

A COHOMOLOGICAL CHARACTERIZATION OF NILPOTENT FUSION SYSTEMS

ANTONIO DÍAZ RAMOS, ARTURO ESPINOSA BARO, AND ANTONIO VIRUEL

(Communicated by Michael A. Mandell)

ABSTRACT. We provide a nilpotency criterion for fusion systems in terms of the vanishing of its cohomology with twisted coefficients.

1. INTRODUCTION

Let G be a finite group and let p be a prime. Then G is said to be p -nilpotent if a Sylow p -subgroup has a complement, i.e., if there exists a *split* short exact sequence

$$1 \rightarrow N \rightarrow G \rightarrow S$$

with $S \in \text{Syl}_p(G)$. It turns out that this property can be characterized solely in terms of the *fusion system* of G over S . This terminology was introduced in [6] and it is straightforward that

$$G \text{ is } p\text{-nilpotent} \Leftrightarrow \mathcal{F}_S(G) = \mathcal{F}_S(S),$$

where $\mathcal{F}_S(G)$ denotes the fusion system of G over S . Consequently, a fusion system \mathcal{F} over the p -group S is termed nilpotent if $\mathcal{F} = \mathcal{F}_S(S)$. Already several authors have provided fusion system counterparts to characterizations of p -nilpotency for finite groups; see [2], [3], [7], [9], [10], [11], [13] and [14]. In this work, we prove the fusion system version of a p -nilpotency criterion from the late 1960's due to Wong [16] and Hoechsmann, Roquette and Zassenhaus [12].

Theorem 1.1. *Let \mathcal{F} be a fusion system. Then the following are equivalent:*

- (1) \mathcal{F} is nilpotent.
- (2) For each $\mathbb{F}_p[\mathcal{F}]$ -module M , if $H^m(\mathcal{F}^c; M) = 0$ for some $m > 0$, then $H^n(\mathcal{F}^c; M) = 0$ for every $n > 0$.

Here, an $\mathbb{F}_p[\mathcal{F}]$ -module is a finitely generated $\mathbb{F}_p[S]$ -module that is \mathcal{F} -invariant, and $H^*(\mathcal{F}^c; M)$ is twisted cohomology over \mathcal{F}^c , i.e., over the \mathcal{F} -centric subgroups. See Definitions 2.1 and 2.2 for full details. We give a topological proof of Theorem 1.1 via the classifying space $B\mathcal{F}$ of \mathcal{F} [8]. Recall that, in the terminology of [6], $B\mathcal{F} \simeq |\mathcal{L}|_p^\wedge$, where \mathcal{L} is the unique centric linking system associated to \mathcal{F} and $(\cdot)_p^\wedge$ denotes p -completion in the sense of Bousfield and Kan [4].

Received by the editors March 31, 2017, and, in revised form, May 24, 2017.

2010 *Mathematics Subject Classification.* Primary 20D15, 20D20, 55N25, 55R40.

The authors were partially supported by MEC grants MTM2013-41768-P and MTM2016-78647-P and Junta de Andalucía grant FQM-213.

The second author was supported by Polish National Science Centre grant 2016/21/P/ST1/03460 within the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 665778.

The original version of Theorem 1.1 was equivalently stated in terms of Tate’s cohomology. Moreover, its proof resorted to dimension shifting. This approach is not suitable here as there are not enough acyclic modules. For instance, if $B\mathcal{F}$ is simply connected, then, for each $\mathbb{F}_p[\mathcal{F}]$ -module M , $H^*(\mathcal{F}^c; M) = H^*(\mathcal{F}; M)$ is cohomology with trivial coefficients. Hence, it will not vanish except in trivial cases.

Notation. Throughout this work, by fusion system we mean a *saturated* fusion system. The unacquainted reader may find an explanation of this terminology and general background on fusion systems in [1].

2. PROOF OF THE THEOREM

We start introducing modules for fusion systems and their cohomology:

Definition 2.1. Let \mathcal{F} be a fusion system over the finite p -group S . An $\mathbb{F}_p[\mathcal{F}]$ -module is a finitely generated $\mathbb{F}_p[S]$ -module M which is \mathcal{F} -invariant, i.e., such that

$$\forall P \leq S, \forall \varphi \in \text{Hom}_{\mathcal{F}}(P, S), \forall p \in P, \forall m \in M : \varphi(p) \cdot m = p \cdot m.$$

It is clear that an $\mathbb{F}_p[\mathcal{F}]$ -module is the same thing as a finitely generated $\mathbb{F}_p[S/\text{foc}(\mathcal{F})]$ -module, where $\text{foc}(\mathcal{F})$ is the focal subgroup of \mathcal{F} :

$$\text{foc}(\mathcal{F}) = \langle [P, \text{Aut}_{\mathcal{F}}(P)], P \leq S \rangle.$$

By inflation, every $\mathbb{F}_p[\mathcal{F}]$ -module is also an $\mathbb{F}_p[S/\text{hfp}(\mathcal{F})]$ -module, where $\text{hfp}(\mathcal{F})$ is the hyperfocal subgroup of \mathcal{F} :

$$\text{hfp}(\mathcal{F}) = \langle [P, O^p(\text{Aut}_{\mathcal{F}}(P))], P \leq S \rangle.$$

Definition 2.2 ([15, Definition 2.3]). Let \mathcal{F} be a saturated fusion system over the finite p -group S and let M be an $\mathbb{F}_p[\mathcal{F}]$ -module. For each $n \geq 0$, define the twisted cohomology group $H^n(\mathcal{F}^c; M)$ as the \mathcal{F}^c -stable elements

$$H^n(\mathcal{F}^c; M) = \{z \in H^n(S; M) \mid \forall P \in \mathcal{F}^c, \forall \varphi \in \text{Hom}_{\mathcal{F}}(P, S) : \text{res}(z) = \varphi^*(z)\}.$$

Here, $\varphi^* : H^n(S; M) \rightarrow H^n(P; M)$ is the homomorphism induced in cohomology by φ , and $\text{res} = \iota^*$ for the inclusion $\iota : P \hookrightarrow S$. By Alperin’s Fusion Theorem, if the action of $S/\text{hfp}(\mathcal{F})$ on M is trivial, then $H^n(\mathcal{F}^c; M)$ coincide with the \mathcal{F} -stable elements

$$H^n(\mathcal{F}; M) = \{z \in H^n(S; M) \mid \forall P \leq S, \forall \varphi \in \text{Hom}_{\mathcal{F}}(P, S) : \text{res}(z) = \varphi^*(z)\}.$$

In general, the abelian group $H^n(\mathcal{F}^c; M)$ may be recovered via topology as the cohomology of the classifying space of \mathcal{F} [15, Corollary 5.4]:

$$H^*(\mathcal{F}^c; M) \cong H^*(B\mathcal{F}; M),$$

where $\pi_1(B\mathcal{F}) = S/\text{hfp}(\mathcal{F})$ by [5, Theorem B]. Now we are ready to prove the main theorem.

Proof of Theorem 1.1. For the implication (1) \Rightarrow (2), note that

$$H^n(\mathcal{F}_S(S)^c; M) = H^n(S; M).$$

From here it is enough to follow the group theoretical proof; see [16, Theorem 1] or [12, Proposition 1a]. For the reverse implication, set $\pi = S/\text{hfp}(\mathcal{F})$ and consider the universal covering space [5, Theorem 4.4]

$$\pi \rightarrow BO^p(\mathcal{F}) \rightarrow B\mathcal{F},$$

where $O^p(\mathcal{F}) \leq \mathcal{F}$ is the unique p -power index fusion subsystem of \mathcal{F} over $\mathfrak{h}\eta\mathfrak{p}(\mathcal{F}) \leq S$. For $M = \mathbb{F}_p[\pi]$ with π acting by left multiplication, we get

$$H^1(\mathcal{F}^c; M) \cong H^1(B\mathcal{F}; M) \cong H^1(BO^p(\mathcal{F}); \mathbb{F}_p) = 0,$$

as $BO^p(\mathcal{F})$ is simply connected. Hence, by hypothesis

$$0 = H^n(\mathcal{F}^c; M) \cong H^n(BO^p(\mathcal{F}); \mathbb{F}_p)$$

for all $n \geq 1$. As $BO^p(\mathcal{F})$ is a p -complete space [6, Proposition 1.11], it must be contractible [4, I.5.5]. Hence, $B\mathcal{F} \simeq K(\pi, 1)$, $\mathfrak{h}\eta\mathfrak{p}(\mathcal{F}) = 1$ and $\mathcal{F} = \mathcal{F}_S(S)$. \square

Remark 2.3. This argument provides an alternative proof of the implication (2) \Rightarrow (1) for finite groups. Namely, consider the fibration of classifying spaces of finite groups,

$$BO^p(G) \rightarrow BG \rightarrow BG/O^p(G),$$

and then p -complete it [4, II.5.1].

REFERENCES

- [1] Michael Aschbacher, Radha Kessar, and Bob Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, vol. 391, Cambridge University Press, Cambridge, 2011. MR2848834
- [2] Adolfo Ballester-Bolínches, Luis M. Ezquerro, Ning Su, and Yanming Wang, *On the focal subgroup of a saturated fusion system*, J. Algebra **468** (2016), 72–79, DOI 10.1016/j.jalgebra.2016.07.034. MR3550858
- [3] David J. Benson, Jesper Grodal, and Ellen Henke, *Group cohomology and control of p -fusion*, Invent. Math. **197** (2014), no. 3, 491–507, DOI 10.1007/s00222-013-0489-5. MR3251827
- [4] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin-New York, 1972. MR0365573
- [5] C. Broto, N. Castellana, J. Grodal, R. Levi, and B. Oliver, *Extensions of p -local finite groups*, Trans. Amer. Math. Soc. **359** (2007), no. 8, 3791–3858, DOI 10.1090/S0002-9947-07-04225-0. MR2302515
- [6] Carles Broto, Ran Levi, and Bob Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), no. 4, 779–856, DOI 10.1090/S0894-0347-03-00434-X. MR1992826
- [7] José Cantarero, Jérôme Scherer, and Antonio Viruel, *Nilpotent p -local finite groups*, Ark. Mat. **52** (2014), no. 2, 203–225, DOI 10.1007/s11512-013-0181-4. MR3255138
- [8] Andrew Chermak, *Fusion systems and localities*, Acta Math. **211** (2013), no. 1, 47–139, DOI 10.1007/s11511-013-0099-5. MR3118305
- [9] Antonio Díaz Ramos, *A spectral sequence for fusion systems*, Algebr. Geom. Topol. **14** (2014), no. 1, 349–378, DOI 10.2140/agt.2014.14.349. MR3158762
- [10] Antonio Díaz, Adam Glessner, Sejong Park, and Radu Stancu, *Tate’s and Yoshida’s theorems on control of transfer for fusion systems*, J. Lond. Math. Soc. (2) **84** (2011), no. 2, 475–494, DOI 10.1112/jlms/jdr019. MR2835340
- [11] Jon González-Sánchez, Albert Ruiz, and Antonio Viruel, *On Thompson’s p -complement theorems for saturated fusion systems*, Kyoto J. Math. **55** (2015), no. 3, 617–626, DOI 10.1215/21562261-3089100. MR3395982
- [12] K. Hoehsmann, P. Roquette, and H. Zassenhaus, *A cohomological characterization of finite nilpotent groups*, Arch. Math. (Basel) **19** (1968), 225–244, DOI 10.1007/BF01899499. MR0227278
- [13] Radha Kessar, Markus Linckelmann, and Gabriel Navarro, *A characterisation of nilpotent blocks*, Proc. Amer. Math. Soc. **143** (2015), no. 12, 5129–5138, DOI 10.1090/proc/12646. MR3411131
- [14] Jun Liao and Jiping Zhang, *Nilpotent fusion systems*, J. Algebra **442** (2015), 438–454, DOI 10.1016/j.jalgebra.2015.03.002. MR3395068
- [15] Rémi Molinier, *Cohomology with twisted coefficients of the classifying space of a fusion system*, Topology Appl. **212** (2016), 1–18, DOI 10.1016/j.topol.2016.09.001. MR3553745
- [16] W. J. Wong, *A cohomological characterization of finite nilpotent groups*, Proc. Amer. Math. Soc. **19** (1968), 689–691, DOI 10.2307/2035865. MR0229728

DEPARTAMENTO DE ÁLGEBRA, GEOMETRÍA Y TOPOLOGÍA, UNIVERSIDAD DE MÁLAGA, APDO
CORREOS 59, 29080 MÁLAGA, SPAIN

E-mail address: adiazramos@uma.es

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, ADAM MICKIEWICZ UNIVERSITY, UMUL-
TOWSKA 87, 61-614 POZNAN, POLAND

E-mail address: arturo.espinosabaro@gmail.com

DEPARTAMENTO DE ÁLGEBRA, GEOMETRÍA Y TOPOLOGÍA, UNIVERSIDAD DE MÁLAGA, APDO
CORREOS 59, 29080 MÁLAGA, SPAIN

E-mail address: viruel@uma.es