Let X be a Banach space with weak*-convex block compact dual ball. Let $Y \subset X$ be a subspace with property $\mathcal{E}(\gamma_1)$ such that X/Y has property $\mathcal{E}(\gamma_2)$. Then X has property $\mathcal{E}(\gamma_1+\gamma_2)$.*

Theorem 2.4. *Let X be a Banach space and $(X_n)_{n\in\mathbb{N}}$ an increasing sequence of subpaces with $X = \overline{\bigcup_{n\in\mathbb{N}} X_n}$. Suppose that each X_n has weak*-sequential dual ball with sequential order α_n. Then X has weak*-sequential dual ball with sequential order $\leq \alpha + 1$, where $\alpha := \sup\{\alpha_n : n \in \mathbb{N}\}$.*

Proof. Set $R_n : X^* \to X_n^*$ as the restriction operator for every $n \in \mathbb{N}$. Since the countable product of sequentially compact spaces is sequentially compact and (B_{X^*}, w^*) is homeomorphic to a subspace of $\prod(B_{X_n^*}, w^*)$, it follows that X has weak*-sequentially compact dual ball. In order to prove the theorem, it is enough to prove that if $F \subset B_{X^*}$ and $0 \in \overline{F}^{w^*}$, then $0 \in S_{\alpha+1}(F)$. Since B_{X^*} is weak*-sequentially compact, we have that $0 \in \overline{R_n(F)}^{w^*} = S_\alpha(R_n(F)) = R_n(S_\alpha(F))$ for every $n \in \mathbb{N}$, where the last equality follows from Lemma 2.2. Thus, we can take a sequence $x_n^* \in S_\alpha(F)$ such that $R_n(x_n^*) = 0$. Now there exists some subsequence of x_n^* converging to a point $x^* \in S_{\alpha+1}(F)$. Since $R_n(x^*) = 0$ for every $n \in \mathbb{N}$, we conclude that $x^* = 0$. □

Corollary 2.5. *Let X be a Banach space and $(X_\alpha)_{\alpha<\gamma}$ an increasing sequence of subspaces with $X = \overline{\bigcup_{\alpha<\gamma} X_\alpha}$, where γ is a countable limit ordinal. Suppose that each X_α has weak*-sequential dual ball with sequential order $\leq \theta_\alpha$. Then X has weak*-sequential dual ball with sequential order $\leq \theta+1$ where $\theta := \sup\{\theta_\alpha : \alpha < \gamma\}$.*

The next theorem follows from combining Theorem 1.1 and Corollary 2.5:

Theorem 2.6. *Let γ be a countable ordinal, X_γ a Banach space and $(X_\alpha)_{\alpha\leq\gamma}$ an increasing sequence of subspaces of X_γ such that:*

(1) *X_0 has weak*-sequential dual ball with sequential order $\leq \theta$;*
(2) *each quotient $X_{\alpha+1}/X_\alpha$ has weak*-angelic dual;*
(3) *$X_\alpha = \overline{\bigcup_{\beta<\alpha} X_\beta}$ if α is a limit ordinal;*
(4) *X_γ has weak*-sequentially compact dual ball.*

Then each X_α has weak-sequential dual ball with sequential order $\leq \theta+\alpha$ if $\alpha < \omega$ and sequential order $\leq \theta + \alpha + 1$ if $\alpha \geq \omega$.*

Proof. It follows from (4) that every X_α has weak*-sequentially compact dual ball. Thus, the result for $\alpha < \omega$ follows by applying inductively Theorem 1.1. Suppose $\alpha \geq \omega$ and X_β has weak*-sequential dual ball with sequential order $\leq \theta + \beta + 1$ for every $\beta < \alpha$. If α is a limit ordinal, then it follows from (3) and from Corollary 2.5 that X_α has weak*-sequential dual ball with sequential order

$$\leq \sup_{\beta<\alpha}\{\theta + \beta + 1\} + 1 = \theta + \alpha + 1.$$

If α is a successor ordinal, then the result is a consequence of Theorem 1.1. □

We also have the following convex equivalent version of the previous theorem:

Theorem 2.7. *Let γ be a countable ordinal, X_γ a Banach space and $(X_\alpha)_{\alpha\leq\gamma}$ an increasing sequence of subspaces of X_γ such that:*

(1) *X_0 has property $\mathcal{E}(\theta)$;*
(2) *each quotient $X_{\alpha+1}/X_\alpha$ has $\mathcal{E}$;*
(3) *$X_\alpha = \overline{\bigcup_{\beta<\alpha} X_\beta}$ if α is a limit ordinal;*
(4) *X_γ has weak*-convex block compact dual ball.*

Then each X_α has property $\mathcal{E}(\theta + \alpha)$ if $\alpha < \omega$ and property $\mathcal{E}(\theta + \alpha + 1)$ if $\alpha \geq \omega$.

3. Applications

As an application of Theorem 1.1, we obtain that the Johnson-Lindenstrauss space JL_2 has weak*-sequential dual ball. Let us recall the definition of JL_2:

Let $\{N_r : r \in \Gamma\}$ be an uncountable maximal almost disjoint family of infinite subsets of $\mathbb{N}$. For each N_r, $\chi_{N_r} \in \ell_\infty$ denotes the characteristic function of N_r. The Johnson-Lindenstrauss space JL_2 is defined as the completion of $\operatorname{span}(c_0 \cup \{\chi_{N_r} : r \in \Gamma\}) \subset \ell_\infty$ with respect to the norm

$$\|x + \sum_{1\leq i\leq k} a_i\chi_{N_{r_i}}\| = \max\left\{\|x + \sum_{1\leq i\leq k} a_i\chi_{N_{r_i}}\|_\infty, \left(\sum_{1\leq i\leq k} |a_i|^2\right)^{\frac{1}{2}}\right\},$$

where $x \in c_0$ and $\|\cdot\|_\infty$ is the supremum norm in ℓ_∞. If we just consider the supremum norm in the definition, then we obtain the space JL_0. We refer the reader to [14] for more information about these spaces.

Theorem 3.1. *The Johnson-Lindenstrauss space JL_2 has weak*-sequential dual ball with sequential order* 2.

Proof. We use the following results proved in [14]:

(i) JL_2 has an equivalent Fréchet differentiable norm;
(ii) JL_2/c_0 is isometric to $\ell_2(\Gamma)$.

It follows from (i) that JL_2 has weak*-sequentially compact dual ball (cf. [12]). It follows from (ii) and Theorem 1.1 that JL_2 has weak*-sequential dual ball with sequential order ≤ 2. Since JL_2 does not have weak*-angelic dual (cf. [9, Proposition 5.12]) we have that JL_2 has weak*-sequential dual ball with sequential order 2. □

Theorem 3.1 provides an example of a Banach space with weak*-sequential dual ball which does not have weakly*-angelic dual, answering a question of Plichko in [20, Question 10].

For a scattered compact space K, we denote by $ht(K)$ the height of K, i.e. the minimal ordinal γ such that the γth Cantor-Bendixson derivative $K^{(\gamma)}$ is discrete. Since every Banach space with weak*-sequential dual ball has the Mazur property, the following theorem improves [17, Theorem 4.1]:

Theorem 3.2. *Let K be an infinite scattered compact space. If $ht(K) < \omega$, then $\mathcal{C}(K)$ has weak*-sequential dual ball with sequential order $\leq ht(K)$. Moreover, if $\omega \leq ht(K) < \omega_1$, then $\mathcal{C}(K)$ has weak*-sequential dual ball with sequential order $\leq ht(K)+1$.*

Proof. It is well-known that if K is scattered, then $\mathcal{C}(K)$ is Asplund and therefore $B_{C(K)^*}$ is weak*-sequentially compact (see, for example, [22]). Denote by $\{K^{(\alpha)} : \alpha \leq \gamma\}$ the Cantor-Bendixson derivatives of K, where $\gamma = ht(K)$. For every $\alpha \leq \gamma$, set

$$X_\alpha = \{f \in \mathcal{C}(K) : f(t) = 0 \text{ for every } t \in K^{(\alpha)}\}.$$

Since $\mathcal{C}(K)$ contains a complemented copy of c_0, every finite-codimensional subspace of $\mathcal{C}(K)$ is isomorphic to $\mathcal{C}(K)$. Therefore, since X_γ is a finite-codimensional subspace of $\mathcal{C}(K)$, it is isomorphic to $\mathcal{C}(K)$. Notice that for every $0 \leq \alpha < \gamma$ we have that $X_{\alpha+1}/X_\alpha$ is isomorphic to $c_0(K^{(\alpha)} \setminus K^{(\alpha+1)})$. Moreover, if $\alpha \leq \gamma$ is a limit ordinal, then $\bigcap_{\beta<\alpha} K^{(\beta)} = K^{(\alpha)}$ and therefore

$$\overline{\bigcup_{\beta<\alpha} X_\beta} = \overline{\{f \in \mathcal{C}(K) : \exists \beta < \alpha \text{ with } f(t) = 0 \ \forall t \in K^{(\beta)}\}} = X_\alpha.$$

Now the conclusion follows from Theorem 2.6. □

R. Haydon [13] and K. Kunen [18] constructed under CH an FU compact space K such that $B_{C(K)^*}$ does not have countable tightness. Thus, it is not true for a general compact space K that if K is sequential, then $B_{C(K)^*}$ is sequential. We refer the reader to [11] for a discussion on this topic.

It can be easily checked that the space JL_0 is isomorphic to a $\mathcal{C}(K)$ space where K is a scattered compact space with $ht(K) = 2$ and sequential order 2. Thus, JL_0 has weak*-sequential dual ball with sequential order 2.

The known examples in ZFC of sequential compact spaces are all of sequential order ≤ 2. Nevertheless, A.I. Baškirov constructed sequential compact spaces of any sequential order $\leq \omega_1$ under the Continuum Hypothesis [3]. A different construction was also given by V. Kannan in [16]. Baškirov's construction is studied in detail in [1] and, as C. Baldovino highlights in [1, Remark 6.8], these constructions are scattered compact spaces such that the sequential order and the scattering height coincide whenever the sequential order is a successor ordinal.

Moreover, A. Dow constructed under the assumption $\mathfrak{b} = \mathfrak{c}$ a scattered compact space K of sequential order 4 such that the sequential order and the scattering height coincide [8].

Corollary 3.3. *Under the Continuum Hypothesis there exist Banach spaces with weak*-sequential dual ball of any sequential order* $< \omega$ *and Banach spaces with weak*-sequential dual ball with arbitrarily large countable sequential order.*

On the other hand, under $\mathfrak{b} = \mathfrak{c}$*, there exist Banach spaces with weak*-sequential dual ball of any sequential order* ≤ 4.

Acknowledgment

I would like to thank José Rodríguez for his helpful suggestions and remarks.

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