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FUGLEDE-PUTNAM THEOREM FOR LOCALLY MEASURABLE OPERATORS

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Abstract. We extend the Fuglede-Putnam theorem from the algebra $B(H)$ of all bounded operators on the Hilbert space H to the algebra of all locally measurable operators affiliated with a von Neumann algebra.

1. Introduction

The (first part of the) following problem was suggested by von Neumann (see pp. 60-61, Appendix 3 in [8]).

Problem 1. *Let $a, b, c \in B(H)$. If a is normal and if $ac = ca$, does it follow that $a^*c = ca^*$? More generally, if a and b are normal and if $ac = cb$, does it follow that $a^*c = cb^*$?*

If the operators a and c belong to a finite factor $\mathcal{M}$, then the first part of the problem was resolved (in the affirmative) by von Neumann himself. In full generality, a problem was resolved by Fuglede [4].

Furthermore, von Neumann mentioned that a "formal" analogue of Problem 1 for unbounded operators can be *non-rigorously* answered in the negative due to the fact that a product of 2 unbounded operators does not always exists. A partial affirmative answer was given by Putnam (see Theorem 1.6.2 in [9]). He proved that if $cb \subset ac$, then $cb^* \subset ac^*$ provided that c is *bounded*.

In what follows, we propose a rigorous analogue of Problem 1 for unbounded operators affiliated with a von Neumann algebra $\mathcal{M}$. We start with a proper framework.

The set of all operators affiliated to a von Neumann algebra $\mathcal{M}$ does not necessarily form an algebra. At the same time, the class of unital $*$-algebras[1] which consist of operators affiliated with $\mathcal{M}$ is vast. In particular, it contains all algebras of measurable operators [12] and those of τ-measurable operators [7].

According to [15], in this class, there is a unique maximal element called $LS(\mathcal{M})$. We call $LS(\mathcal{M})$ the algebra of all locally measurable operators affiliated with $\mathcal{M}$. An equivalent constructive definition of $LS(\mathcal{M})$ is given in Section 2.

We now properly restate Problem 1 for unbounded operators affiliated with $\mathcal{M}$.

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[1]The operations in these algebras are strong sum, strong product, the scalar multiplication and the usual adjoint of operators. For precise definitions, see Section 2.

Problem 2. *Let $\mathcal{M}$ be a von Neumann algebra and let $a, b, c \in LS(\mathcal{M})$. If a and b are normal and if $ac = cb$, does it follow that $a^*c = cb^*$?*

Theorem 5 in [2] delivers the positive answer to Problem 2 for the case when a, b and c are measurable operators affiliated with a von Neumann algebra $\mathcal{M}$ of type I (see also [1]). In the case of an arbitrary finite von Neumann algebra $\mathcal{M}$, Problem 2 is resolved in the affirmative in [5] (see Corollary 3.6 there).

We answer Problem 2 in the affirmative in full generality. Our methods are stronger than those of [1], [2], [4], [5], [9] and are of independent interest.

The following theorem is the main result of the paper.

Theorem 3. *Let $\mathcal{M}$ be an arbitrary von Neumann algebra and let a, b, c be locally measurable operators affiliated with $\mathcal{M}$. If a and b are normal and if $ac = cb$, then $a^*c = cb^*$.*

The corollary below extends the classical spectral theorem for normal operator (see e.g. [10, Ch. 13, Theorem 13.33]) to the setting of locally measurable operators.

Corollary 4. *Let $\mathcal{M}$ be an arbitrary von Neumann algebra and let a, b be locally measurable operators affiliated with $\mathcal{M}$. If a is normal and $ab = ba$, then $eb = be$ for every spectral projection e of the operator a. If a and b are normal, then the following conditions are equivalent:*

(a) *$ab = ba$;*
(b) *$ef = fe$ for every spectral projection e of the operator a and for every spectral projection f of the operator b;*
(c) *$\phi(a)\psi(b) = \psi(b)\phi(a)$ for every Borel complex function ϕ and ψ on $\mathbb{C}$, which are bounded on compact subsets.*

2. Preliminaries

Let H be a Hilbert space, let $B(H)$ be the $*$-algebra of all bounded linear operators on H, and let $\mathbf{1}$ be the identity operator on H. Given a von Neumann algebra $\mathcal{M}$ acting on H, denote by $\mathcal{Z}(\mathcal{M})$ the centre of $\mathcal{M}$ and by $\mathcal{P}(\mathcal{M}) = \{p \in \mathcal{M} : p = p^2 = p^*\}$ the lattice of all projections in $\mathcal{M}$. Let $\mathcal{P}_{fin}(\mathcal{M})$ be the set of all finite projections in $\mathcal{M}$.

A linear operator $a : \mathfrak{D}(a) \to H$, where the domain $\mathfrak{D}(a)$ of a is a linear subspace of H, is said to be *affiliated* with $\mathcal{M}$ if $ba \subseteq ab$ for all b from the commutant $\mathcal{M}'$ of algebra $\mathcal{M}$.

A densely-defined closed linear operator a (possibly unbounded) affiliated with $\mathcal{M}$ is said to be *measurable* with respect to $\mathcal{M}$ if there exists a sequence $\{p_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathcal{M})$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathfrak{D}(a)$ and $p_n^{\perp} = \mathbf{1} - p_n \in \mathcal{P}_{fin}(\mathcal{M})$ for every $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers. Let us denote by $S(\mathcal{M})$ the set of all measurable operators.

Let $a, b \in S(\mathcal{M})$. It is well known that $a + b$, ab and a^* are densely-defined and preclosed operators. Moreover, the closures $\overline{a+b}$ (strong sum), $\overline{ab}$ (strong product) and a^* are also measurable, and equipped with these operations $S(\mathcal{M})$ is a unital $*$-algebra over the field $\mathbb{C}$ of complex numbers [12]. It is clear that $\mathcal{M}$ is a $*$-subalgebra of $S(\mathcal{M})$.

A densely-defined linear operator a affiliated with $\mathcal{M}$ is called *locally measurable* with respect to $\mathcal{M}$ if there is a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in $\mathcal{M}$ such that $z_n \uparrow \mathbf{1}$, $z_n(H) \subset \mathfrak{D}(a)$ and $az_n \in S(\mathcal{M})$ for all $n \in \mathbb{N}$.

The set $LS(\mathcal{M})$ of all locally measurable operators is a unital $*$-algebra over the field $\mathbb{C}$ with respect to the same algebraic operations as in $S(\mathcal{M})$ [14], and $S(\mathcal{M})$ is a $*$-subalgebra of $LS(\mathcal{M})$. It is clear that if $\mathcal{M}$ is finite, the algebras $S(\mathcal{M})$ and $LS(\mathcal{M})$ coincide. If von Neumann algebra $\mathcal{M}$ is of type III and $\dim(\mathcal{Z}(\mathcal{M})) = \infty$, then $S(\mathcal{M}) = \mathcal{M}$, but $LS(\mathcal{M}) \neq \mathcal{M}$.

For every subset $E \subset LS(\mathcal{M})$, the sets of all self-adjoint (resp., positive) operators in E will be denoted by E_h (resp. E_+). The partial order in $LS(\mathcal{M})$ is defined by its cone $LS_+(\mathcal{M})$ and is denoted by $\leq$.

Let a be a closed operator with dense domain $\mathfrak{D}(a)$ in H and let $a = u|a|$ be the polar decomposition of the operator a, where $|a| = (a^*a)^{\frac{1}{2}}$ and u is a partial isometry in $B(H)$ such that u^*u (respectively, uu^*) is the right (left) support $r(a)$ (respectively, $l(a)$) of a. It is known that $a = |a^*|u$ and $a \in LS(\mathcal{M})$ (respectively, $a \in S(\mathcal{M})$) if and only if $|a| \in LS(\mathcal{M})$ (respectively, $|a| \in S(\mathcal{M})$) and $u \in \mathcal{M}$ [6, §§2.2, 2.3]. If a is a self-adjoint operator affiliated with $\mathcal{M}$, then the spectral family of projections $e_\lambda(a) = e_{(-\infty,\lambda]}(a)$, $\lambda \in \mathbb{R}$, for a belongs to $\mathcal{M}$ [6, §2.1]. A locally measurable operator a is measurable if and only if $e_\lambda^\perp(|a|) \in \mathcal{P}_{fin}(\mathcal{M})$ for some $\lambda > 0$ [6, § 2.2].

In what follows, we use the notation $n(a) = \mathbf{1} - r(a)$ for the projection onto the kernel of the operator a.

Assume now that $\mathcal{M}$ is a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. A densely-defined closed linear operator a affiliated with $\mathcal{M}$ is called τ-measurable if for each $\varepsilon > 0$ there exists $e \in \mathcal{P}(\mathcal{M})$ with $\tau(e^\perp) \leq \varepsilon$ such that $e(H) \subset \mathfrak{D}(a)$. Let us denote by $S(\mathcal{M},\tau)$ the set of all τ-measurable operators. It is well known [7] that $S(\mathcal{M},\tau)$ is a $*$-subalgebra of $S(\mathcal{M})$ and $\mathcal{M} \subset S(\mathcal{M},\tau)$. It is clear that if $\mathcal{M}$ is a semifinite factor, the algebras $S(\mathcal{M},\tau)$ and $S(\mathcal{M})$ coincide. Note also that for every $a \in S(\mathcal{M},\tau)$ there exists $\lambda > 0$ such that $\tau(e_\lambda^\perp(|a|)) < \infty$ (see [7] and [6, §2.6]).

Measure topology is defined in $S(\mathcal{M},\tau)$ by the family $V(\varepsilon,\delta)$, $\varepsilon > 0, \delta > 0$, of neighborhoods of zero:

$$V(\varepsilon,\delta) = \{a \in S(\mathcal{M},\tau) : \|ae\|_{\mathcal{M}} \leq \delta \ \text{ for some } e \in \mathcal{P}(\mathcal{M}) \text{ with } \tau(e^\perp) \leq \varepsilon\}.$$

Convergence of the sequence $\{a_n\} \subset S(\mathcal{M},\tau)$ in measure topology is called *convergence in measure*. When equipped with measure topology, $S(\mathcal{M},\tau)$ is a complete metrizable topological $*$-algebra (see [7]). For basic properties of the measure topology, see [7]. We remark only that $e_n \to 0$ in measure, $e_n \in \mathcal{P}(\mathcal{M})$, if and only if $\tau(e_n) \to 0$.

Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful normal semifinite trace τ. We set

$$(L_1 \cap L_\infty)(\mathcal{M},\tau) = \Big\{x \in \mathcal{M} : \ \tau(|x|) < \infty\Big\}.$$

The following property is standard.

Property 5. *Let $\mathcal{M}$ be a semifinite von Neumann algebra and let τ be a faithful normal semifinite trace on $\mathcal{M}$. If $x, y \in (L_1 \cap L_\infty)(\mathcal{M},\tau)$, then $\tau(xy) = \tau(yx)$.*

3. The Fuglede-Putnam theorem in $S(\mathcal{M},\tau)$

The proof of Theorem 3 in full generality is based on its special case for the $*$-algebra $S(\mathcal{M},\tau)$.

Theorem 6. *Let $\mathcal{M}$ be a semifinite von Neumann algebra and let τ be a faithful normal semifinite trace on $\mathcal{M}$. Let $a, b, c \in S(\mathcal{M}, \tau)$, and let a and b be normal. If $ac = cb$, then $a^*c = cb^*$.*

Our strategy for proving Theorem 6 relies on a number of auxiliary lemmas. Whereas some of them look similar to those used in [5], the lemmas below appear to be stronger than their counterparts from [5].

Lemma 7. *If $a \in \mathcal{M}$ is normal, $p \in \mathcal{P}(\mathcal{M})$ and $\tau(p) < \infty$ with $ap = pap$, then $ap = pa$.*

Proof. Denote, for brevity,

$$a_1 = pap, \quad a_2 = pa(\mathbf{1} - p).$$

Due to the normality of a and using the equality $ap = pap$, we have

$$a_1^* a_1 = (ap)^*(ap) = pa^*ap = paa^*p = (pa)(pa)^* = (a_1 + a_2)(a_1 + a_2)^* = a_1a_1^* + a_2a_2^*.$$

Since $\tau(p) < \infty$, it follows that $a_1, a_2 \in (L_1 \cap L_\infty)(\mathcal{M}, \tau)$. Taking the trace and using Property 5, we conclude that $\tau(a_2a_2^*) = 0$. Since τ is faithful, it follows that $a_2 = 0$. This completes the proof. □

Lemma 8. *Let $a, b \in \mathcal{M}$ be normal, $c \in S_+(\mathcal{M}, \tau)$ and $ac = cb$. Let $\lambda > 0$ be such that $\tau(e_{(\lambda,+\infty)}(c)) < \infty$. Setting $p_1 = e_{[0,\lambda]}(c)$ and $p_2 = e_{(\lambda,+\infty)}(c)$, we obtain*

$$p_i a = ap_i, \ p_i b = bp_i, \ i = 1, 2.$$

Proof. Set $a_{ij} = p_i a p_j$ and $b_{ij} = p_i b p_j$ for $i, j = 1, 2$.

Step 1. We claim that

$$\tau(a_{12}^* a_{12}) = \tau(a_{21}^* a_{21}), \quad \tau(b_{12}^* b_{12}) = \tau(b_{21}^* b_{21}).$$

Indeed, using the equality $a^*a = aa^*$ and Property 5, we have

$$\tau(a_{12}^* a_{12}) = \tau(p_2 a^* p_1 a p_2) = \tau(p_2 a^* a p_2) - \tau(p_2 a^* p_2 a p_2)$$
$$= \tau(p_2 a a^* p_2) - \tau(p_2 a p_2 a^* p_2) = \tau(p_2 a p_1 a^* p_2) = \tau(a_{21} a_{21}^*) = \tau(a_{21}^* a_{21}).$$

The proof of the second equality in the claim is identical.

Step 2. We claim that

$$\tau(a_{12}^* a_{12}) \le \tau(b_{12}^* b_{12})$$

and

$$\tau(b_{21}^* b_{21}) \le \tau(a_{21}^* a_{21}).$$

Since $p_2 c^2 p_2 \ge \lambda^2 p_2$, it follows that

$$a_{12} a_{12}^* = p_1 a \cdot p_2 \cdot a^* p_1 \le \lambda^{-2} \cdot p_1 a \cdot p_2 c^2 p_2 \cdot a^* p_1 = \lambda^{-2} (a_{12} c)(a_{12} c)^*.$$

By assumption,

$$a_{12} c = p_1 a p_2 \cdot c = p_1 \cdot ac \cdot p_2 = p_1 \cdot cb \cdot p_2 = c \cdot p_1 b p_2 = c b_{12}. \tag{1}$$

Thus,

$$a_{12} a_{12}^* \le \lambda^{-2} (c b_{12})(c b_{12})^*. \tag{2}$$

Since $cp_1 \in \mathcal{M}$ and since $b_{12} \in (L_1 \cap L_\infty)(\mathcal{M}, \tau)$, it follows that

$$c b_{12} = c p_1 \cdot b_{12} \in (L_1 \cap L_\infty)(\mathcal{M}, \tau).$$

Hence (see Property 5),

$$\tau(a_{12}^*a_{12}) = \tau(a_{12}a_{12}^*) \overset{(2)}{\le} \lambda^{-2}\tau((cb_{12})(cb_{12})^*) = \lambda^{-2}\tau((cb_{12})^*(cb_{12})) < \infty.$$

Using now the inequality $p_1c^2p_1 \le \lambda^2 p_1$, we have that

$$(cb_{12})^*(cb_{12}) = b_{12}^*c^2b_{12} = b_{12}^* \cdot p_1c^2p_1 \cdot b_{12} \le \lambda^2 \cdot b_{12}^*p_1b_{12} = \lambda^2 b_{12}^*b_{12}$$

and

$$\tau(a_{12}^*a_{12}) \le \tau(b_{12}^*b_{12}). \tag{3}$$

Let $a' = b^*$ and $b' = a^*$. Taking the adjoints in the equality $ac = cb$, we obtain $a'c = cb'$. In addition

$$a'_{12} \overset{def}{=} p_1a'p_2 = b_{21}^*, \quad b'_{12} \overset{def}{=} p_1b'p_2 = a_{21}^*.$$

Applying (3) to the triple (a', b', c), we obtain

$$\tau(b_{21}^*b_{21}) = \tau(b_{21}b_{21}^*) = \tau((a'_{12})^*a'_{12}) \le \tau((b'_{12})^*b'_{12}) = \tau(a_{21}a_{21}^*) = \tau(a_{21}^*a_{21}).$$

This proves the claim.

Step 3. Using Steps 1, 2, we obtain

$$\tau(a_{12}^*a_{12}) \le \tau(b_{12}^*b_{12}) = \tau(b_{21}^*b_{21}) \le \tau(a_{21}^*a_{21}) = \tau(a_{12}^*a_{12}).$$

Thus,

$$\tau(a_{12}^*a_{12}) = \tau(a_{21}^*a_{21}) = \tau(b_{12}^*b_{12}) = \tau(b_{21}^*b_{21}).$$

Step 4. We claim that $ap_2 = p_2a$ and $bp_2 = p_2b$.

By (1), we have

$$a_{12}c = cb_{12}.$$

Now, using Property 5, we obtain

$$\tau((a_{12}c)(a_{12}c)^*) = \tau((cb_{12})(cb_{12})^*) = \tau((cb_{12})^*(cb_{12})).$$

The definition of p_1 now yields

$$(cb_{12})^*(cb_{12}) = b_{12}^*c^2b_{12} = b_{12}^* \cdot p_1c^2p_1 \cdot b_{12} \le \lambda^2 b_{12}^* \cdot p_1 \cdot b_{12} = \lambda^2 b_{12}^*b_{12}.$$

It follows from Step 3 that

$$\tau((a_{12}c)(a_{12}c)^*) \le \lambda^2\tau(a_{12}^*a_{12}).$$

In other words,

$$\tau(a_{12} \cdot p_2c^2p_2 \cdot a_{12}^*) = \tau((a_{12}c)(a_{12}c)^*) \le \lambda^2\tau(a_{12}a_{12}^*) = \tau(a_{12} \cdot \lambda^2p_2 \cdot a_{12}^*).$$

Hence,

$$\tau(a_{12} \cdot p_2(c^2 - \lambda^2\mathbf{1})p_2 \cdot a_{12}^*) \le 0.$$

Since τ is faithful and since

$$a_{12} \cdot p_2(c^2 - \lambda^2\mathbf{1})p_2 \cdot a_{12}^* \ge 0,$$

it follows that

$$a_{12} \cdot p_2(c^2 - \lambda^2\mathbf{1})p_2 \cdot a_{12}^* = 0. \tag{4}$$

For every $\varepsilon > 0$ and $p_\varepsilon = e_{(\lambda+\varepsilon,+\infty)}(c)$ we have $c^2p_\varepsilon = p_\varepsilon c^2p_\varepsilon \ge (\lambda+\varepsilon)^2p_\varepsilon$. Therefore,

$$c^2p_2 \ge \lambda^2p_2 + \varepsilon^2p_\varepsilon, \quad p_2(c^2 - \lambda^2\mathbf{1})p_2 \ge \epsilon^2p_\varepsilon.$$

We now infer from (4) that

$$a_{12} \cdot p_\varepsilon \cdot a_{12}^* = 0.$$

Since $p_\varepsilon \to p_2$ in measure as $\epsilon \to 0$, we obtain $a_{12}a_{12}^* = 0$. Thus, $a_{12} = 0$ and

$$ap_2 = p_1 ap_2 + (\mathbf{1} - p_1)ap_2 = (\mathbf{1} - p_1)ap_2 = p_2 ap_2.$$

Hence, we infer from Lemma 7 that $ap_2 = p_2 a$. Similarly, $bp_2 = p_2 b$. It follows immediately that

$$ap_1 = a - ap_2 = a - p_2 a = p_1 a, \quad bp_1 = b - bp_2 = b - p_2 b = p_1 b.$$

□

Lemma 9. *Let $a, b \in \mathcal{M}$ be normal, $c \in S_+(\mathcal{M}, \tau)$ and $ac = cb$. Then $a^*c = cb^*$.*

Proof. The assumption $c \in S_+(\mathcal{M}, \tau)$ guarantees that there exists $\lambda > 0$ such that $\tau(\mathbf{1} - e_\lambda(c)) < \infty$. Set $p_2 = e_{(\lambda,+\infty)}(c)$, $p_1 = (\mathbf{1} - p_2)$ and $a_j = ap_j$, $b_j = bp_j$, $c_j = cp_j$, $j = 1, 2$. By Lemma 8, the operators a and b commute with p_j; in particular, a_j and b_j are normal $j = 1, 2$. By the same lemma, the operator a commutes with projections $(\mathbf{1} - e_\nu(c))$ for all $\nu \geq \lambda$. Since finite linear combinations of projections $(\mathbf{1} - e_\nu(c))$, $\nu \geq \lambda$, converge to operator c_2 in the measure topology and since multiplication in $S(\mathcal{M}, \tau)$ is continuous in that topology, it follows that

$$c_2 a = ac_2 \text{ and, similarly, } c_2 b = bc_2. \tag{5}$$

Appealing now to Lemma 8, we obtain

$$c_2 a = ac_2 = ac \cdot p_2 = cb \cdot p_2 = c \cdot bp_2 \overset{L.8}{=} c \cdot p_2 b = cp_2 \cdot b = c_2 b. \tag{6}$$

Combining (6) and (5) now yields

$$a^* c_2 = (c_2 a)^* = (c_2 b)^* = (bc_2)^* = c_2 b^*.$$

Taking (5) into account, we rewrite (6) as $ac_2 = c_2 b$. Combining this with the assumption $ac = cb$, we infer $ac_1 = c_1 b$. Taking into account that $c_1 \in \mathcal{M}$ and applying the classical Fuglede-Putnam theorem we derive that

$$a^* c_1 = c_1 b^*.$$

Thus,

$$a^* c = a^* c_1 + a^* c_2 = c_1 b^* + c_2 b^* = cb^*.$$

□

Lemma 10. *Let $a, b \in \mathcal{M}$ be normal and let $c \in S(\mathcal{M}, \tau)$ be such that $ac = cb$. If $n(c^*) \preceq n(c)$ or $n(c) \preceq n(c^*)$, then $a^*c = cb^*$.*

Proof. We only consider the first case (the second case can be reduced to the first one by considering the triple (b^*, a^*, c^*) instead of the triple (a, b, c)).

Let $c = v|c|$ be a polar decomposition of c so that $v^*v = r(c)$ and $vv^* = r(c^*)$. Let w be a partial isometry such that $w^*w = n(c^*)$ and $ww^* \leq n(c)$. Define an isometry $u = v^* + w$ (that is, $u^*u = \mathbf{1}$). It is immediate that $u^*|c| = c$ and $uc = |c|$. Thus,

$$(uau^*) \cdot |c| = ua \cdot c = u \cdot ac = u \cdot cb = uc \cdot b = |c| \cdot b.$$

Since $u^*u = \mathbf{1}$ and since a is normal, it follows that uau^* is also normal. Applying Lemma 9 to the triple $(uau^*, b, |c|)$, we obtain

$$(ua^*u^*) \cdot |c| = |c| \cdot b^*.$$

Therefore,

$$a^* c = a^* \cdot u^*|c| = u^* \cdot (ua^*u^*) \cdot |c| = u^* \cdot |c| \cdot b^* = cb^*.$$

This completes the proof. □

We now give the proof of Theorem 6 in the case of arbitrary semifinite von Neumann algebra $\mathcal{M}$ with a faithful normal semifinite trace τ.

Proof of Theorem 6. Let us suppose at first that $a, b \in \mathcal{M}$. By [11, Theorem 2.1.3] there exist central projections $z_1, z_2 \in \mathcal{Z}(\mathcal{M})$ such that

$$z_1 + z_2 = \mathbf{1}, \quad n(c^*)z_1 \preceq n(c)z_1, \ n(c)z_2 \preceq n(c^*)z_2.$$

It is immediate that

$$az_1 \cdot cz_1 = a \cdot z_1 c \cdot z_1 = a \cdot cz_1 \cdot z_1 = ac \cdot z_1^2 = cb \cdot z_1^2 = c \cdot bz_1 \cdot z_1 = c \cdot z_1 b \cdot z_1 = cz_1 \cdot bz_1,$$

$$az_2 \cdot cz_2 = a \cdot z_2 c \cdot z_2 = a \cdot cz_2 \cdot z_2 = ac \cdot z_2^2 = cb \cdot z_2^2 = c \cdot bz_2 \cdot z_2 = c \cdot z_2 b \cdot z_2 = cz_2 \cdot bz_2.$$

Clearly, $n(cz_k) = n(c)z_k$ and $n(c^* z_k) = n(c^*)z_k$, $k = 1, 2$, where the left hand side is taken in the algebra $z_k\mathcal{M}$. Applying Lemma 10 to the triples (az_1, bz_1, cz_1) and (az_2, bz_2, cz_2), we obtain

$$a^* z_1 \cdot cz_1 = cz_1 \cdot b^* z_1, \quad a^* z_2 \cdot cz_2 = cz_2 \cdot b^* z_2.$$

Summing these equalities, we obtain that $a^* c = cb^*$. This proves the assertion for the case $a, b \in \mathcal{M}$.

Now let a, b be arbitrary normal operators in $S(\mathcal{M}, \tau)$ and $ac = cb$. Let q_n (respectively, r_n) be the spectral projection for a (respectively, b) corresponding to the set $\{z : |z| \leq n\}$. It is clear that $\{q_n\}$ and $\{r_n\}$ are increasing sequences of projections with $\sup_{n\geq 1} q_n = \mathbf{1}$ and $\sup_{n\geq 1} r_n = \mathbf{1}$. In addition (see e.g. [10, Ch. 13, Theorems 13.24, 13.33]),

$$aq_n = q_n a, \quad a^* q_n = q_n a^*, \quad br_n = r_n b, \quad b^* r_n = r_n b^*, \quad n \in \mathbb{N}.$$

Multiplying the equality $ac = cb$ by q_n on the left and by r_n on the right, we obtain

$$(q_n a) \cdot (q_n c r_n) = (q_n c r_n) \cdot (r_n b), \quad n \in \mathbb{N}.$$

Clearly, $q_n a \in \mathcal{M}$ and $r_n b \in \mathcal{M}$ are normal operators for every $n \in \mathbb{N}$. It follows from the preceding paragraph that

$$q_n \cdot a^* c \cdot r_n = (q_n a)^* \cdot (q_n c r_n) = (q_n c r_n) \cdot (r_n b)^* = q_n \cdot cb^* \cdot r_n.$$

Thus,

$$q_n(a^* c - cb^*)r_n = 0, \quad n \in \mathbb{N}.$$

Since $a, b \in S(\mathcal{M}, \tau)$, it follows that $\tau(1 - q_n) \to 0$, $\tau(1 - r_n) \to 0$ as $n \to \infty$. Thus $q_n \to \mathbf{1}$, $r_n \to \mathbf{1}$ in measure. Therefore, for every $x \in S(\mathcal{M}, \tau)$, we have $q_n x r_n \to x$ in measure as $n \to \infty$. Taking $x = a^* c - cb^*$, we complete the proof. □

4. The Fuglede-Putnam theorem in the $*$-algebra $LS(\mathcal{M})$

Lemma 11 below is the key tool used to extend the Fuglede-Putnam theorem from $\tau-$measurable operators to measurable ones.

Lemma 11. *Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $q \in \mathcal{P}(\mathcal{M})$ be a finite projection. Then there exists partition of unity $\{z_i\}_{i\in I} \subset \mathcal{P}(\mathcal{Z}(\mathcal{M}))$, such that every von Neumann algebra $z_i\mathcal{M}$, $i \in I$, has a faithful normal semifinite trace τ_i with $\tau_i(z_i q) < \infty$.*

Proof. It is well known that a commutative von Neumann algebra $\mathcal{Z}(\mathcal{M})$ is $*$-isomorphic to the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$ of all essentially bounded measurable complex-valued functions defined on a measure space (Ω, Σ, μ) with the measure μ satisfying the direct sum property (we identify functions that are equal almost everywhere) (see e.g. [3, Ch. 7, §7.3]). The direct sum property of a measure μ means that the Boolean algebra of all projections of the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$ is order complete, and for any non-zero $p \in \mathcal{P}(\mathcal{M})$ there exists a non-zero projection $r \leq p$ such that $\mu(r) < \infty$. The direct sum property of a measure μ is equivalent to the fact that the functional $\nu(f) := \int_\Omega f\, d\mu$ is a semifinite normal faithful trace on the algebra $L^\infty(\Omega, \Sigma, \mu)$. Therefore there exists partition of unity $\{r_j\}_{j \in j} \subset \mathcal{P}(L^\infty(\Omega, \Sigma, \mu))$, such that $\nu_j(f) = \nu(r_j f)$ is faithful normal finite trace on $r_j L^\infty(\Omega, \Sigma, \mu)$ for every $j \in J$.

Let φ be a $*$-isomorphism from $\mathcal{Z}(\mathcal{M})$ onto the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$. Denote by $L^+(\Omega, \Sigma, m)$ the set of all measurable real-valued functions defined on (Ω, Σ, μ) and taking values in the extended half-line $[0, \infty]$ (functions that are equal almost everywhere are identified).

By [13, Ch. V, §2, Theorem 2.34 and Proposition 2.35] there exists a faithful semifinite normal extended center valued trace T,

$$T \colon \mathcal{M}_+ \to L^+(\Omega, \Sigma, \mu),$$

such that $\mu(\{\omega \in \Omega : T(q)(\omega) = +\infty\}) = 0$. Thus characteristic functions $q_n = \chi_{A_n}$ corresponding to sets $A_n = \{\omega \in \Omega : n - 1 \leq T(q)(\omega) < n\}$, $n \in \mathbb{N}$, partition the unit element χ_Ω of Boolean algebra $\mathcal{P}(L^\infty(\Omega, \Sigma, \mu))$. In addition

$$T(q\varphi^{-1}(q_n)) = \varphi^{-1}(q_n)T(q) \leq nq_n$$

for all $n \in \mathbb{N}$.

It is clear that $\{z_n^j = \varphi^{-1}(r_j q_n),\ j \in J,\ n \in \mathbb{N}\}$ is a partition of unity in $\mathcal{P}(\mathcal{Z}(\mathcal{M}))$. In addition, the functional $\tau_{j,n} : z_n^j \mathcal{M}_+ \to [0, \infty]$, given by the formula

$$\tau_{j,n}(x) = \nu_j(T(x)),\ x \in z_n^j \mathcal{M}_+,$$

is a faithful normal finite trace on $z_n^j \mathcal{M}$. In particular,

$$\tau_{j,n}(z_n^j q) = \nu_j(T(z_n^j q)) \leq n\nu_j(\varphi^{-1}(q_n)r_j) \leq n\nu_j(r_j) < \infty \text{ for all } j \in J, \quad n \in \mathbb{N}.$$

Setting $i = (j, n)$ and $I = J \times \mathbb{N}$, we complete the proof. □

Lemma 12. *Let $\mathcal{M}$ be a von Neumann algebra and let $\{z_i\}_{i \in I} \subset \mathcal{Z}(\mathcal{M})$ be a partition of unity. If $x \in LS(\mathcal{M})$ is such that $xz_i = 0$ for every $i \in I$, then $x = 0$.*

Proof. Since $z_i \leq n(x)$ for all $i \in I$, it follows that $\mathbf{1} = \sup_{i \in I} z_i \leq n(x)$. Thus $n(x) = \mathbf{1}$, i.e. $x = 0$. □

The following lemma extends the result of Theorem 6 to the setting of measurable operators.

Lemma 13. *Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $a, b, c \in S(\mathcal{M})$. If a and b are normal and if $ac = cb$, then $a^*c = cb^*$.*

Proof. Choose n so large that projections $e_{|a|}(n, +\infty)$, $e_{|b|}(n, +\infty)$ and $e_{|c|}(n, +\infty)$ are finite. Let q be a finite projection given by the formula

$$q = e_{|a|}(n, +\infty) \vee e_{|b|}(n, +\infty) \vee e_{|c|}(n, +\infty).$$

Let $\{z_i\}_{i\in I}$ be the partition of unity constructed in Lemma 11. We have

$$az_i \cdot cz_i = cz_i \cdot bz_i, \quad i \in I.$$

It follows from Lemma 11 that, for a given $i \in I$,

$$\tau_i(e_{|a|}(n,+\infty)z_i), \tau_i(e_{|b|}(n,+\infty)z_i), \tau_i(e_{|c|}(n,+\infty)z_i) < \infty.$$

A standard argument yields

$$e_{|a|}(n,+\infty)z_i = e_{|az_i|}(n,+\infty),$$

where the right hand side is taken in the algebra $z_i\mathcal{M}$. It follows that az_i, bz_i and cz_i are τ_i−measurable operators for every $i \in I$. Theorem 6 implies that

$$a^*z_i \cdot cz_i = cz_i \cdot b^*z_i.$$

The assertion follows now from Lemma 12. □

Lemma 14 extends the Fuglede-Putnam theorem to the setting of locally measurable operators affiliated with a semifinite von Neumann algebra $\mathcal{M}$.

Lemma 14. *Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $a, b, c \in LS(\mathcal{M})$. If a and b are normal and if $ac = cb$, then $a^*c = cb^*$.*

Proof. By the (constructive) definition of the algebra $LS(\mathcal{M})$, there exist central projections $\{p_k\}_{k\geq 1}$, $\{q_l\}_{l\geq 1}$ and $\{r_m\}_{m\geq 1}$ such that $p_k \uparrow \mathbf{1}$, $q_l \uparrow \mathbf{1}$ and $r_m \uparrow \mathbf{1}$ and such that

$$ap_k, bq_l, cr_m \in S(\mathcal{M}), \quad k, l, m \geq 1.$$

Denote the triple (k, l, m) by n and set $P_n = p_k q_l r_m$. Since

$$aP_n \cdot cP_n = cP_n \cdot bP_n, \quad n \in \mathbb{N}^3,$$

it follows from Lemma 13 that

$$a^*P_n \cdot cP_n = cP_n \cdot b^*P_n, \quad n \in \mathbb{N}^3.$$

In other words (here, we let $r_0 = 0$),

$$(a^*c - cb^*)p_k q_l \cdot (r_m - r_{m-1}) = 0, \quad m \in \mathbb{N}.$$

Since $\{r_m - r_{m-1}\}_{m\geq 1}$ is a partition of unity which consists of central projections, it follows from Lemma 12 that

$$(a^*c - cb^*)p_k q_l = 0, \quad k, l \in \mathbb{N}.$$

Repeating the argument for l and, after that, for k, we complete the proof. □

The following assertion can be found in [1] (see Theorem 1 there). We provide a short proof for the convenience of the reader.

Lemma 15. *Let $\mathcal{M}$ be a purely infinite von Neumann algebra and let $a, b, c \in LS(\mathcal{M})$. If a and b are normal and if $ac = cb$, then $a^*c = cb^*$.*

Proof. Recall that $S(\mathcal{M}) = \mathcal{M}$. Choose central projections $\{p_k\}_{k\geq 1}$, $\{q_l\}_{l\geq 1}$ and $\{r_m\}_{m\geq 1}$ such that $p_k \uparrow \mathbf{1}$, $q_l \uparrow \mathbf{1}$ and $r_m \uparrow \mathbf{1}$ and such that

$$ap_k, bq_l, cr_m \in \mathcal{M}, \quad k, l, m \geq 1.$$

Denote the triple (k, l, m) by n and let $P_n = p_k q_l r_m$. We have

$$aP_n \cdot cP_n = cP_n \cdot bP_n, \quad n \in \mathbb{N}^3.$$

By the classical Fuglede-Putnam theorem, we have

$$a^* P_n \cdot c P_n = c P_n \cdot b^* P_n, \quad n \in \mathbb{N}^3.$$

The same argument as in Lemma 14 yields the assertion. □

Proof of Theorem 3. It is well known that for every von Neumann algebra $\mathcal{M}$ there exist central projections $z_1, z_2 \in \mathcal{Z}(\mathcal{M})$ such that $z_1 + z_2 = \mathbf{1}$, $\mathcal{M}z_1$ is the semifinite von Neumann algebra and $\mathcal{M}z_2$ is the purely infinite von Neumann algebra (see, for example, [11, Ch. 2, §2.2]). We have

$$a z_k \cdot c z_k = c z_k \cdot b z_k, \quad k = 1, 2.$$

Lemmas 14 and 15 imply that

$$a^* z_k \cdot c z_k = c z_k \cdot b^* z_k, \quad k = 1, 2.$$

Summing these equalities, we complete the proof. □

We need the following useful property of locally measurable operators.

Lemma 16. *Let $\mathcal{M}$ be a von Neumann algebra and let $x \in LS(\mathcal{M})$. Let $\{p_n\}_{n \geq 1} \subset \mathcal{P}(\mathcal{M})$ be such that $p_n \uparrow \mathbf{1}$. If $p_n x p_n = 0$ for every $n \geq 1$, then $x = 0$.*

Proof. Fix $m \in \mathbb{N}$. For every $n \geq m$, we have

$$p_m x p_n = p_m \cdot p_n x p_n = 0.$$

Thus, $p_n \leq \mathbf{1} - r(p_m x)$ for every $n \geq 1$. Since $p_n \uparrow \mathbf{1}$, it follows that $r(p_m x) = 0$ and, therefore, $p_m x = 0$.

Hence, $x^* p_m = 0$ for every $m \geq 1$. Thus, $p_m \leq \mathbf{1} - r(x^*)$ for every $m \geq 1$. Since $p_m \uparrow \mathbf{1}$, it follows that $r(x^*) = 0$ and, therefore, $x = 0$. □

Lemma 17. *Let $\mathcal{M}$ be a von Neumann algebra and let $a, b \in LS(\mathcal{M})$. If a is normal and if $ab = ba$, then $eb = be$ for every spectral projection e of the operator a.*

Proof. Let $b_1 = \Re(b) = \frac{b+b^*}{2}$ and $b_2 = \Im(b) = \frac{b-b^*}{2i}$. By Theorem 3 we have that $ab^* = b^* a$. Thus $ab_j = b_j a$, $j = 1, 2$. Let a Borel function ϕ be given by the formula $\phi(t) = (t+i)^{-1}$, $t \in \mathbb{R}$, and let $c_j = \phi(b_j)$, $j = 1, 2$. Since $b_j^* = b_j$ and since $|\phi(t)| \leq 1$, $t \in \mathbb{R}$, it follows from the Spectral Theorem that $c_j \in \mathcal{M}$, $j = 1, 2$. Since $ab_j = b_j a$, it follows that

$$a(b_j + i)^{-1} - (b_j + i)^{-1} a = (b_j + i)^{-1} \cdot ((b_j + i)a - a(b_j + i)) \cdot (b_j + i)^{-1} = 0,$$

that is, $ac_j = c_j a$. Theorem 13.33 in [10] yields that $ec_j = c_j e$, $j = 1, 2$, for every spectral projection e of the operator a. Thus, $eb_1 = b_1 e$ and $eb_2 = b_2 e$. Summing these equalities, we obtain $eb = be$. □

Proof of Corollary 4. (a) ⇒ (b). Lemma 17 states that $eb = be$ for every spectral projection e of the operator a. Again applying Lemma 17 to the couple (b, e), we obtain that $ef = fe$ for every spectral projection e of the operator a and for every spectral projection f of the operator b.

(b) ⇒ (c). Let q_n (respectively, r_n) be the spectral projection for a (respectively, b) corresponding to the set $D_n = \{z : |z| \leq n\}$, $n \in \mathbb{N}$. Denote $\phi_n = \phi \cdot \chi_{D_n}$ and $\psi_n = \psi \cdot \chi_{D_n}$. By the Spectral Theorem, we have

$$q_n \cdot \phi(a) = \phi(a) \cdot q_n = \phi_n(a q_n), \quad r_m \cdot \psi(b) = \psi(b) \cdot r_m = \psi_m(b r_m).$$

Bounded operators aq_n and br_m are normal, and their spectral projections commute. By the Spectral Theorem for bounded operators, these operators commute and, therefore,

$$\phi_n(aq_n) \cdot \psi_m(br_m) = \psi_m(br_m) \cdot \phi_n(aq_n).$$

Thus,

$$q_n r_m \cdot \phi(a)\psi(b) \cdot q_n r_m = q_n r_n \cdot \phi_n(aq_n)\psi_m(br_m) \cdot q_n r_m$$

$$= q_n r_m \cdot \psi_m(br_m)\phi_n(aq_n) \cdot q_n r_m = q_n r_m \cdot \psi(b)\phi(a) \cdot q_n r_m.$$

Taking into account that $r_m \uparrow \mathbf{1}$ and using Lemma 16, we obtain

$$q_n \cdot \phi(a)\psi(b) \cdot q_n = q_n \cdot \psi(b)\phi(a) \cdot q_n.$$

Again appealing to Lemma 16, we obtain (c).

Taking $\phi(z) = z$ and $\psi(z) = z$ in (c), we obtain the implication (c) $\Rightarrow$ (a). □

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