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CONGRUENCES MODULO POWERS OF 11 FOR SOME PARTITION FUNCTIONS

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Dedicated to Professor Heng Huat Chan on the occasion of his 50th birthday

ABSTRACT. Let $R_0(N)$ be the Riemann surface of the congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$. Using some properties of the field of meromorphic functions on $R_0(11)$, we confirm a conjecture of H.H. Chan and P.C. Toh [J. Number Theory 130 (2010), pp. 1898–1913] about the partition function $p(n)$. Moreover, we prove three infinite families of congruences modulo arbitrary powers of 11 for other partition functions, including 11-regular partitions and 11-core partitions.

1. INTRODUCTION

A partition of an integer n is a sequence of non-increasing positive integers whose sum equals n. Let $p(n)$ denote the number of unrestricted partitions of n. It is well known that the generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Here and throughout the paper, we use the following standard q-series notation:

$$(a;q)_\infty = \prod_{k=1}^{\infty}(1-aq^{k-1}).$$

Let $[x]$ denote the integer part of x. For $\ell \in \{5,7,11\}$, let $\delta_{\ell,j}$ be the reciprocal of 24 modulo ℓ^j, i.e., $24\delta_{\ell,j} \equiv 1 \pmod{\ell^j}$. For $n \geq 0$, it is known that

$$p(5^j n + \delta_{5,j}) \equiv 0 \pmod{5^j}, \tag{1.1}$$

$$p(7^j n + \delta_{7,j}) \equiv 0 \pmod{7^{[j/2]+1}}, \tag{1.2}$$

$$p(11^j n + \delta_{11,j}) \equiv 0 \pmod{11^j}. \tag{1.3}$$

These are known as Ramanujan congruences [24]. Congruences (1.1) and (1.2) were first proved by G.N. Watson using the modular equations of degrees 5 and 7, respectively. Using the modular equation of degree 11, A.O.L. Atkin [3] proved (1.3). Later M. Hirschhorn and D.C. Hunt [15], and F. Garvan [9] gave simple proofs of (1.1) and (1.2), respectively, without using the theory of modular functions.

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The ideas for Watson's proof of (1.1)–(1.2) and Atkin's proof of (1.3) are similar. Let

$$(1.4) \qquad L_{n,\ell} = \begin{cases} (q^\ell;q^\ell)_\infty \sum\limits_{m=0}^{\infty} p(\ell^n m + \delta_{\ell,n})q^{m+1} & \text{if } n \text{ is odd,} \\ (q;q)_\infty \sum\limits_{m=0}^{\infty} p(\ell^n m + \delta_{\ell,n})q^{m+1} & \text{if } n \text{ is even.} \end{cases}$$

One can show that $L_{n,\ell}$ are modular functions on $\Gamma_0(\ell)$ for $\ell \in \{5,7,11\}$. Therefore, we can express them using linear basis for the space of modular functions on $\Gamma_0(\ell)$. Examining the ℓ-adic orders of the coefficients will lead to (1.1)–(1.3).

Let

$$\Delta = q(q;q)_\infty^{24}, \quad E_8 = 1 + 480\sum_{n=1}^{\infty} \frac{n^7 q^n}{1-q^n}.$$

H.H. Chan and P.C. Toh [8] observed that there exist integers a_n, b_n and c_n with $(5,a_n) = (7,b_n) = (11,c_n) = 1$ such that

$$(1.5) \qquad L_{n,5} \equiv 5^n a_n \Delta \pmod{5^{n+1}},$$

$$(1.6) \qquad L_{n,7} \equiv 7^{[n/2]+1} b_n \Delta \pmod{7^{[n/2]+2}},$$

and

$$(1.7) \qquad L_{n,11} \equiv 11^n c_n \Delta E_8 \pmod{11^{n+1}}.$$

It is clear that both (1.5) and (1.6) follow immediately from Watson's work (see [17]). Chan and Toh [8] commented that "it is very likely that one can obtain a rigorous proof of (1.7) using Atkin's method given in [3]." In this paper, our first goal is to show that (1.7) indeed follows from Atkin's work [3]. So we can rewrite it as

Theorem 1. *For any integer $n \ge 1$, there exists an integer c_n with $(11,c_n) = 1$ such that*

$$L_{n,11} \equiv 11^n c_n \Delta E_8 \pmod{11^{n+1}}.$$

Motivated by Ramanujan's work [24], arithmetic properties of various types of partition functions have been studied. For example, the t-regular partitions and t-core partitions have drawn much attention. Let t be a positive integer. A partition of n is called a t-core partition if it has no hook numbers divisible by t. We denote the number of t-core partitions of n by $a_t(n)$ with the convention that $a_t(0) = 1$. The generating function of $a_t(n)$ is given by (see [10], for example)

$$\sum_{n=0}^{\infty} a_t(n) q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}.$$

A partition is called t-regular if none of its parts are divisible by t. For example, $4+3+2+1$ is a 5-regular partition of 10, but $5+3+1+1$ is not 5-regular. We denote by $b_t(n)$ the number of t-regular partitions of n and agree that $b_t(0) = 1$. It is easy to see that the generating function of $b_t(n)$ is

$$(1.8) \qquad \sum_{n=0}^{\infty} b_t(n) q^n = \frac{(q^t;q^t)_\infty}{(q;q)_\infty}.$$

If we follow the notation of Chan and Toh [8], we define $p_{[1^ct^d]}(n)$ by

$$\sum_{n=0}^{\infty} p_{[1^ct^d]}(n)q^n = \frac{1}{(q;q)_\infty^c (q^t;q^t)_\infty^d}, \quad c,d,t \in \mathbb{Z}.$$

It is then clear that in this notation, we have $a_t(n) = p_{[1^1t^{-t}]}(n)$ and $b_t(n) = p_{[1^1t^{-1}]}(n)$.

For some particular integer triples (c,d,t), arithmetic properties of $p_{[1^ct^d]}(n)$ have been extensively investigated. See [1], [5]-[8], [10], [11], [14], [16], [19]-[21], [23] and [27]-[29]. For more comprehensive reference lists about t-core partitions and t-regular partitions, we refer the reader to [27] and [28].

It should be noted that so far almost all works have concentrated on discovering congruences modulo small powers of primes for those partition functions. There are only a few works where congruences modulo arbitrary prime powers appear; see [4, 6–8, 13, 18, 21, 23, 28, 29] for example. By using Ramanujan's cubic continued fraction, H.C. Chan [6] proved that

$$p_{[1^12^1]}(3^jn+c_j) \equiv 0 \pmod{3^{2[j/2]+1}},$$

where $c_j \equiv 1/8 \pmod{3^j}$. Similarly, letting $d_j \equiv 1/8 \pmod{5^j}$, Chan and Toh [8] showed that for any integer $n \geq 0$,

$$p_{[1^12^1]}(5^jn+d_j) \equiv 0 \pmod{5^{[j/2]}}.$$

Recently, using the modular equation of fifth order, L. Wang [28] proved that for any integers $k \geq 1$ and $n \geq 0$,

$$b_5\Big(5^{2k-1}n + \frac{5^{2k}-1}{6}\Big) \equiv 0 \pmod{5^k}.$$

Wang [29] also proved that

$$p_{[1^15^1]}\Big(5^kn + \frac{3\cdot 5^k+1}{4}\Big) \equiv 0 \pmod{5^k}.$$

While congruences modulo arbitrary powers of 2, 3, 5 or 7 have appeared in the literature, we observed that after the work of Atkin [3], people seldom discover congruences modulo powers of 11 for partition functions other than $p(n)$. One of the few examples known to us is the work of B. Gordon [12], where Gordon established many congruences modulo arbitrary powers of 11 for the function $p_k(n)$ defined by

$$\sum_{n=0}^{\infty} p_k(n)q^n = (q;q)_\infty^k. \tag{1.9}$$

In view of this phenomenon, the second goal of this paper is to provide more partition congruences modulo arbitrary powers of 11. We will follow the strategy of Atkin [3] and Gordon [12] to establish those congruences for three different types of partition functions.

Theorem 2. *For any integers $n \geq 0$ and $k \geq 1$, we have*

$$a_{11}\Big(11^kn + 11^k - 5\Big) \equiv 0 \pmod{11^k}.$$

Theorem 3. *For any integers $n \geq 0$ and $k \geq 1$, we have*

$$b_{11}\Big(11^{2k-1}n + \frac{7\cdot 11^{2k-1}-5}{12}\Big) \equiv 0 \pmod{11^k}.$$

Theorem 4. *For any integers $n \geq 0$ and $k \geq 1$, we have*

$$p_{[1^1 11^1]}\Big(11^k n+\frac{11^k+1}{2}\Big) \equiv 0 \pmod{11^k}.$$

We remark here that Theorem 2 was discovered by F. Garvan [11, eq. (1.9)]. To prove Theorem 2, Garvan used Hecke operators on spaces of cusp forms, and we will give a new proof by applying Atkin's approach of U-operators on modular functions.

The method used in this paper can be applied to obtain similar results for $p_{[1^c 11^d]}(n)$ for other values of $c,d \in \mathbb{Z}$. Since the partition functions in Theorems 2–4 are more popular, we will illustrate the method by studying these examples.

2. Preliminary results

In this section, we collect some facts which are essential in proving our results. We will follow the notation of Gordon [12].

Let $\mathbb{H}$ be the upper half complex plane. Recall that the Dedekind eta function is

$$\eta(\tau)=q^{1/24}(q;q)_\infty, \quad q=e^{2\pi i\tau}, \quad \tau\in \mathbb{H}.$$

For any positive integer N, the congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ is defined as

$$\Gamma_0(N):=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\Big| a,b,c,d\in\mathbb{Z}, ad-bc=1, c\equiv 0 \ (\mathrm{mod}\ N)\right\}.$$

Let $R_0(N)$ be the Riemann surface of $\Gamma_0(N)$. Let $K_0(N)$ be the field of meromorphic functions on $R_0(N)$. It is known that $R_0(N)$ has a cusp at $\tau=i\infty$ and $q=e^{2\pi i\tau}$ is a uniformizing parameter there. If $f(\tau)\in K_0(N)$, then the Laurent expansion about $\tau=i\infty$ has the form

$$f(\tau)=\sum_{n\geq n_0} a_n q^n.$$

By abuse of notation, we also denote $f(\tau)$ by $f(q)$. For example, let

$$\phi(q)=\frac{\eta(121\tau)}{\eta(\tau)}=q^5\frac{(q^{121};q^{121})_\infty}{(q;q)_\infty}.$$

It is known that $\phi(q)\in K_0(121)$. This function will play a key role in our proofs.

We define the U-operator as

$$Uf(\tau)=\sum_{11n\geq n_0} a_{11n}q^n.$$

It is known (see [2, pp. 80–82], for example) that if $f(q)\in K_0(121)$, then $Uf(q)\in K_0(11)$.

If $f(\tau)\in K_0(11)$ and p is a point of $R_0(11)$, we use $\mathrm{ord}_p f(\tau)$ to denote the order of $f(\tau)$ at p.

Let V be the vector space of functions $g(\tau)\in K_0(11)$ which are holomorphic except possibly at 0 and ∞. Atkin [3] has constructed a basis for V. Following the notation of Gordon [12], for $k\neq 0,-1$, let $J_k(\tau)$ be the element of Atkin's basis whose order at ∞ is k. We define $J_0(\tau) = 1$ and $J_{-1}(\tau) = J_{-6}(\tau)J_5(\tau)$. In terms of the notation of Atkin, we have for $k \geq 1$,

$$J_k(\tau) = \begin{cases} g_k(\tau) & \text{if } k \equiv 0 \pmod 5, \\ g_{k+2}(\tau) & \text{if } k \equiv 4 \pmod 5, \\ g_{k+1}(\tau) & \text{otherwise,} \end{cases} \tag{2.1}$$

and $J_k(\tau) = G_k(\tau)$ for $k \leq -2$. Explicit expressions of $J_k(\tau)$ $(-6 \leq k \leq 5)$ could be found in [3, Appendix A]. For example, $J_5(\tau) = (\eta(11\tau)/\eta(\tau))^{12}$ and

$$J_1(\tau) = \frac{1}{10} \cdot \frac{1}{(q;q)_\infty^5} \Big(-\sum_{n=0}^{\infty} \Big(1 + \big(\frac{n-3}{11}\big)\Big) p_5(n) q^n + 11^2 q^{25} (q^{121}; q^{121})_\infty^5 \Big), \tag{2.2}$$

where $p_5(n)$ was defined in (1.9).

Lemma 2.1 (Cf. [12, Lemma 3])**.** *For all $k \in \mathbb{Z}$, we have*

(i) $J_{k+5}(\tau) = J_k(\tau)J_5(\tau)$,

(ii) $\{J_k(\tau) | k \in \mathbb{Z}\}$ *is a basis of* V,

(iii) $\mathrm{ord}_\infty J_k(\tau) = k$,

(iv) $\mathrm{ord}_0 J_k(\tau) = \begin{cases} -k & \text{if } k \equiv 0 \pmod 5, \\ -k-1 & \text{if } k \equiv 1, 2 \text{ or } 3 \pmod 5, \\ -k-2 & \text{if } k \equiv 4 \pmod 5, \end{cases}$

(v) *the Fourier series of $J_k(\tau)$ has integer coefficients and is of the form* $J_k(q) = q^k + \cdots$.

From [12] we know that V is mapped into itself by the linear transformation

$$T_\lambda : g(q) \to U(\phi(q)^\lambda g(q))$$

for any integer λ. Following Atkin, we write the elements of V as row vectors and let matrices act on the right. Let $C^{(\lambda)} = (c_{\mu,\nu}^{(\lambda)})$ be the matrix of T_λ with respect to the basis $\{J_k\}$ of V. We have

$$U(\phi(q)^\lambda J_\mu(q)) = \sum_{\nu \in \mathbb{Z}} c_{\mu,\nu}^{(\lambda)} J_\nu(q). \tag{2.3}$$

For any integer n, let $\pi(n)$ be the 11-adic order of n with the convention that $\pi(0) = \infty$. As shown in [12], we have

$$\pi(c_{\mu,\nu}^{(\lambda)}) \geq [(11\nu - \mu - 5\lambda + \delta)/10], \tag{2.4}$$

where $\delta = \delta(\mu, \nu)$ depends on the residues of μ and ν (mod 5) according to Table 1.

Table 1

μ \ ν	0	1	2	3	4
0	-1	8	7	6	15
1	0	9	8	2	11
2	1	10	4	3	12
3	2	6	5	4	13
4	3	7	6	5	9

From Table 1, we see that $\delta(\lambda,\mu)\geq -1$ for any λ, μ. Therefore, (2.4) implies that

$$\pi(c_{\mu,\nu}^{(\lambda)}) \geq [(11\nu-\mu-5\lambda-1)/10]. \tag{2.5}$$

By Lemma 2.1(v) and (2.3) we know that the Fourier series of $U(\phi^{\lambda}J_{\mu})$ has all coefficients divisible by 11 if and only if

$$c_{\mu,\nu}^{(\lambda)} \equiv 0 \pmod{11} \quad \text{for all } \nu. \tag{2.6}$$

We define a function $\theta(\lambda,\mu)$ as follows. If (2.6) holds we put $\theta(\lambda,\mu)=1$ and $\theta(\lambda,\mu)=0$ otherwise. From [12] we know that

$$\theta(\lambda,\mu)=\theta(\lambda-11,\mu), \quad \theta(\lambda+12,\mu-5)=\theta(\lambda,\mu). \tag{2.7}$$

This implies that $\theta(\lambda,\mu)$ is completely determined by its values in the range $0\leq \lambda\leq 10$, $0\leq\mu\leq 4$, which are listed in Table 2.

Table 2

μ \ λ	0	1	2	3	4	5	6	7	8	9	10
0	0	1	0	1	0	1	0	1	1	0	0
1	1	1	0	1	0	0	0	1	1	0	0
2	1	1	1	0	0	0	0	1	1	0	0
3	1	0	1	0	1	0	0	1	1	0	0
4	1	0	1	0	1	0	1	1	0	0	0

Let $M_k(\Gamma_0(N),\chi)$ denote the space of modular forms of weight k on $\Gamma_0(N)$ with Dirichlet character χ (see [22]). In particular, if χ is the trivial Dirichlet character, we also write $M_k(\Gamma_0(N),\chi)$ as $M_k(\Gamma_0(N))$. The following result, known as Sturm's criterion [25], will be used in proving Theorem 1.

Lemma 2.2. *Let p be a prime and $f(z)=\sum_{n=0}^{\infty}a(n)q^n \in M_k(\Gamma_0(N))$ where $a_n\in\mathbb{Q}$ for all $n\geq 0$. If $a_n\equiv 0 \pmod p$ for*

$$n\leq \frac{kN}{12}\prod_{d|N}\Big(1+\frac{1}{d}\Big),$$

where the product is over the distinct prime divisors of N, then $f(z)\equiv 0 \pmod p$, i.e., $a_n\equiv 0 \pmod p$ for any $n\geq 0$.

3. Proofs of the theorems

Proof of Theorem 1. As in [3, p. 20], we define $a(1)=0, a(2)=1$ and for $n\geq 3$,

$$a(n)=\begin{cases} n-1 & \text{if } n\equiv 4 \pmod 5, \\ n-2 & \text{otherwise.}\end{cases}$$

Similarly, let $b(1)=0, b(2)=1$ and $b(n)=a(n)+1$ $(n\geq 3)$. We denote by X^0 the class of functions $f(\tau)$ with

$$f(\tau)=\sum_{n=1}^{N}\lambda_n 11^{a(n)}J_n(\tau), \quad \pi(\lambda_1)=0, \quad N\geq 1,$$

and by Y^0 the class of functions $f(\tau)$ with

$$f(\tau)=\sum_{n=1}^{M}\mu_n 11^{b(n)}J_n(\tau), \quad \pi(\mu_1)=0, \quad M\geq 1.$$

Note here that we have changed Atkin's original definitions in terms of $g_n(\tau)$ to expressions involving $J_n(\tau)$ according to (2.1). We also change the sequences $\xi(n)$ and $\eta(n)$ in [3] to $a(n)$ and $b(n)$ accordingly.

In the proof of (1.3), Atkin [3, p. 26] showed that

$$11^{1-2n}L_{2n-1,11}(\tau)\in X^0, \quad 11^{-2n}L_{2n,11}(\tau)\in Y^0. \tag{3.1}$$

For $n\geq 2$, we have $a(n)\geq 1$ and $b(n)\geq 1$. By Lemma 2.1, the Fourier expansion of $J_n(\tau)$ has integer coefficients. We deduce from (3.1) that

$$\begin{aligned} 11^{1-2n}L_{2n-1,11}(\tau)&\equiv \lambda_1 J_1(\tau) \pmod{11},\\ 11^{-2n}L_{2n,11}(\tau)&\equiv \mu_1 J_1(\tau) \pmod{11}\end{aligned} \tag{3.2}$$

for some integers λ_1 and μ_1 which depend on n and are relatively prime with 11. Thus we have shown that there exist integers c_n such that $(11,c_n)=1$ and

$$11^{-n}L_{n,11}(\tau)\equiv c_n J_1(\tau) \pmod{11}.$$

To prove (1.7), it suffices to show that

$$J_1(\tau)\equiv \Delta E_8 \pmod{11}. \tag{3.3}$$

By Lemma 2.1 we know $\mathrm{ord}_\infty J_1(\tau)=1$ and $\mathrm{ord}_0 J_1(\tau)=-2$. Let

$$f(\tau)=\frac{\eta^{11}(\tau)}{\eta(11\tau)}.$$

From [22, Theorems 1.64 and 1.65], we know that $f(\tau)\in M_5(\Gamma_0(11),\left(\frac{-11}{\cdot}\right))$. Moreover, $\mathrm{ord}_0 f(\tau)=5$. Hence $f^4(\tau)J_1(\tau)\in M_{20}(\Gamma_0(11))$.

Note that $\Delta E_8\in M_{20}(\Gamma_0(11))$, hence $f^4(\tau)J_1(\tau)-\Delta E_8\in M_{20}(\Gamma_0(11))$. Write

$$f^4(\tau)J_1(\tau)-\Delta E_8=\sum_{n=0}^{\infty}c(n)q^n, \quad c(n)\in\mathbb{Z}, \quad \forall n\geq 0.$$

Using (2.2), it is easy to verify that $c(n)\equiv 0 \pmod{11}$ for $n\leq 20$. Hence by Lemma 2.2 we deduce that

$$f^4(\tau)J_1(\tau)\equiv \Delta E_8 \pmod{11}. \tag{3.4}$$

By the binomial theorem, we have $f(\tau)\equiv 1 \pmod{11}$. Therefore, (3.4) implies (3.3), and we complete the proof of Theorem 1. □

Before we proceed to proofs of Theorems 2-4, note that by setting $j=1$ in (1.3), we have

$$p(11n+6)\equiv 0 \pmod{11}. \tag{3.5}$$

It is then clear that Theorems 2-4 are true for the case $k=1$. Therefore, we only need to give proofs for $k\geq 2$.

Proof of Theorem 2. Recall that

$$\sum_{n=0}^{\infty}a_{11}(n)q^n=\frac{(q^{11};q^{11})_\infty^{11}}{(q;q)_\infty}.$$

Let

$$L_0(\tau) := \frac{\eta^{11}(11\tau)\eta(121\tau)}{\eta(\tau)\eta^{11}(1331\tau)} = \frac{(q^{121};q^{121})_\infty}{(q^{1331};q^{1331})_\infty^{11}} \sum_{n\ge 0} a_{11}(n)q^{n-600}.$$

We have

$$UL_0(\tau) = \frac{(q^{11};q^{11})_\infty}{(q^{121};q^{121})_\infty^{11}} \sum_{n\ge 0} a_{11}(11n+6)q^{n-54}.$$

Let

$$L_1(\tau) := U^2L_0(\tau) = \frac{(q;q)_\infty}{(q^{11};q^{11})_\infty^{11}} \sum_{n\ge 0} a_{11}(11^2n+116)q^{n-4}.$$

Note that $L_0(\tau) \in K_0(1331)$, hence $UL_0(\tau) \in K_0(121)$ and $L_1(\tau) \in K_0(11)$. For $r \ge 2$, we define

$$L_r(\tau) := U(\phi(\tau)^{\lambda_{r-1}} L_{r-1}(\tau)), \tag{3.6}$$

where λ_r is 1 if r is odd and -11 if r is even. By induction on r we can show that for $r \ge 1$, $L_r(\tau) \in V$ and

$$L_r(\tau) = \begin{cases} (q;q)_\infty (q^{11};q^{11})_\infty^{-11} \sum\limits_{n\ge 0} a_{11}(11^{r+1}n + 11^{r+1} - 5)q^{n-4} & \text{if } r \text{ is odd,} \\ (q^{11};q^{11})_\infty (q;q)_\infty^{-11} \sum\limits_{n\ge 0} a_{11}(11^{r+1}n + 11^{r+1} - 5)q^{n+1} & \text{if } r \text{ is even.} \end{cases} \tag{3.7}$$

Let

$$\mu_r = \begin{cases} -4 & \text{if } r \text{ is odd,} \\ 1 & \text{if } r \text{ is even.} \end{cases}$$

For any integer $r \ge 1$, since $L_r(\tau) \in V$, from (3.7) we may write

$$L_r(\tau) = \sum_{\nu \ge \mu_r} a_{r,\nu} J_\nu(\tau), \quad a_{r,\nu} \in \mathbb{Z}. \tag{3.8}$$

We will prove that for any $r \ge 1$,

$$\pi(a_{r,\nu}) \ge r + 1 + [\frac{\nu - \mu_r}{2}], \quad \forall \nu \ge \mu_r. \tag{3.9}$$

If $r = 1$, with the help of *Mathematica*, we find that

$$L_1(\tau) = 167948J_{-4}(\tau) + 3529812J_{-3}(\tau) + 19501812J_{-2}(\tau) + 214358881J_0(\tau).$$

Therefore, we have

$$\pi(a_{1,-4}) = 2, \quad \pi(a_{1,-3}) = 3, \quad \pi(a_{1,-2}) = 4, \quad \pi(a_{1,-1}) = \infty, \quad \pi(a_{1,0}) = 8$$

and $\pi(a_{1,\nu}) = \infty$ for any $\nu \ge 1$. Hence (3.9) is true for $r = 1$.

Now suppose (3.9) holds for $r-1$ ($r \ge 2$). From (2.3) we see that

$$a_{r,\nu} = \sum_{\mu=\mu_{r-1}}^{\infty} a_{r-1,\mu} c_{\mu,\nu}^{(\lambda_{r-1})}.$$

Thus

$$\pi(a_{r,\nu}) \ge \min_{\mu \ge \mu_{r-1}} \big(\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})})\big). \tag{3.10}$$

To complete the induction, it suffices to prove that

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \ge r + 1 + [\frac{\nu - \mu_r}{2}], \quad \text{for all } \mu \ge \mu_{r-1}, \nu \ge \mu_r. \tag{3.11}$$

By induction hypothesis and (2.5), we deduce that

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + \left[\frac{\mu-\mu_{r-1}}{2}\right] + \left[\frac{11\nu-\mu-5\lambda_{r-1}-1}{10}\right]. \tag{3.12}$$

Note that if we increase μ by 2, the value of the right hand side cannot decrease. Therefore, its minimum value occurs when $\mu = \mu_{r-1}+1$. Thus

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + \left[\frac{11\nu-\mu_{r-1}-5\lambda_{r-1}-2}{10}\right]. \tag{3.13}$$

If r is odd, then $\mu_{r-1}=1$ and $\lambda_{r-1}=-11$. For $\nu \geq -3$, we have

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + 1 + \left[\frac{11\nu+42}{10}\right] \geq r+1+\left[\frac{\nu+4}{2}\right]. \tag{3.14}$$

For $\nu=-4$, (3.11) reduces to

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r+1, \quad \mu \geq \mu_{r-1}. \tag{3.15}$$

This inequality holds for $\mu=\mu_{r-1}$ since $\pi(a_{r-1,\mu_{r-1}}) \geq r$ and

$$\pi(c_{\mu_{r-1},\nu}^{(\lambda_{r-1})}) \geq \theta(\lambda_{r-1},\mu_{r-1}) = \theta(-11,1) = 1.$$

Similarly it holds for $\mu=\mu_{r-1}+1$, as $\theta(-11,2)=\theta(0,2)=1$. If $\mu \geq \mu_{r-1}+2$, then we have

$$\pi(a_{r-1,\mu}) \geq r + \left[\frac{\mu-\mu_{r-1}}{2}\right] \geq r+1.$$

Thus (3.15) holds.

Combining (3.14) with (3.15), we see that (3.11) holds for r.

If r is even, then $\mu_{r-1}=-4$ and $\lambda_{r-1}=1$. For $\nu \geq 2$, from (3.13) we have

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + 1 + \left[\frac{11\nu-13}{10}\right] \geq r+1+\left[\frac{\nu-1}{2}\right]. \tag{3.16}$$

For $\nu=1$, (3.11) reduces to

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r+1, \quad \mu \geq \mu_{r-1}. \tag{3.17}$$

This inequality holds for $\mu=\mu_{r-1}$ since $\pi(a_{r-1,\mu_{r-1}}) \geq r$ and

$$\pi(c_{\mu_{r-1},\nu}^{(\lambda_{r-1})}) \geq \theta(\lambda_{r-1},\mu_{r-1}) = \theta(1,-4) = 1.$$

Similarly it holds for $\mu=\mu_{r-1}+1$, as $\theta(1,-3)=\theta(0,2)=1$. If $\mu \geq \mu_{r-1}+2$, then we have

$$\pi(a_{r-1,\mu_{r-1}}) \geq r + \left[\frac{\mu-\mu_{r-1}}{2}\right] \geq r+1.$$

Thus (3.17) holds.

Combining (3.16) with (3.17), we see that (3.11) holds for r.

By induction on r, we complete the proof of (3.9) and hence the theorem. □

Proof of Theorem 3. Recall that

$$\sum_{n=0}^{\infty} b_{11}(n)q^n = \frac{(q^{11};q^{11})_\infty}{(q;q)_\infty}.$$

Let

$$L_0(\tau) := \frac{\eta(11\tau)\eta(121\tau)}{\eta(\tau)\eta(1331\tau)} = \frac{(q^{121};q^{121})_\infty}{(q^{1331};q^{1331})_\infty} \sum_{n\geq 0} b_{11}(n)q^{n-50}.$$

We have

$$UL_0(\tau) = \frac{(q^{11};q^{11})_\infty}{(q^{121};q^{121})_\infty}\sum_{n\ge 0} b_{11}(11n+6)q^{n-4}.$$

Let

$$L_1(\tau) := U^2L_0(\tau) = \frac{(q;q)_\infty}{(q^{11};q^{11})_\infty}\sum_{n\ge 0} b_{11}(11^2n+50)q^n.$$

Note that $L_0(\tau)\in K_0(1331)$, hence $UL_0(\tau)\in K_0(121)$ and $L_1(\tau)\in K_0(11)$. For $r\ge 2$, we define

$$L_r(\tau) := U(\phi(\tau)^{\lambda_{r-1}}L_{r-1}(\tau)), \tag{3.18}$$

where λ_r is 1 if r is odd and -1 if r is even. By induction on r we can show that for $r\ge 1$, $L_r(\tau)\in V$ and

$$L_r(\tau) = \begin{cases} (q;q)_\infty(q^{11};q^{11})_\infty^{-1}\sum\limits_{n\ge 0} b_{11}(11^{r+1}n+\frac{5\cdot 11^{r+1}-5}{12})q^n & \text{if } r \text{ is odd,} \\ (q^{11};q^{11})_\infty(q;q)_\infty^{-1}\sum\limits_{n\ge 0} b_{11}(11^{r+1}n+\frac{7\cdot 11^{r+1}-5}{12})q^{n+1} & \text{if } r \text{ is even.}\end{cases} \tag{3.19}$$

Let

$$\mu_r = \begin{cases} 0 & \text{if } r \text{ is odd,} \\ 1 & \text{if } r \text{ is even.}\end{cases}$$

For any integer $r\ge 1$, since $L_r(\tau)\in V$, we may write

$$L_r(\tau) = \sum_{\nu\ge\mu_r} a_{r,\nu}J_\nu, \quad a_{r,\nu}\in\mathbb{Z}. \tag{3.20}$$

We will prove that for any $r\ge 1$,

$$\pi(a_{r,\nu}) \ge 1 + \left[\frac{r}{2}\right] + \left[\frac{\nu-\mu_r}{2}\right], \quad \forall \nu\ge\mu_r. \tag{3.21}$$

If $r=1$, with the help of *Mathematica*, we find that

$$L_1(\tau) = \sum_{\nu=0}^{50} a_{1,\nu}J_\nu(\tau).$$

We have $\pi(a_{1,0})=1$, and the 11-adic orders of $a_{1,\nu}$ $(1\le\nu\le 50)$ are given in Table 3, from which it is easy to verify that (3.21) holds for $r=1$.

TABLE 3

ν	1	2	3	4	5	6	7	8	9	10
$\pi(a_{1,\nu})$	3	4	4	7	6	8	9	10	12	12
ν	11	12	13	14	15	16	17	18	19	20
$\pi(a_{1,\nu})$	14	14	15	17	17	19	21	22	24	24
ν	21	22	23	24	25	26	27	28	29	30
$\pi(a_{1,\nu})$	26	26	27	29	29	31	32	34	36	36
ν	31	32	33	34	35	36	37	38	39	40
$\pi(a_{1,\nu})$	37	38	39	41	41	43	44	45	49	48
ν	41	42	43	44	45	46	47	48	49	50
$\pi(a_{1,\nu})$	50	51	52	55	54	56	57	58	∞	58

Now suppose (3.21) holds for $r-1$ ($r\geq 2$). For the same reason as in the proof of Theorem 2, to complete the induction, it suffices to prove that

$$\pi(a_{r-1,\mu})+\pi(c_{\mu,\nu}^{(\lambda_{r-1})})\geq 1+\left[\frac{r}{2}\right]+\left[\frac{\nu-\mu_r}{2}\right],\quad \text{for all } \mu\geq\mu_{r-1},\nu\geq\mu_r. \tag{3.22}$$

By the induction hypothesis and (2.5), we deduce that

$$\pi(a_{r-1,\mu})+\pi(c_{\mu,\nu}^{(\lambda_{r-1})})\geq 1+\left[\frac{r-1}{2}\right]+\left[\frac{\mu-\mu_{r-1}}{2}\right]+\left[\frac{11\nu-\mu-5\lambda_{r-1}-1}{10}\right].$$

Note that if we increase μ by 2, the value of the right hand side cannot decrease. Therefore, its minimum value occurs when $\mu=\mu_{r-1}+1$. Thus

$$\pi(a_{r-1,\mu})+\pi(c_{\mu,\nu}^{(\lambda_{r-1})})\geq 1+\left[\frac{r-1}{2}\right]+\left[\frac{11\nu-\mu_{r-1}-5\lambda_{r-1}-2}{10}\right]. \tag{3.23}$$

If r is odd, then $\mu_{r-1}=1$ and $\lambda_{r-1}=-1$. We have

$$\pi(a_{r-1,\mu})+\pi(c_{\mu,\nu}^{(\lambda_{r-1})})\geq 1+\left[\frac{r-1}{2}\right]+\left[\frac{11\nu+2}{10}\right]\geq 1+\left[\frac{r}{2}\right]+\left[\frac{\nu}{2}\right],\quad \forall\nu\geq 0. \tag{3.24}$$

Thus (3.22) holds for r.

If r is even, then $\mu_{r-1}=0$ and $\lambda_{r-1}=1$. For $\nu\geq 2$, by (3.23) we have

$$\pi(a_{r-1,\mu})+\pi(c_{\mu,\nu}^{(\lambda_{r-1})})\geq 1+\left[\frac{r-1}{2}\right]+1+\left[\frac{11\nu-17}{10}\right]\geq 1+\left[\frac{r}{2}\right]+\left[\frac{\nu-1}{2}\right]. \tag{3.25}$$

For $\nu=1$, (3.22) reduces to

$$\pi(a_{r-1,\mu})+\pi(c_{\mu,\nu}^{(\lambda_{r-1})})\geq 1+\left[\frac{r}{2}\right],\quad \mu\geq\mu_{r-1}. \tag{3.26}$$

This inequality holds for $\mu=\mu_{r-1}$ since $\pi(a_{r-1,\mu_{r-1}})\geq 1+\left[\frac{r-1}{2}\right]$ and

$$\pi(c_{\mu_{r-1},\nu}^{(\lambda_{r-1})})\geq\theta(\lambda_{r-1},\mu_{r-1})=\theta(1,0)=1.$$

Similarly it holds for $\mu=\mu_{r-1}+1$, as $\theta(1,1)=1$. If $\mu\geq\mu_{r-1}+2$, then by the induction hypothesis we have

$$\pi(a_{r-1,\mu})\geq 1+\left[\frac{r-1}{2}\right]+\left[\frac{\mu-\mu_{r-1}}{2}\right]\geq 1+\left[\frac{r}{2}\right].$$

Thus (3.26) holds.

Combining (3.25) with (3.26) we see that (3.22) holds for r.

By induction on r, we complete the proof of (3.21) and hence the theorem. □

Remark 1. Since the progression of the odd case $r=2m+1$ is always a subprogression of the even case $r=2m$ in (3.19), the statement in Theorem 3 is only for the progressions of even r in (3.19).

Remark 2. It is only the case $\nu=0$ that causes the expression in (3.21) to be $1+\left[\frac{r}{2}\right]$ rather than $1+r$ as in (3.9).

Proof of Theorem 4. Let

$$L_0(\tau):=\frac{\eta(121\tau)\eta(1331\tau)}{\eta(\tau)\eta(11\tau)}=q^{60}\frac{(q^{121};q^{121})_\infty(q^{1331};q^{1331})_\infty}{(q;q)_\infty(q^{11};q^{11})_\infty}.$$

We have

$$L_0(\tau)=(q^{121};q^{121})_\infty(q^{1331};q^{1331})_\infty\sum_{n\geq 0}p_{[1^1 11^1]}(n)q^{n+60}.$$

Applying the U-operator twice, we get

$$L_1(\tau) := U^2 L_0(\tau) = (q;q)_\infty (q^{11};q^{11})_\infty \sum_{n\geq 0} p_{[1^1 11^1]}(11^2 n + 61) q^{n+1}.$$

Since $L_0(\tau) \in K_0(1331)$, we have $UL_0(\tau) \in K_0(121)$ and $L_1(\tau) \in K_0(11)$. For $r \geq 2$, we define

$$L_r(\tau) := U(\phi^{\lambda_{r-1}}(\tau) L_{r-1}(\tau)), \tag{3.27}$$

where $\lambda_r = 1$ for any $r \geq 1$. By induction on r we can show that for $r \geq 1$, $L_r(\tau) \in V$ and

$$L_r(\tau) = (q;q)_\infty (q^{11};q^{11})_\infty \sum_{n\geq 0} p_{[1^1 11^1]}\Big(11^{r+1} n + \frac{11^{r+1}+1}{2}\Big) q^{n+1}. \tag{3.28}$$

Let $\mu_r = 1$ for all $r \geq 1$. For any integer $r \geq 1$, since $L_r(\tau) \in V$ we can write

$$L_r(\tau) = \sum_{\nu \geq \mu_r} a_{r,\nu} J_\nu(\tau), \quad a_{r,\nu} \in \mathbb{Z}.$$

We will prove that for any $r \geq 1$,

$$\pi(a_{r,\nu}) \geq r + 1 + [\frac{\nu - \mu_r}{2}], \quad \forall \nu \geq \mu_r. \tag{3.29}$$

If $r = 1$, with the help of *Mathematica*, we find that

$$L_1(\tau) = \sum_{\nu=1}^{60} a_{1,\nu} J_\nu(\tau).$$

The 11-adic orders of $a_{1,\nu}$ $(1 \leq \nu \leq 60)$ are given in Table 4, from which it is easy to verify that (3.29) holds for $r = 1$.

TABLE 4

ν	1	2	3	4	5	6	7	8	9	10
$\pi(a_{1,\nu})$	2	3	3	5	5	8	8	10	11	11
ν	11	12	13	14	15	16	17	18	19	20
$\pi(a_{1,\nu})$	13	14	14	17	16	18	19	20	22	22
ν	21	22	23	24	25	26	27	28	29	30
$\pi(a_{1,\nu})$	24	25	25	27	27	29	31	32	34	34
ν	31	32	33	34	35	36	37	38	39	40
$\pi(a_{1,\nu})$	36	36	37	39	39	41	42	44	46	46
ν	41	42	43	44	45	46	47	48	49	50
$\pi(a_{1,\nu})$	47	48	49	51	51	53	54	55	59	57
ν	51	52	53	54	55	56	57	58	59	60
$\pi(a_{1,\nu})$	59	60	61	64	63	65	66	67	∞	68

Now suppose (3.29) holds for $r-1$ $(r \geq 2)$. For the same reason as in the proof of Theorem 2, to complete the induction, it suffices to prove that

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + 1 + \big[\frac{\nu - \mu_r}{2}\big], \quad \text{for all } \mu \geq \mu_{r-1}, \nu \geq \mu_r. \tag{3.30}$$

By the induction hypothesis and (2.5), we deduce that

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + \Big[\frac{\mu - \mu_{r-1}}{2}\Big] + \Big[\frac{11\nu - \mu - 5\lambda_{r-1} - 1}{10}\Big]. \tag{3.31}$$

Note that if we increase μ by 2, the value of the right hand side cannot decrease. Therefore, its minimum value occurs when $\mu = \mu_{r-1} + 1$. Thus

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + \left[\frac{11\nu - \mu_{r-1} - 5\lambda_{r-1} - 2}{10}\right].$$

Since $\mu_{r-1} = \lambda_{r-1} = 1$, for $\nu \geq 2$ we have

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + 1 + \left[\frac{11\nu - 18}{10}\right] \geq r + 1 + \left[\frac{\nu - 1}{2}\right]. \tag{3.32}$$

For $\nu = 1$, (3.30) reduces to

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + 1. \tag{3.33}$$

By the induction hypothesis, we have $\pi(a_{r-1,\mu_{r-1}}) \geq r$. Since

$$\pi(c_{\mu_{r-1},\nu}^{(\lambda_{r-1})}) \geq \theta(\lambda_{r-1}, \mu_{r-1}) = \theta(1,1) = 1,$$

we see that (3.33) holds for $\mu = \mu_{r-1}$. Similarly it holds for $\mu = \mu_{r-1} + 1$, as $\theta(1,2) = 1$. If $\mu \geq \mu_{r-1} + 2$, then we have

$$\pi(a_{r-1,\mu}) \geq r + \left[\frac{\mu - \mu_{r-1}}{2}\right] \geq r + 1.$$

Thus (3.33) holds for r.

Combining (3.32) with (3.33), we see that (3.30) holds for r.

By induction on r, we complete the proof of (3.29) and hence the theorem. □

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