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ON PELLER'S CHARACTERIZATION OF TRACE CLASS HANKEL OPERATORS AND SMOOTHNESS OF KDV SOLUTIONS

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(Communicated by Joachim Krieger)

This paper is dedicated to the memory of Ludwig Faddeev, one of the founders of soliton theory

Abstract. In the context of the Cauchy problem for the Korteweg-de Vries equation we put forward a new effective method to link smoothness of the solution with the rate of decay of the initial data. Our approach is based on the Peller characterization of trace class Hankel operators.

1. Introduction

In the recent article [11] we extended the inverse scattering transform (IST) for the Korteweg-de Vries (KdV) equation

$$\begin{cases} \partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \\ u(x,0) = q(x) \end{cases} \tag{1.1}$$

to initial data $q(x)$ rapidly decaying at $+\infty$ but having almost unrestricted behavior at $-\infty$. Note that this setting is very different from classical (rapidly decaying or periodic initial data) due to a much more complicated spectral situation. Our approach is based upon Hankel operators, and it was some subtle results from the theory of Hankel operators that allowed us to remove nearly all conditions on $q(x)$ at $-\infty$. In the present note we show how the famous characterization of trace class Hankel operators due to Peller [18] applies to the study of the effect of $+\infty$ on the smoothness of $u(x,t)$. Our goal here is not to achieve optimal results (this will be done elsewhere) but rather to introduce a new effective approach to study delicate relations between decay of the data and smoothness of the corresponding solutions.

To state our main result we need some preliminary information. Let q be a real function such that the differential expression $-\partial_x^2 + q(x)$ is in the limit point case at $-\infty$ and $q(x) = O\left(x^{-2-\varepsilon}\right)$ as $x \to +\infty$. It is well-known that densely defined (symmetric) $-\partial_x^2 + q(x)$ can be extended to the self-adjoint Schrödinger operator $\mathbb{L}_q$ on $L^2(\mathbb{R})$. The corresponding Schrödinger equation

$$\mathbb{L}_q u = k^2 u$$

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has the right Jost solution $\psi(x,k)$, i.e. the solution subject to

$$\psi(x,k) = e^{ikx} + o(1), \quad x \to \infty,$$

and the (essentially unique) left Weyl solution $\Psi\left(x,k^2\right)$, i.e. the solution subject to

$$\Psi\left(x,k^2\right) \in L^2(-\infty,0) \text{ for any } \operatorname{Im} k^2 > 0.$$

Then as in [11] we can define the right reflection coefficient

$$R(k) = \frac{W(\overline{\psi}(\cdot,k), \Psi\left(\cdot,k^2+i0\right))}{W(\Psi(\cdot,k^2+i0), \psi(\cdot,k))}, \quad \operatorname{Im} k = 0, \tag{1.2}$$

where $W(f,g) = fg' - f'g$ is the Wronskian. It can be shown [11] that R is well defined for a.e. $k \in \mathbb{R}$ and

$$R(-k) = \overline{R(k)}, \quad |R(k)| \leq 1.$$

Note that in our setting the left reflection coefficient need not exist.

Assume now that $\operatorname{Spec}(\mathbb{L}_q)$ is bounded below. Then [11], shifting the origin to the right if needed, R admits the analytic split

$$R(k) = R_0(k) - T_0(k)/\psi(0,k) + A(k). \tag{1.3}$$

Here $R_0(k)$ and $T_0(k)$ are respectively the right reflection and transmission coefficients from $q_0 := q|_{\mathbb{R}_+}$. Without loss of generality the origin can be moved to the right so that $T_0(k)/\psi(0,k)$ is meromorphic in $\mathbb{C}^+$ with only one simple pole $i\kappa_0$. Then $T_0(k)/\psi(0,k)$ is uniformly bounded in $\mathbb{C}^+$ away from $i\kappa_0$. The function $A(k)$ is bounded on $\mathbb{R}$ and can be analytically continued into $\mathbb{C}^+ \diagdown i\Delta$ where

$$i\Delta = \left\{k \in i\mathbb{R}_+ : k^2 \in \operatorname{Spec}(\mathbb{L}_q) \cap \mathbb{R}_-\right\}.$$

Furthermore, its jump across $i\Delta$,

$$d\rho(s) = i\left(A(is-0) - A(is+0)\right) ds/2\pi, \tag{1.4}$$

defines a non-negative, finite measure ρ supported on Δ. Outside $i\Delta$ the function $A(k)$ is uniformly bounded in $\mathbb{C}^+$. The exact formula for $A(k)$ is not essential to us. Thus $R_0(k)$ is the only term in (1.3) that need not in general admit an analytic continuation into $\mathbb{C}^+$.

The analytic split (1.3) is the main reason why the IST works for the KdV equation with unrestricted behavior at $-\infty$. More precisely, the set

$$S_q = \{R, d\rho\}$$

forms scattering data for $\mathbb{L}_q$; i.e. S_q determines q uniquely.

We next recall [17] that given function $\varphi \in L^\infty(\mathbb{R})$, the operator $\mathbb{H}(\varphi)$ defined on $H^2(\mathbb{C}^+)$ by

$$\mathbb{H}(\varphi)f = \mathbb{J}\mathbb{P}_-\varphi f, \quad f \in H^2\left(\mathbb{C}^+\right), \tag{1.5}$$

is called the Hankel operator with symbol φ. Here $H^2(\mathbb{C}^\pm)$ stands for the standard Hardy space of analytic on $\mathbb{C}^\pm$ function respectively, $\mathbb{P}_\pm$ is the orthogonal (Riesz) projection in $L^2(\mathbb{R})$ onto $H^2(\mathbb{C}^\pm)$, and $\mathbb{J}$ is the operator of reflection, i.e. $(\mathbb{J}f)(x) = f(-x)$. Apparently, $\mathbb{J}H^2(\mathbb{C}^-) = H^2(\mathbb{C}^+)$.

We are now ready to state our main result.

Theorem 1.1 (Main Theorem)**.** *Suppose that a (real) initial profile q in* (1.1) *satisfies*

$$\inf \operatorname{Spec}(\mathbb{L}_q) = -h^2 > -\infty \qquad \textit{(boundedness from below)}; \tag{1.6}$$

$$\int^{\infty} x^N |q(x)|\, dx < \infty, \quad N > 9/2 \quad \textit{(decay at} +\infty). \tag{1.7}$$

Then (1.1) *has a global-in-time classical*[1] *solution given by*

$$u(x,t) = -2\partial_x^2 \log\det(1 + \mathbb{H}(\varphi_{x,t})), \quad t > 0, \tag{1.8}$$

where

$$\varphi_{x,t}(k) = R(k)\, e^{i(8k^3 t + 2kx)} + \int_0^h e^{8s^3 t - 2sx}\, \frac{d\rho(s)}{s + ik}, \tag{1.9}$$

such that if $u_b(x,t)$ is the (necessarily unique) classical solution with data $q_b = q|_{(b,\infty)}$, then $u_b(x,t)$ converges to $u(x,t)$ uniformly on compacts in $\mathbb{R} \times \mathbb{R}_+$ as $b \to -\infty$. Furthermore, the map $(x,t) \to \mathbb{H}(\varphi_{x,t})$ is continuously differentiable in trace norm n times in x and m times in t where $n \leq 2(N-2)$ and $m \leq 2(N-2)/3$. Consequently, $\partial_x^j \partial_t^k u(x,t)$ is continuous on $\mathbb{R} \times \mathbb{R}_+$ if $0 \leq j + 3k \leq 2N - 6$.

The condition (1.6) is optimal, but (1.7) is not. The best bound known to us is $N = 11/4$ [2], but as will be easily seen from our arguments, this bound is not optimal either. Note that the approach of [2] requires a rapid decay at $-\infty$ whereas ours doesn't.

The paper is organized as follows. In Section 2 we give brief background information on trace class Hankel operators. Section 3 is devoted to the proof of Theorem 1.1, and the final section, Section 4, is reserved for relevant discussions.

2. Trace class Hankel Operators

We refer the reader to [17] and [18] for the details on the facts given in this section.

It directly follows from definition (1.5) that the Hankel operator $\mathbb{H}(\varphi)$ is bounded if its symbol φ is bounded and $\mathbb{H}(\varphi + h) = \mathbb{H}(\varphi)$ for any $\varphi \in H^\infty(\mathbb{C}^+)$ (analytic and bounded on the $\mathbb{C}^+$ function). The latter means that only the part of φ analytic in $\mathbb{C}^-$ matters. More specifically,

$$\mathbb{H}(\varphi) = \mathbb{H}(\widetilde{\mathbb{P}}_- \varphi),$$

where

$$\begin{aligned}(\widetilde{\mathbb{P}}_- \varphi)(x) &= (x+i)\left(\mathbb{P}_- \frac{1}{\cdot + i}\varphi\right)(x) \\ &= -\frac{1}{2\pi i}\int_{\mathbb{R}} \left(\frac{1}{s - (x - i0)} - \frac{1}{s+i}\right)\varphi(s)ds, \quad \varphi \in L^\infty(\mathbb{R}).\end{aligned}$$

We note that the condition $\varphi \in L^\infty(\mathbb{R})$ guarantees only that $\widetilde{\mathbb{P}}_- \varphi \in BMOA(\mathbb{C}^-)$, but the Hankel operator $\mathbb{H}(\varphi)$ is still well defined by (1.5) and bounded.

A much more subtle fact, the Nehari Theorem, says that $\mathbb{H}(\varphi)$ is compact iff $\widetilde{\mathbb{P}}_- \varphi \in C(\mathbb{R})$. We will crucially use the following delicate theorem.

[1] That is, at least three times continuously differentiable in x and once in t.

Theorem 2.1 (Peller, 1980)**.** *Let* $\varphi \in L^\infty(\mathbb{R})$. *Then* $\mathbb{H}(\varphi)$ *is trace class iff* $\left(\widetilde{\mathbb{P}}_-\varphi\right)'' \in L^1(\mathbb{C}^-)$ *and* $\sup_{\operatorname{Im} z \le -1}\left|\widetilde{\mathbb{P}}_-\varphi(z)\right| < \infty$.

In general, the membership of $\mathbb{H}(\varphi)$ in any Shatten-von Neumann class $\mathfrak{S}_p, 0 < p < \infty$, is characterized by the membership of $\widetilde{\mathbb{P}}_-\varphi$ in the Besov classes of smooth functions. If $\varphi \in C^\infty\left(\overline{\mathbb{R}}\right)$, then $\mathbb{H}(\varphi) \in \mathfrak{S}_p$ for any $0 < p \le \infty$.

3. Proof of the Main Theorem

Take $b > 0$ and consider the problem (1.1) with initial data $q_b = q|_{(b,\infty)}$. This problem [2] is well-posed, and its solution u_b can be written as

$$u_b(x,t) = -2\partial_x^2 \log\det\left(1 + \mathbb{H}(\varphi_{x,t}^b)\right), \tag{3.1}$$

where

$$\varphi_{x,t}^b(k) = R_b(k)\, e^{i(8k^3t+2kx)} + \int_0^h e^{8s^3t-2sx}\,\frac{d\rho_b(s)}{s+ik}. \tag{3.2}$$

Here R_b is the right reflection coefficient off q_b and

$$d\rho_b(s) = \sum_n c_n^b \delta\left(s - \kappa_n^b\right) ds, \tag{3.3}$$

where c_n^b is the norming constant of the bound state $-\left(\kappa_n^b\right)^2$. Split R_b by (1.3) as follows

$$R_b(k) = R_0(k) - T_0(k)/\psi(0,k) + A_b(k), \tag{3.4}$$

where A_b can be analytically continued into $\mathbb{C}^+$ as a meromorphic function having simple poles at $\left(i\kappa_n^b\right)$ with residues $\left(c_n^b\right)$. If follows from (3.2), (3.3) and (3.4) that

$$\varphi_{x,t}^b(k) = \left(R_0(k) - \frac{T_0(k)}{\psi(0,k)}\right)\xi_{x,t}(k) + \Phi_{x,t}^b(k),$$

where

$$\Phi_{x,t}^b(k) = A_b(k)\,\xi_{x,t}(k) - \sum_n \frac{ic_n^b\xi_{x,t}(i\kappa_n^b)}{k - i\kappa_n^b},$$
$$\xi_{x,t}(k) := \exp i(8k^3t + 2kx).$$

Consider the part of $\varphi_{x,t}^b$ analytic in $\mathbb{C}^-$:

$$\begin{aligned}\widetilde{\mathbb{P}}_-\varphi_{x,t}^b &= \widetilde{\mathbb{P}}_-\Phi_{x,t}^b - \widetilde{\mathbb{P}}_-\left(\frac{T_0}{\psi(0,\cdot)}\xi_{x,t}\right) + \widetilde{\mathbb{P}}_-\left(R_0\xi_{x,t}\right)\\ &= \phi_1^b + \phi_2 + \phi_3,\end{aligned} \tag{3.5}$$

only ϕ_1^b being dependent on b. Let us treat it first. One clearly has that for any $\beta \ge 0$,

$$\begin{aligned}|\xi_{x,t}(\alpha + i\beta)| &= \xi_{x,t}(i\beta)\exp\left\{-24t\left(\sqrt{\beta}\alpha\right)^2\right\}\\ &= e^{8\beta^3t-2\beta x}\exp\left\{-24t\left(\sqrt{\beta}\alpha\right)^2\right\}.\end{aligned} \tag{3.6}$$

That is, for every fixed $\beta > 0$ the function $\xi_{x,t}(\alpha + i\beta)$ shows a rapid decay as $|\alpha| \to \infty$. Observe that as opposed to $A_b \xi_{x,t}$, the function $\Phi^b_{x,t}$ has only removable singularities in $\mathbb{C}^+$ and therefore for any $h_0 > h$ we clearly have

$$\begin{aligned}\phi_1^b &= -\frac{k+i}{2\pi i}\int_{\mathbb{R}} \frac{\Phi^b_{x,t}(s)}{s+i}\ \frac{ds}{s-(k-i0)}\\ &= \frac{k+i}{2\pi i}\int_{\mathbb{R}+ih_0} \frac{A_b\,(s)\,\xi_{x,t}(s)}{s+i}\ \frac{ds}{k-s}.\end{aligned}$$

It is proven in [11] that $A_b\,(s) \to A\,(s)$ uniformly on compacts in $\mathbb{C}^+$, and hence we have

$$\begin{aligned}\lim_{b\to\infty} \phi_1^b\,(k) &= \frac{k+i}{2\pi i}\int_{\mathbb{R}+ih_0} \frac{A\,(s)\,\xi_{x,t}(s)}{s+i}\ \frac{ds}{k-s}\\ &=: \phi_1\,(k)\,.\end{aligned}$$

Since the last equation holds for $h_0 > h$, we immediately conclude that $\phi_1\,(k)$ is an entire function with the property that for any $n \geq 1$,

$$\lim \partial_k^n \phi_1\,(k) = 0, \quad \operatorname{Im} k = 0, \quad k \to \pm\infty.$$

Moreover, for every n, m,

$$\lim_{b\to\infty} \partial_x^n \partial_t^m \phi_1^b\,(k) = \partial_x^n \partial_t^m \phi_1\,(k)$$

and $\partial_x^n \partial_t^m \phi_1\,(k)$ is an entire function such that $\lim \partial_k^n \phi_1\,(k) = 0$, $\operatorname{Im} k = 0$, $k \to \pm\infty$. We can now conclude that for every n, m,

$$\left\|\partial_x^n \partial_t^m \left[\mathbb{H}\left(\phi_1^b\right) - \mathbb{H}\left(\phi_1\right)\right]\right\|_{\mathfrak{S}_1} \to 0, \quad b \to \infty. \tag{3.7}$$

For the symbol ϕ_2 in (3.5) we notice that $T_0\,(k)\,/\psi\,(0,k)$ is meromorphic in $\mathbb{C}^+$ with only one simple pole, call it $i\kappa_0$. Therefore, merely repeating the same arguments as before, we have

$$\phi_2\,(k) = -\frac{k+i}{2\pi i}\int_{\mathbb{R}+ih_0} \frac{T_0\,(s)\,\xi_{x,t}(s)}{(s+i)\,\psi\,(0,s)}\ \frac{ds}{k-s}, \tag{3.8}$$

with any $0 < h_0 < \kappa_0$ and for every n, m,

$$\partial_x^n \partial_t^m \mathbb{H}\,(\phi_2) \in \mathfrak{S}_1. \tag{3.9}$$

The symbol ϕ_3 in (3.5) is the most difficult as $R_0\,(k)$ need not extend into $\mathbb{C}^+$ as an analytic function. Our analysis is based on the representation [4]

$$R_0\,(k) = \frac{T_0\,(k)}{2ik}\int_0^\infty e^{-2iks} g\,(s)\,ds, \tag{3.10}$$

where g is some function for which we only need the bound

$$|g\,(s)| \leq |q\,(s)| + const \int_s^\infty |q|\,,$$

which implies that under the condition (1.7) (i.e. $\int^\infty x^N\,|q\,(x)|\,dx < \infty$)

$$\int^\infty x^{N-1}\,|g\,(x)|\,dx < \infty. \tag{3.11}$$

By construction, q_0 is supported on a half-line, and hence [4] $T_0\,(k)$ is meromorphic in the entire complex plane with one simple pole $i\kappa_0$ in $\mathbb{C}^+$. Generically $T_0\,(0) = 0$ (otherwise we shift the origin), and hence $f\,(k) := T_0\,(k)\,/2ik$ is analytic and bounded on the strip $S_{h_0} = \{0 \le \operatorname{Im} k \le h_0\}$ for any $h_0 < \kappa_0$. Thus for ϕ_3 in (3.5) we now have

$$\phi_3 = \widetilde{\mathbb{P}}_- \left(\xi_{x,t} f G\right),$$

where

$$G(k) := \int_0^\infty e^{-2iks} g(s)\, ds.$$

It follows from (3.11) that

$$\partial_k^n G(k) \in H^\infty\left(\mathbb{C}^-\right) \cap C(\mathbb{R}) \tag{3.12}$$

for any integer $n \le N-1$, but it doesn't in general extend analytically into $\mathbb{C}^+$, and we can no longer deform the contour into the upper half plane. Let us now consider instead its pseudoanalytic extension into $\mathbb{C}^+$. Following [5] we call $F(x,y)$ a pseudoanalytic extension of $f(x)$ into $\mathbb{C}$ if

$$F(x,0) = f(x) \quad \text{and} \quad \overline{\partial} F(x,y) \to 0, y \to 0,$$

where $\overline{\partial} := (1/2)(\partial_x + i\partial_y)$. Note that due to (3.12) for any $n \le N-1$ the Taylor formula

$$G\left(\lambda, \overline{\lambda}\right) = \sum_{m=0}^{n-1} \frac{G^{(n)}\left(\overline{\lambda}\right)}{m!} \left(\lambda - \overline{\lambda}\right)^m, \quad \lambda \in \mathbb{C}^+, \tag{3.13}$$

defines such a continuation, as $G\left(\lambda, \overline{\lambda}\right)$ clearly agrees with G on the real line, and for $\lambda \in \mathbb{C}^+$,

$$\overline{\partial} G\left(\lambda, \overline{\lambda}\right) = \frac{G^{(n)}\left(\overline{\lambda}\right)}{(n-1)!} \left(\lambda - \overline{\lambda}\right)^{n-1}, \quad n \le N-1. \tag{3.14}$$

Now evaluate ϕ_3 by the Cauchy-Green formula applied to the strip S_{h_0}. We have $(\lambda = \alpha + i\beta)$

$$\begin{aligned} \phi_3(k) &= \widetilde{\mathbb{P}}_- \phi_{x,t}(k) \\ &= \frac{k+i}{2\pi i} \int_{\mathbb{R}} \frac{\xi_{x,t}(\lambda)(fG)(\lambda)}{\lambda + i} \frac{d\lambda}{\lambda - (k - i0)} \\ &= \frac{k+i}{2\pi i} \int_{\mathbb{R}+ih_0} \frac{\xi_{x,t}(\lambda) f(\lambda) G\left(\lambda, \overline{\lambda}\right)}{\lambda + i} \frac{d\lambda}{k - \lambda} \\ &+ \frac{k+i}{\pi} \int_{S_{h_0}} \frac{f(\lambda)\xi_{x,t}(\lambda)\overline{\partial} G\left(\lambda, \overline{\lambda}\right)}{\lambda + i} \frac{d\alpha d\beta}{\lambda - k} \\ &= \phi_4(k) + \phi_5(k). \end{aligned} \tag{3.15}$$

The function ϕ_4 is similar to ϕ_2 given by (3.8), and hence as for ϕ_2 we have

$$\partial_x^n \partial_t^m \mathbb{H}(\phi_4) \in \mathfrak{S}_1. \tag{3.16}$$

It remains to treat

$$\phi_5(k) = \frac{k+i}{\pi} \int_{S_{h_0}} \frac{f(\lambda)\xi_{x,t}(\lambda)\overline{\partial} G\left(\lambda, \overline{\lambda}\right)}{\lambda + i} \frac{d\alpha d\beta}{\lambda - k}.$$

Since $\partial_t \xi_{x,t} = -\partial_x^3 \xi_{x,t}$ without loss of generality we can consider x-derivatives only. We define the x-derivatives of $\mathbb{H}(\phi_5)$ by the formula

$$\partial_x^j \mathbb{H}(\phi_5) = \mathbb{H}\left(\partial_x^j \phi_5\right). \tag{3.17}$$

We now show that

$$\mathbb{H}\left(\partial_x^j\phi_5\right)\in\mathfrak{S}_1 \text{ for any } j<2\left(N-2\right). \tag{3.18}$$

By Theorem 2.1 we need to demonstrate that $\left(\partial_x^j\phi_5\right)''\in L^1\left(\mathbb{C}^-\right)$. One has

$$\left(\partial_x^j\phi_5\right)''(k)=\frac{2}{\pi}\int_{S_{h_0}}\frac{\left(2i\lambda\right)^j f\left(\lambda\right)\xi_{x,t}\left(\lambda\right)\overline{\partial}G\left(\lambda,\overline{\lambda}\right)}{\left(\lambda-k\right)^3}\,d\alpha d\beta,$$

and thus we need to prove convergence of the following integral ($k=u+iv$):

$$\begin{aligned}&\int_{\mathbb{C}^-}\left|\int_{S_{h_0}}\frac{\lambda^{j-1}T_0\left(\lambda\right)\xi_{x,t}\left(\lambda\right)\overline{\partial}G\left(\lambda,\overline{\lambda}\right)}{\left(\lambda-k\right)^3}\,d\alpha d\beta\right|dudv\\&\leq\int_{S_{h_0}}\left|\lambda^j f\left(\lambda\right)\xi_{x,t}\left(\lambda\right)\right|\left(\int_{\mathbb{C}^-}\frac{dudv}{\left|\lambda-k\right|^3}\right)d\alpha d\beta:=I.\end{aligned} \tag{3.19}$$

By a direct computation

$$\begin{aligned}\int_{\mathbb{C}^-}\frac{dudv}{\left|\lambda-k\right|^3}&=\int_{\mathbb{C}^-}\frac{dudv}{\left|u-\alpha+i\left(v-\beta\right)\right|^3}=\int_{v\leq-\beta}\int_{\mathbb{R}}\frac{dudv}{\left|u+iv\right|^3}\\&=\int_{\beta}^{\infty}\int_{\mathbb{R}}\frac{dudv}{\left|u+iv\right|^3}=\int_{\beta}^{\infty}\frac{dv}{v^2}\int_{\mathbb{R}}\frac{du}{\left|u+i\right|^3}=const\ \beta^{-1}.\end{aligned} \tag{3.20}$$

Next, it follows from (3.14) that

$$\left|\overline{\partial}G\left(\lambda,\overline{\lambda}\right)\right|\leq\frac{2^{n-1}}{(n-1)!}\left\|G^{(n)}\right\|_\infty\ \beta^{n-1}. \tag{3.21}$$

Let us now estimate $\xi_{x,t}\left(\lambda\right)$. By (3.20), (3.21), and (3.6), the inequality (3.19) can be continued:

$$\begin{aligned}I&\leq const\int_{S_{h_0}}\left(|\alpha|+h_0\right)^{j-1}\xi_{x,t}(i\beta)\exp\left\{-24t\left(\sqrt{\beta}\alpha\right)^2\right\}\beta^{n-1}d\alpha d\beta\\&=const\int_0^{h_0}\xi_{x,t}(i\beta)\beta^{n-2}\left[\int_{\mathbb{R}}\left(|\alpha|+h_0\right)^{j-1}\exp\left\{-24t\left(\sqrt{\beta}\alpha\right)^2\right\}d\alpha\right]d\beta.\end{aligned}$$

Substituting $\omega=\sqrt{\beta}\alpha$ we have

$$\begin{aligned}&\int_{\mathbb{R}}\left(|\alpha|+h\right)^{j-1}\exp\left\{-24t\left(\sqrt{\beta}\alpha\right)^2\right\}d\alpha\\&=\frac{1}{\sqrt{\beta}}\int_{\mathbb{R}}\left(\frac{|\omega|}{\sqrt{\beta}}+h_0\right)^{j-1}\exp\left\{-24t\omega^2\right\}d\omega.\end{aligned}$$

This integral is clearly convergent since

$$\begin{aligned}&\frac{1}{\sqrt{\beta}}\int_{\mathbb{R}}\left(\frac{|\omega|}{\sqrt{\beta}}\right)^{j-1}\exp\left\{-24t\omega^2\right\}d\omega\\&=\beta^{-\frac{j}{2}}\int_{\mathbb{R}}|\omega|^{j-1}\exp\left\{-24t\omega^2\right\}d\omega<\infty.\end{aligned}$$

But $\xi_{x,t}(i\beta) = \exp\{8\beta^3 t - 2\beta x)\}$, and we finally have

$$I \leq const \int_0^{h_0} \xi_{x,t}(i\beta)\beta^{-\frac{j}{2}}\beta^{n-2}d\beta$$
$$\leq const \int_0^{h_0} \beta^{-\frac{j}{2}}\beta^{n-2}d\beta = const \int_0^{h_0} \beta^{-\frac{j}{2}+n-2}d\beta.$$

The last integral converges if $-j/2 + n - 2 > -1$ or

$$j < 2\,(n-1) \leq 2\,(N-2)\,,$$

which proves (3.18). It follows form (3.5) and (3.15) that

$$\widetilde{\mathbb{P}}_- \varphi^b_{x,t} = \phi^b_1 + \phi_2 + \phi_4 + \phi_5.$$

The latter combined with (3.18), (3.7), (3.9), and (3.16) immediately yields

$$\left\|\partial^n_x \partial^m_t \left[\mathbb{H}\left(\varphi^b_{x,t}\right) - \mathbb{H}\left(\varphi_{x,t}\right)\right]\right\|_{\mathfrak{S}_1} \to 0, \quad b \to \infty, \tag{3.22}$$

for any n, m subject to

$$n + 3m = j < 2\,(N-2)\,.$$

It remains to show that

$$u_b(x,t) \to u(x,t) = -2\partial^2_x \log\det\left(1 + \mathbb{H}(\varphi_{x,t})\right), \ b \to \infty,$$

and $u\,(x,t)$ solves (1.1). As in [11], [20] we rewrite $u = u_b + \Delta u_b$. For Δu_b we have

$$\Delta u_b\,(x,t) = -2\partial^2_x \log\det\left(I - \left\{\left(I + \mathbb{H}(\varphi_{x,t})\right)^{-1}\left[\mathbb{H}(\varphi_{x,t}) - \mathbb{H}(\varphi^b_{x,t})\right]\right\}\right) \tag{3.23}$$

where $\left(I + \mathbb{H}(\varphi_{x,t})\right)^{-1}$ is bounded [11]. By the well-known differentiation formula

$$\left(\log\det\left(1 + A\right)\right)' = \operatorname{tr}\left(1 + A\right)^{-1} A'$$

it follows from (3.22) and (3.23) that

$$\partial^n_x \Delta u_b \to 0 \ (n = 1, 2, 3), \quad \partial_t \Delta u_b \to 0, \ b \to \infty,$$

and therefore

$$\begin{aligned} &\partial_t u - 6u\partial_x u + \partial^3_x u \\ &= \partial_t \Delta u_b + 3\partial_x\left[\left(\Delta u_b - 2u\right)\Delta u_b\right] + \partial^3_x \Delta u_b \to 0, \ b \to \infty. \end{aligned} \tag{3.24}$$

Thus $u(x,t)$ solves (1.1) for all x and $t > 0$ and Theorem 1.1 is proven.

4. Discussion

We conclude our short note with some comments.

1. Theorem 1.1 says if $q\,(x)$ is subject to conditions (1.6)-(1.7), then (1.1) is globally well-posed in the following strong sense: classical solutions $u_n\,(x,t)$ with compactly supported initial data $q_n\,(x)$ converge to a classical solution $u\,(x,t)$ uniformly on any compact x-domain for any $t > 0$ and independently of the choice of $q_n\,(x)$ approximating $q\,(x)$. Note that [6] the condition

$$\operatorname*{Sup}_{|I|=1} \int_I \max\left(-q\,(x)\,, 0\right)\ dx < \infty \ \text{(essential boundedness from below)} \tag{4.1}$$

is sufficient for the condition (1.6) to hold and is also necessary if $q \leq 0$. Therefore, any q subject to (1.6)-(1.7) is essentially bounded from below, decays sufficiently fast at $+\infty$ but is arbitrary otherwise. Thus we don't assume any kind of pattern of behavior of $q\,(x)$ at $-\infty$.

2. We note that the problems of the well-posedness and related regularity of (1.1) have been extensively studied since about the same time when the IST was first introduced. The literature on the subject is truly enormous, and we make no attempt to give a comprehensive review here. Besides the already discussed [2], we only mention a few relevant papers where much more literature on the subject can be found. In [16] the existence and uniqueness of a weak solution is proven for $L^2(\mathbb{R})$ data subject to the additional condition

$$\int^{\infty} x^N q(x)^2\, dx < \infty, \quad N > 3/2. \tag{4.2}$$

In [9] the latter condition is improved to $N > 3/4$. In the famous [1] the global well-posedness is proven without the extra condition (4.2). This paper drew much attention. In particular, its results were generalized to singular initial data from the Sobolev space $H^s(\mathbb{R})$ with negative index s. In [13] the well-posedness result was extended locally in time to $s > -3/4$ and then globally in time in [3]. In [15] and [7] the limiting case $s = -3/4$ was finally included. Moreover, it was shown in [14] that $s = -3/4$ is in a certain sense optimal. Note however that $s = -3/4$ is the threshold for the harmonic analytical methods used in above mentioned papers. It was shown in [12] and [11] that IST techniques can push well-posedness to some classes of singular functions from $H^{-1}(\mathbb{R})$.

3. Most of the papers discussed in item 2 also provide some norm estimates for the solution $u(x,t)$. For this reason the referee posed the question: "Does one have any control of the 'growth' in time of the norms of the solutions?" We believe that our explicit formula (1.8) for $u(x,t)$ should yield new types of norm estimates. However, at the moment we don't know even the basic form of trace formulas for potentials subject to our conditions (1.6)-(1.7). The referee also asked if "for a solution $u(x;t)$ corresponding to a data (potential) satisfying the hypothesis (1.6)-(1.7) can one say anything about the decay property of $u(x;1)$ for $x > 0$?" It is yet another good question of practical importance. Our explicit formula (1.8) is not convenient for subtle asymptotic analysis of $u(x,t)$. Methods based upon the Riemann-Hilbert problem have been proven best for such analysis. The main issue is that these methods break down in a serious way, and it is far from being clear how to modify them. We note that the basic theory guarantees only that the spectrum of $\mathbb{L}_{u(x,t)}$ is independent of t, which is insufficient to make meaningful conclusions about asymptotic behavior of $u(x,t)$ even for fixed t.

4. The negative spectrum of $\mathbb{L}_q$ has multiplicity one but could be of any type (including absolutely continuous (a.c.)), and the positive spectrum has a.c. component filling $\mathbb{R}_+$ but need not be uniform (however no embedded bound states). Thus Theorem 1.1 says that the IST for the KdV equation works smoothly without boundary condition at $-\infty$, which is a very strong manifestation of spatial anisotropy of the KdV equation. Our [21] appears to be the first rigorous paper to this effect. We however still needed some extra (technical) assumptions which were further relaxed in [20] and [19], culminating in [11], where techniques of Hankel operators allowed us to get rid of inessential conditions.

5. The bounded invertibility of $I + \mathbb{H}(\varphi_{x,t})$ for all x and positive t is the reason why a blow-up solution doesn't develop over time. It is one of the main results of our [11].

6. If $q(x) = O(x^{-\infty})$, $x \to \infty$, then the solution is (infinitely) smooth. A precise description of smoothness in the scale $q(x) = O\left(e^{-Cx^{\delta}}\right)$, $x \to \infty$, is given in [19]. We proved that if $\delta > 1/2$, then $u(x,t)$ is meromorphic in x on the entire complex plane (with no real poles). If $\delta = 1/2$, then $u(x,t)$ is meromorphic in x in a strip around the real line whose width is increasing as $\sqrt{t}$. If $0 < \delta < 1/2$, then $u(x,t)$ is Gevrey smooth. In all these cases $u(x,t)$ is smooth in t. It is worth mentioning that the smoothing effect of the KdV flow is so strong that even strong (non-integrable) singularities instantaneously disappear [10].

7. As we have already mentioned the condition (1.6) is optimal, but the condition (1.7) is not. The bound on N in (1.7) can be lowered to 7/2. It can be achieved by a different-from-(3.10) representation for R_0, which follows from our [22]. The main loss of accuracy however comes from the estimate (3.19), where the oscillatory nature of the integral was lost. In fact, the integral on the left hand side of (3.19) admits a sharp estimate based upon the steepest descent approximation significantly lowering the bound on N. We also expect some improvements related to pseudoanalytic continuations. The very interesting paper [5] contains a number of if-and-only-if statements linking smoothness of an analytic in $\mathbb{C}^-$ function and the rate of decay of the $\overline{\partial}$ derivative of its pseudoanalytic continuation into $\mathbb{C}^+$. We hope that all this will result in optimal statements relating decay of $q(x)$ at $-\infty$ and smoothness of $u(x,t)$. We will return to it elsewhere.

8. We emphasize the importance of the analytic split (1.3) in our consideration. It allows us to effectively separate the influence of the behavior of initial data at $-\infty$ from that at $+\infty$, which effect the solution in profoundly different ways.

9. The formula (1.8) can also be written as

$$u(x,t) = 2\,\mathrm{tr}\left\{(I + \mathbb{H}(\varphi_{x,t}))^{-2}\left[(\partial_x \mathbb{H}(\varphi_{x,t}))^2 - \partial_x^2 \mathbb{H}(\varphi_{x,t}) - \mathbb{H}(\varphi_{x,t})\partial_x^2 \mathbb{H}(\varphi_{x,t})\right]\right\}.$$

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