MONOCHROMATIC FINSLER SURFACES AND A LOCAL ELLIPSOID CHARACTERIZATION

SERGEI IVANOV

(Communicated by Lei Ni)

ABSTRACT. We prove the following localized version of a classical ellipsoid characterization: Let $B \subset \mathbb{R}^3$ be a convex body with a smooth strictly convex boundary and 0 in the interior, and suppose that there is an open set of planes through 0 such that all sections of B by these planes are linearly equivalent. Then all these sections are ellipses and the corresponding part of B is a part of an ellipsoid.

We apply this to differential geometry of Finsler surfaces in normed spaces and show that in certain cases the intrinsic metric of a surface imposes restrictions on its extrinsic geometry similar to implications of Gauss' Theorema Egregium. As a corollary we construct 2-dimensional Finsler metrics that do not admit local isometric embeddings to dimension 3.

1. INTRODUCTION

Although this paper is motivated by Finsler geometry, one of the main results is the following theorem about convex sets.

Theorem 1. *Let $B \subset \mathbb{R}^n$, $n \geq 3$, be a convex body with a smooth strictly convex boundary and 0 in the interior. Let $H \subset \mathbb{R}^n$ be a 2-dimensional plane through 0 and suppose that there is a neighborhood of H in the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^n)$ such that for every plane H' from this neighborhood the cross-section $H' \cap B$ is linearly equivalent to $H \cap B$.*

Then $H \cap B$ is an ellipse centered at 0 and furthermore B coincides with an ellipsoid in a neighborhood of H.

Here by *linear equivalence* of two cross-sections we mean the existence of a linear bijection between their planes sending one cross-section to the other. The word "smooth" in this paper always means C^∞, and strict convexity means quadratic strict convexity, i.e., positivity of appropriate second derivatives. By $\mathrm{Gr}_k(V)$ we denote the Grassmannian of k-dimensional linear subspaces of a vector space V.

Theorem 1 is a "local version" of the following classical ellipsoid characterization due to Auerbach, Mazur and Ulam [1]: *If all planar cross-sections through 0 of a convex body are linearly equivalent, then the body is a centered ellipsoid.* (Note that in this context local statements are stronger than global ones.)

Remark 1.1. The following more general question goes back to Banach [2] and remains only partially solved: For given k and n $(k < n)$, are Euclidean spaces the only n-dimensional Banach spaces with the property that all k-dimensional linear subspaces are isometric? The above mentioned Auerbach-Mazur-Ulam theorem answers this question for $k = 2$. Gromov [9] proved that the answer is affirmative if k is even or $n \geq k + 2$. The case when k is odd and $n = k + 1$ remains open. See also [8, Note 7.2] for discussion and related results.

The proofs in [1] and [9] are based on global obstructions arising from algebraic topology of Grassmannians. These methods do not work in the local version of the problem.

Remark 1.2. It is plausible that the smoothness assumption in Theorem 1 can be relaxed. However (some form of) strict convexity is necessary. Indeed, pick any convex body $B_0 \subset \mathbb{R}^2$ and let $B \subset \mathbb{R}^2 \times \mathbb{R}$ be a convex body which coincides with $B_0 \times \mathbb{R}$ in a neighborhood of the plane $H = \mathbb{R}^2 \times \{0\}$. Then all cross-sections of B by planes sufficiently close to H are linearly equivalent to B_0 but B_0 is not necessarily an ellipse.

Every convex body with 0 in the interior is the unit ball of a norm. We use the word "norm" in a slightly generalized meaning, namely a norm is not required to be symmetric. Smooth strictly convex bodies correspond to especially nice norms called Banach-Minkowski ones. By definition, a *Banach-Minkowski norm* on a vector space V is a (possibly non-symmetric) norm $\Phi\colon V \to \mathbb{R}_+$ which is smooth outside 0 and such that the function Φ^2 is (quadratically) strictly convex. A *Banach-Minkowski space* is a finite-dimensional vector space equipped with a Banach-Minkowski norm. These spaces are often referred to as Minkowski spaces but we use the longer term to avoid confusion with special relativity.

A norm is called *Euclidean* if it is associated with an inner product. The following is a slightly more detailed reformulation of Theorem 1 in terms of norms.

Theorem 1'. *Let $V = (V^n, \Phi)$ be a Banach-Minkowski space, $n \geq 3$, and assume that $\mathcal{U} \subset \mathrm{Gr}_2(V)$ is a connected open set such that for all $H, H' \in \mathcal{U}$ the normed planes $(H, \Phi|_H)$ and $(H', \Phi|_{H'})$ are isometric.*

Then $\Phi|_H$ is a Euclidean norm for every $H \in \mathcal{U}$. Moreover there exists a Euclidean norm on V whose restriction to every plane from $\mathcal{U}$ coincides with the restriction of Φ.

Remark 1.3. The assumption that $\mathcal{U}$ is connected is not necessary if $n = 3$. In higher dimensions one can construct examples where Φ coincides with one Euclidean norm near one plane and with another Euclidean norm near some other plane. The details of this construction are left to the reader.

If B is the unit ball of a norm Φ and two planar cross-sections $H \cap B$ and $H' \cap B$ are linearly equivalent, then the normed planes $(H, \Phi|_H)$ and $(H', \Phi|_{H'})$ are isometric (and vice versa). Thus Theorem 1' implies Theorem 1.

Theorem 1' easily follows from its 3-dimensional case (see Section 4). Finslerian results discussed below work only in 3-dimensional spaces. For these reasons in what follows we mainly restrict our attention to dimension 3.

Finsler surfaces. For a smooth surface $M^2 \subset \mathbb{R}^3$ and a point $p \in M$, the celebrated Gauss' Theorema Egregium implies that local intrinsic geometry of M near p determines the type of the second-order extrinsic geometry at p. Namely the sign

of the Riemannian curvature of M at p determines whether the second fundamental form of M at p is definite, semi-definite, or degenerate. (We call a quadratic form *definite* if it is either positive definite or negative definite.)

The distinction between the three types of the second fundamental form is affine invariant. Hence one may ask whether there are similar relations between intrinsic and extrinsic geometry for Finsler surfaces in normed 3-spaces. More precisely, we have the following question.

Question 1.4 (cf. [6, Remark 1.6]). Let M be a 2-dimensional Finsler manifold and $f_i \colon M \to V_i$, $i = 1, 2$, smooth isometric embeddings to 3-dimensional Banach-Minkowski spaces. For $p \in M$, is it always true that the second fundamental forms of f_i at p are of the same type: definite, semi-definite, or degenerate?

Remark 1.5. If the dimension of the ambient space V is greater than 3, then the answer to any sensible variant of Question 1.4 is negative. Indeed, as shown in [6], locally any 2-dimensional Finsler manifold is isometric to a strictly saddle surface in $\mathbb{R}^4$ equipped with some Banach-Minkowski norm. In contrast, in the Euclidean case this is possible only for negatively curved metrics.

In this paper we show that at least for some Finsler metrics the answer to Question 1.4 is affirmative. Namely this is the case if the metric is second-order flat at the point in question or monochromatic (see below for definitions). In both cases the second fundamental form is necessarily degenerate, unless in the monochromatic case the metric is actually Riemannian.

Recall that a *Finsler manifold* is a smooth manifold M equipped with a Finsler metric, that is, a continuous function $\varphi \colon TM \to \mathbb{R}_+$ such that φ is smooth on $TM \setminus 0$ and $\varphi|_{T_x M}$ is a Banach-Minkowski norm for every $x \in M$. Every Banach-Minkowski space is naturally a Finsler manifold. An isometric immersion is a norm-preserving immersion from one Finsler manifold to another. Since our set-up is local, we consider isometric embeddings rather than immersions.

The standard definition of the second fundamental form requires an inner product, which we do not have in our set-up. We use the following affine invariant version of the definition. Let M be a smooth manifold, V a vector space, and $f \colon M \to V$ a smooth immersion. Then the second fundamental form of f at $p \in M$ is a symmetric bilinear form on $T_p M$ with values in the quotient vector space $V / \operatorname{Im} d_p f$ where $d_p f \colon T_p M \to V$ is the differential of f at p. This form is defined as the Hessian of $\pi \circ f$ at p where $\pi \colon V \to V / \operatorname{Im} d_p f$ is the quotient projection. The Hessian is well-defined since p is a critical point of $\pi \circ f$. If the quotient space is 1-dimensional, one may regard the second fundamental form as a real-valued form defined up to a multiplication by a constant.

Following S.-S. Chern and D. Bao (see [4, §3.3]), we call a Finsler manifold $M = (M, \varphi)$ *monochromatic* if all its tangent spaces are isometric as normed vector spaces. Note that Riemannian manifolds are monochromatic.

Theorem 2. *Let M^2 be a monochromatic non-Riemannian Finsler manifold, let V^3 be a Banach-Minkowski space and let $f \colon M \to V$ be a smooth isometric embedding. Then the second fundamental form of f is degenerate at every point.*

In a sense, Theorem 2 is a reformulation of Theorem 1; see Remark 3.7.

The property that the second fundamental form is degenerate at every point is a strong restriction. We use implications of this property and Theorem 2 to construct

examples of Finsler metrics in $\mathbb{R}^2$ that do not admit local isometric embeddings to 3-dimensional Banach-Minkowski spaces. See Example 5.1 and Proposition 5.2.

As already mentioned in Remark 1.5, all Finsler surfaces admit local isometric embeddings to dimension 4. Thus $n = 4$ is the minimal n such that all Finsler surfaces admit local isometric embeddings to dimension n. Note that there is no such universal dimension for *global* isometric embeddings and moreover non-compact Finsler manifolds generally do not admit isometric embeddings to Banach-Minkowski spaces; see [5] and [14].

Definition 1.6. Recall that a Finsler metric is called *flat* if it is locally isometric to a Banach-Minkowski space.

Let $M = (M, \varphi)$ be a Finsler manifold. We say that the metric φ is *second-order flat* at a point $p \in M$ if there exists a flat Finsler metric φ_0 defined in a neighborhood of p such that for $x \in M$ near p and $v \in T_x M \setminus \{0\}$ one has

$$(1.1) \qquad \frac{\varphi(x, v)}{\varphi_0(x, v)} = 1 + o(|x - p|^2), \quad x \to p,$$

where $|x - p|$ is the distance from x to p in an arbitrary local coordinate system.

If φ is Riemannian, i.e., $\varphi = \sqrt{g}$ where g is a Riemannian metric tensor, then (1.1) is equivalent to the existence of local coordinates in which the second derivatives of g vanish at p. In dimension 2 this is equivalent to $K(p) = 0$ where K is the curvature of the metric. By Gauss' Theorem this implies that every smooth isometric embedding of M to $\mathbb{R}^3$ has a degenerate second fundamental form at p. In the next theorem we generalize this implication to Finsler surfaces.

Theorem 3. *Let $M = (M^2, \varphi)$ be a Finsler manifold whose metric is second-order flat at a point $p \in M$. Let V^3 be a Banach-Minkowski space and $f \colon M \to V$ a smooth isometric embedding. Then the second fundamental form of f at p is degenerate.*

Theorems 2 and 3 imply that the answer to Question 1.4 is affirmative if the Finsler metric is second-order flat at p or monochromatic (Riemannian or not); see Corollary 4.4.

The rest of the paper is organized as follows. We begin with the proof of Theorem 3 in Section 2. Subsequent arguments do not depend on Theorem 3 directly but they use similar ideas. In Section 3 we prove the first part of Theorem 1' (see Proposition 3.1) and deduce Theorem 2. In Section 4 we finish the proof of Theorem 1' and obtain some corollaries. Finally in Section 5 we construct examples of metrics that are not locally embeddable to dimension 3.

2. Proof of Theorem 3

Let (M, φ) and p be as in Theorem 3 and φ_0 as in Definition 1.6. Since the statement of the theorem is local, we may restrict ourselves to a small coordinate neighborhood U of p. Thus we may assume that $M = U \subset \mathbb{R}^2$ and use the standard identification $TU = U \times \mathbb{R}^2$. Then φ is a function of two variables $x \in U$ and $v \in \mathbb{R}^2$.

We further assume that our local coordinates are chosen so that φ_0 is the restriction of a Banach-Minkowski norm. In other words, $\varphi_0(x, v) = \varphi_0(v)$ does not depend on x. Then (1.1) implies that for every $v \in \mathbb{R}^2$ we have $\varphi(p, v) = \varphi_0(v)$,

$$(2.1) \qquad d_p\big(\varphi(\cdot, v)\big) = 0,$$

and

$$d_p^2\big(\varphi(\cdot, v)\big) = 0. \tag{2.2}$$

Here and below d_p denotes the differential of a function at p, d_p^2 the second differential at p, etc.

Let $V = (V, \Phi)$ be a 3-dimensional Banach-Minkowski space and let $f\colon (U, \varphi) \to V$ be a smooth isometric embedding. Then

$$\Phi(d_x f(v)) = \varphi(x, v) \tag{2.3}$$

for all $x \in U$ and $v \in \mathbb{R}^2$. Define $H = \operatorname{Im} d_p f$, that is, H is the tangent plane to $f(U)$ at $f(p)$, regarded as a linear subspace of V. The map $d_p f$ is a linear isometry between $(\mathbb{R}^2, \varphi_0)$ and $(H, \Phi|_H)$. Fix an isomorphism between V/H and $\mathbb{R}$ and denote by S the second fundamental form of f at p composed with this isomorphism. That is, S is a symmetric real-valued bilinear form on $\mathbb{R}^2$ given by $S(v, w) = \pi(d_p^2 f(v, w))$ for all $v, w \in \mathbb{R}^2$, where $\pi\colon V \to V/H \cong \mathbb{R}$ is the quotient map. Our goal is to prove that S is degenerate. We say that vectors $v, w \in \mathbb{R}^2$ are S-orthogonal if $S(v, w) = 0$.

Let $v \in \mathbb{R}^2$ and $\tilde{v} = d_p f(v)$. Differentiating (2.3) with respect to x at $x = p$ in the direction $w \in \mathbb{R}^2$ and taking into account (2.1) yields

$$d_{\tilde{v}} \Phi(d_p^2 f(v, w)) = 0. \tag{2.4}$$

Here $d_{\tilde{v}} \Phi$ denotes the differential of Φ at $\tilde{v}$, this differential is a linear map from V to $\mathbb{R}$, and $d_p^2 f(v, w) \in V$ is an argument of this linear map. Subsequent formulas involving derivatives should be read in a similar way.

Differentiating (2.3) twice with respect to x at $x = p$ in directions $w, w_1 \in \mathbb{R}^2$ and taking into account (2.2) yields

$$d_{\tilde{v}} \Phi(d_p^3 f(v, w, w_1)) + d_{\tilde{v}}^2 \Phi(d_p^2 f(v, w), d_p^2 f(v, w_1)) = 0. \tag{2.5}$$

We fix the notation $\tilde{v} = d_p f(v)$ for the rest of the proof.

Let Σ, γ and $\tilde{\gamma}$ denote the unit spheres of Φ, φ_0 and $\Phi|_H$, respectively. That is,

$$\Sigma = \{v \in V : \Phi(v) = 1\},$$
$$\gamma = \{v \in \mathbb{R}^2 : \varphi_0(v) = 1\},$$
$$\tilde{\gamma} = \Sigma \cap H = d_p f(\gamma).$$

Note that γ and $\tilde{\gamma}$ are smooth strictly convex curves enclosing the origin in their respective planes. For $v \in \gamma$, we denote by $\ell_v \subset \mathbb{R}^2$ the tangent direction of γ at v (that is, the 1-dimensional linear subspace of $\mathbb{R}^2$ parallel to the tangent to γ at v) and define $\tilde{\ell}_v = d_p f(\ell_v)$.

Lemma 2.1. *There exists a vector $\tau \in V \setminus H$ such that*

$$d_p^2 f(v, w) = S(v, w) \cdot \tau \tag{2.6}$$

for all $v, w \in \mathbb{R}^2$.

Proof. Let $v, w \in \gamma$ be linearly independent S-orthogonal vectors. We are going to show that $d_p^2(v, w) = 0$.

First assume that $\ell_v \neq \ell_w$. Let $\xi = d_p^2 f(v, w)$. Since $S(v, w) = 0$, we have $\xi \in H$ by the definition of the second fundamental form. By (2.4) we have $d_{\tilde{v}} \Phi(\xi) = 0$ where $\tilde{v} = d_p f(v)$. This means that ξ is a tangent vector to Σ at $\tilde{v}$. Since $\xi \in H$,

it follows that $\xi \in \tilde{\ell}_v$. Interchanging v and w yields that $\xi \in \tilde{\ell}_w$, hence $\xi \in \tilde{\ell}_v \cap \tilde{\ell}_w$. Since $\tilde{\ell}_v \neq \tilde{\ell}_w$, it follows that $\xi = 0$ as claimed.

Now consider the case $\ell_v = \ell_w$. In this case one can apply the above argument to the opposite unit vector $v' = \frac{-v}{\varphi_0(-v)}$ in place of v. Indeed, $\ell_{v'} \neq \ell_w$ since otherwise $\ell_v = \ell_w = \ell_{v'}$, contrary to the strict convexity of γ. Therefore $d_p^2 f(v', w) = 0$ and hence $d_p^2 f(v, w) = 0$.

Thus $d_p^2 f(v, w) = 0$ for all pairs of vectors $v, w \in \mathbb{R}^2$ that are linearly independent and S-orthogonal. To finish the proof we need the following fact, which is left as an exercise to the reader.

Fact. Let S and S_1 be symmetric bilinear forms on $\mathbb{R}^2$ such that $S_1(v, w) = 0$ for all pairs of linearly independent S-orthogonal vectors $v, w \in \mathbb{R}^2$. Then $S_1 = \lambda S$ for some $\lambda \in \mathbb{R}$.

Pick a basis e_1, e_2, e_3 of V such that $e_1, e_2 \in H$ and e_3 is mapped to $1 \in \mathbb{R}$ by the quotient map $\pi \colon V \to V/H \cong \mathbb{R}$ (recall that we have fixed an isomorphism between V/H and $\mathbb{R}$). Decompose $d_p^2 f$ into coordinate components with respect to this basis: $d_p^2 f = S_1 e_1 + S_2 e_2 + S_3 e_3$. Applying the above fact to S_1 and S_2 yields that $S_1 = \lambda_1 S$ and $S_2 = \lambda_2 S_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. And $S_3 = S$ by the choice of e_3. Thus (2.6) holds for $\tau = \lambda_1 e_1 + \lambda_2 e_2 + e_3$. $\qquad\square$

From now on we assume that $S \neq 0$ (otherwise the statement of Theorem 3 is trivial). Let τ be the vector from Lemma 2.1 and let $P \colon V \to H$ be the projector such that $P(\tau) = 0$. Substituting (2.6) into (2.4) yields

$$S(v, w) \cdot d_{\tilde{v}} \Phi(\tau) = 0.$$

Since $S \neq 0$, it follows that

$$(2.7) \qquad\qquad d_{\tilde{v}} \Phi(\tau) = 0$$

for almost all $\tilde{v} \in \tilde{\gamma}$ and hence for all $\tilde{v} \in \tilde{\gamma}$.

For $v \in \gamma$ define $\kappa(v) = d_{\tilde{v}}^2 \Phi(\tau, \tau)$. Due to strict convexity of Φ we have $\kappa(v) > 0$ for all $v \in \gamma$. Define a map $T \colon \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to H$ by $T = P \circ d_p^3 f$. This map is symmetric and linear in each argument. Due to (2.7), the first term in (2.5) equals

$$d_{\tilde{v}} \Phi(d_p^3 f(v, w, w_1)) = d_{\tilde{v}} \Phi(T(v, w, w_1)).$$

The second term in (2.5) equals

$$d_{\tilde{v}}^2 \Phi(d_p^2 f(v, w), d_p^2 f(v, w_1)) = \kappa(v) S(v, w) S(v, w_1)$$

by (2.6) and the definition of $\kappa(v)$. These identities and (2.6) imply that

$$(2.8) \qquad\qquad d_{\tilde{v}} \Phi(T(v, w, w_1)) + \kappa(v) S(v, w) S(v, w_1) = 0$$

for all $v, w, w_1 \in \mathbb{R}^2$.

Lemma 2.2. *If $v, w \in \gamma$ are linearly independent and S-orthogonal, then*

$$T(v, w, w_1) = 0$$

for all $w_1 \in \mathbb{R}^2$.

Proof. Similarly to the proof of Lemma 2.1 we may assume that $\ell_v \neq \ell_w$, otherwise replace v by $v' = \frac{-v}{\varphi_0(-v)}$. Since $S(v, w) = 0$, the second term in (2.8) vanishes, hence $d_{\tilde{v}} \Phi(T(v, w, w_1)) = 0$. Since $T(v, w, w_1) \in H$, it follows that $T(v, w, w_1) \in \tilde{\ell}_v$.

Interchanging v and w yields that $T(v, w, w_1) = T(w, v, w_1) \in \tilde{\ell}_w$. Since $\tilde{\ell}_v \neq \tilde{\ell}_w$, it follows that $T(v, w, w_1) \in \tilde{\ell}_v \cap \tilde{\ell}_w = \{0\}$. $\qquad\square$

Fix v and w as in Lemma 2.2 and let w_1 range over $\mathbb{R}^2$. By Lemma 2.2 the first term in (2.8) vanishes. Since $\kappa(v) \neq 0$, it follows that

$$S(v, w_1)S(w, w_1) = 0$$

for all $w_1 \in \mathbb{R}^2$. This implies that one of the linear functions $S(v, \cdot)$ and $S(w, \cdot)$ vanishes everywhere on $\mathbb{R}^2$. Equivalently, either v or w belong to the kernel of S. Therefore S is degenerate. This finishes the proof of Theorem 3.

3. Linearly equivalent sections

In this section we consider 3-dimensional Banach-Minkowski spaces with many linearly equivalent cross-sections. Our main goal is to prove the first claim of Theorem 1' in dimension 3. We restate it as the following proposition.

Proposition 3.1. *Let $V = (V^3, \Phi)$ be a Banach-Minkowski space and $\mathcal{U} \subset \mathrm{Gr}_2(V)$ an open set such that all normed planes $(H, \Phi|_H)$, $H \in \mathcal{U}$, are isometric. Then $\Phi|_H$ is a Euclidean norm for every $H \in \mathcal{U}$.*

Theorem 1 follows from Proposition 3.1 and Lemma 4.1; see the next section. In this section we prove Proposition 3.1 and deduce Theorem 2 from it. The proof of Proposition 3.1 is based on differential-geometric analysis of Φ near a plane $H \in \mathcal{U}$. The key result of this analysis is Lemma 3.5.

Let (V, Φ) and $\mathcal{U}$ be as in Proposition 3.1. Fix $H \in \mathcal{U}$ and a vector $\nu \in V \setminus H$. First we define a convenient local parametrization of $\mathrm{Gr}_2(V)$. Namely a neighborhood of H in $\mathrm{Gr}_2(V)$ is parametrized by a neighborhood of 0 in the dual space H^* as follows. For $\alpha \in H^*$ let $\pi_\alpha \colon V \to \mathbb{R}$ be the unique linear function such that $\pi_\alpha|_H = \alpha$ and $\pi_\alpha(\nu) = 1$, and let $H_\alpha = \ker \pi_\alpha$. Then $H_0 = H$ and the map $\alpha \mapsto H_\alpha$ is a diffeomorphism from H^* onto the set of all planes from $\mathrm{Gr}_2(V)$ that do not contain ν. We restrict this diffeomorphism to a neighborhood U of 0 in H^* such that $H_\alpha \in \mathcal{U}$ for all $\alpha \in U$. Throughout the proof we keep replacing this neighborhood by smaller ones but use the same notation U for all these neighborhoods.

Remark 3.2. Let x, y, z be coordinates in V associated to a basis e_1, e_2, e_3 where $e_1, e_2 \in H$ and $e_3 = \nu$. Then the map $\alpha \mapsto H_\alpha$ can be described as follows: H_α is the plane defined by the equation $ax + by + z = 0$ where (a, b) are the coordinates of $\alpha \in H^*$ with respect to the dual basis.

The map $\alpha \mapsto \pi_\alpha$ is affine. Its linear component is the map $I \colon H^* \to V^*$ defined by the relations $I(\alpha)|_H = \alpha$ and $I(\alpha)(\nu) = 0$ for all $\alpha \in H^*$. We will need the following calculation: If $t \mapsto \alpha(t)$ is a smooth path in H^* with $\alpha(0) = 0$ and $\dot{\alpha}(0) = h$, and $t \mapsto v(t)$ is a smooth path in V with $v(0) = v_0 \in H$, then

$$(3.1) \qquad \frac{d}{dt}\Big|_{t=0} \pi_{\alpha(t)}(v(t)) = \pi_0(\dot{v}(0)) + I(\dot{\alpha}(0))(v(0)) = \pi_0(\dot{v}(0)) + h(v_0).$$

Let φ_α denote the restriction of Φ to H_α. Similarly to the previous section we denote by Σ the unit sphere of Φ, by γ the unit circle of (H, φ_0), and by ℓ_v the tangent direction to γ at $v \in \gamma$. We extend the notation ℓ_v to all $v \in H \setminus \{0\}$ by homogeneity: $\ell_{\lambda v} = \ell_v$ for all $\lambda > 0$. For vector spaces X and Y we denote by $\mathrm{Lin}(X, Y)$ the space of all linear maps from X to Y.

Our assumptions imply that $(H_\alpha, \varphi_\alpha)$ is isometric to (H, φ_0) for all $\alpha \in U$. Arguing by contradiction, suppose that the norm φ_0 is not Euclidean. We need the following technical lemma.

Lemma 3.3. *If U is sufficiently small, then there exists a smooth family $\{L_\alpha\}_{\alpha \in U}$ of linear maps from H to V such that $L_0 = \mathrm{id}_H$, $L_\alpha(H) = H_\alpha$, and L_α is an isometry from (H, φ_0) to $(H_\alpha, \varphi_\alpha)$ for each $\alpha \in U$.*

This lemma follows from the 2-dimensional case of the main result of [3]. To keep the argument self-contained, we give a short proof here.

Proof of Lemma 3.3. Since φ_0 is not Euclidean, the group of self-isometries of (H, φ_0) is discrete. It follows that (possibly for a smaller neighborhood U) there is a unique continuous family $\{L_\alpha\}_{\alpha \in U}$ of isometries $L_\alpha \colon (H, \varphi_0) \to (H_\alpha, \varphi_\alpha)$ such that $L_0 = \mathrm{id}_H$. We have to prove that this family is smooth.

It suffices to show that the set $\{L_\alpha\}$ is a 2-dimensional smooth submanifold of $\mathrm{Lin}(H, V)$ and this submanifold is transverse to $\mathrm{Lin}(H, H)$. Let us show that the maps L_α are solutions of four equations $\{f_i = 0\}_{i=1}^4$ where the functions $f_i \colon \mathrm{Lin}(H, V) \to \mathbb{R}$ are smooth in a neighborhood of L_0 and satisfy

$$(3.2) \qquad \bigcap_{i=1}^4 \ker(d_{L_0} f_i) \cap \mathrm{Lin}(H, H) = \{0\}.$$

The desired fact follows from the existence of such f_i's and the implicit function theorem.

We define the functions f_i by

$$(3.3) \qquad f_i(L) = \Phi(L(v_i)) - 1, \qquad L \in \mathrm{Lin}(H, V),$$

for suitably chosen vectors $v_1, v_2, v_3, v_4 \in \gamma$. Such functions satisfy $f_i(L_\alpha) = 0$ since L_α are isometries. They are smooth in a neighborhood of L_0 since Φ is smooth outside 0. It remains to prove that one can choose $v_1, v_2, v_3, v_4 \in H$ so that the maps f_i defined by (3.3) satisfy (3.2).

Suppose the contrary. Fix $v_1, v_2, v_3 \in \gamma$ such that they are pairwise linearly independent and the lines ℓ_{v_i}, $i = 1, 2, 3$, are pairwise different. Let v_4 range over γ. If $A \in \mathrm{Lin}(H, H)$ belongs to the kernel of $d_{L_0} f_i$, then $d_{v_i} \varphi_0(A(v_i)) = 0$ and hence $A(v_i) \in \ell_{v_i}$. Observe that the conditions $A(v_i) \in \ell_{v_i}$ for $i = 1, 2, 3$ determine a linear map A uniquely up to a scalar factor. Hence there exists a non-zero $A \in \mathrm{Lin}(H, H)$ such that $d_v \varphi_0(A(v)) = 0$ for all $v = v_4 \in \gamma$ and hence for all $v \in H$. This implies that φ_0 is preserved by the flow generated by the linear vector field $v \mapsto A(v)$ on H. Thus (H, φ_0) admits a one-parameter group of self-isometries, a contradiction. $\square$

Let $\{L_\alpha\}$ be the family from Lemma 3.3. Let R denote the differential of this family at 0, that is, $R \colon H^* \to \mathrm{Lin}(H, V)$ is a linear map given by $R = d_0(\alpha \mapsto L_\alpha)$.

Let $\alpha \in U$, $h \in H^*$ and $v \in H$. Since $L_\alpha(H) = H_\alpha$, we have $\pi_\alpha(L_\alpha(v)) = 0$. Differentiating this identity with respect to α at $\alpha = 0$ in the direction h and taking into account (3.1) we obtain

$$(3.4) \qquad \pi_0(R(h)(v)) + h(v) = 0.$$

Since L_α is an isometry, we have $\Phi(L_\alpha(v)) = \varphi_0(v)$. Differentiating this identity with respect to α at $\alpha = 0$ in the direction h yields

$$(3.5) \qquad d_v \Phi(R(h)(v)) = 0$$

provided that $v \neq 0$.

Lemma 3.4. *If $h \in H^*$ and $v \in \ker h$, then $R(h)(v) = 0$.*

Proof. If $v \in \ker h$, then the second term in (3.4) vanishes and therefore

$$R(h)(v) \in \ker \pi_0 = H.$$

This and (3.5) imply that

$$(3.6) \qquad R(h)(v) \in \ell_v \quad \text{for all } h \in H^* \text{ and } v \in \ker h \setminus \{0\}.$$

Pick a linear isomorphism $J \colon H \to H^*$ such that $v \in \ker J(v)$ for all $v \in H$. (An example of such J is given by $J(x, y) = (-y, x)$ in coordinates from Remark 3.2.) Define

$$Q(v) = R(J(v))(v).$$

The map $v \mapsto Q(v)$ is an H-valued quadratic form on H. We regard Q as a vector field on H. By (3.6), Q is tangent to level sets of φ_0.

We are to show that $Q = 0$. First observe that Q vanishes at some point $v_0 \in \gamma$ (and hence on the entire line generated by v_0). Indeed, if this is not the case, then Q has a constant orientation along γ (i.e., it is directed either "clockwise" or "counterclockwise" everywhere). On the other hand, for opposite points $v \in \gamma$ and $v' = \frac{-v}{\varphi_0(-v)}$ the vectors $Q(v)$ and $Q(v')$ are positively proportional due to quadratic homogeneity of Q, hence they have opposite orientations on γ, a contradiction.

Since Q is quadratic and has a line of zeros, it can be decomposed into a product

$$Q(v) = f(v)W(v),$$

where $f \colon H \to \mathbb{R}$ is a linear function and W is a linear vector field. The trajectories of W are contained in level sets of φ_0 and hence bounded. Since W is linear, it follows that either $W = 0$ or these trajectories are ellipses centered at 0. The latter contradicts our standing assumption that φ_0 is not a Euclidean norm. Thus $W = 0$ and hence $Q = 0$.

Thus $R(J(v))(v) = 0$ for all $v \in H$. To finish the proof of the lemma, observe that if $h \in H^*$ and $v \in \ker h \setminus \{0\}$, then h is a scalar multiple of $J(v)$. $\square$

Lemma 3.4 implies the following strong property of Φ (compare with (2.7)).

Lemma 3.5. *There exists a vector $\tau \in V \setminus \{0\}$ which is tangent to Σ at every point of $\gamma = H \cap \Sigma$.*

Proof. Pick a basis e_1, e_2, e_3 of V and decompose R into coordinate components: $R = R_1 e_1 + R_2 e_2 + R_3 e_3$ where $R_i \in \mathrm{Lin}(H^*, H^*)$ for $i = 1, 2, 3$. By Lemma 3.4 we have $\ker R_i(h) \supset \ker h$ and hence $R_i(h)$ is a scalar multiple of h for every $h \in H^*$. Therefore $R_i = \lambda_i \, \mathrm{id}_{H^*}$ for some $\lambda_i \in \mathbb{R}$. Equivalently,

$$(3.7) \qquad R_i(h)(v) = \lambda_i h(v)$$

for all $h \in H^*$ and $v \in H$. Define $\tau = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$. By (3.7) we have

$$(3.8) \qquad R(h)(v) = h(v) \cdot \tau$$

for all $h \in H^*$ and $v \in H$.

Let $v \in \gamma$ and pick $h \in H^*$ such that $h(v) \neq 0$. Substituting (3.8) into (3.4) and dividing by $h(v)$ yields that $\pi_0(\tau) + 1 = 0$, therefore $\tau \neq 0$. Substituting (3.8) into (3.5) and dividing by $h(v)$ yields $d_v\Phi(\tau) = 0$, hence τ is tangent to Σ at v. $\square$

The statement of Lemma 3.5 is equivalent to the property that there exists a norm non-increasing linear projector from V to H. If this property held for *all* planes $H \subset V$, then the Blaschke-Kakutani characterization of ellipsoids ([11]; see also [13, §3.4] or [10, §12.3]) would imply that Φ is a Euclidean norm. The next proposition generalizes the Blaschke-Kakutani characterization to our localized setting. It is independent of the previous arguments.

Proposition 3.6. *Let (V^3, Φ) be a Banach-Minkowski space and let Σ be its unit sphere. Suppose that $\mathcal{U} \subset \mathrm{Gr}_2(V)$ is an open set such that for every $H \in \mathcal{U}$ there exists a vector $\tau = \tau_H \in V \setminus \{0\}$ which is tangent to Σ at every point of $H \cap \Sigma$. Then $\Phi|_H$ is a Euclidean norm for every $H \in \mathcal{U}$.*

Proof. Let $v \in H \cap \Sigma$ for some $H \in \mathcal{U}$ and let $v' = -v/\Phi(-v)$ be the opposite Φ-unit vector. Since $\mathcal{U}$ is open, there exists $H_1 \in \mathcal{U}$ such that $v \in H_1$ and $H_1 \neq H$. The strict convexity of Φ implies that τ_H and τ_{H_1} are linearly independent vectors. Both tangent planes to Σ at v and v' contain these two vectors, hence these two tangent planes are parallel.

Let $U_1 = \bigcup_{H \in \mathcal{U}} \mathrm{Gr}_1(H)$, then U_1 is an open subset of $\mathrm{Gr}_1(V)$. Define a map $f \colon U_1 \to \mathrm{Gr}_2(V)$ as follows. For a line $\ell \in U$ let $f(\ell)$ be the direction of the tangent plane to Σ at a point $v \in \ell \cap \Sigma$. By the above argument, this direction is the same for v and the opposite vector v', thus f is well-defined. The strict convexity of Φ implies that f is a diffeomorphism from U_1 onto an open subset of $\mathrm{Gr}_2(V)$.

We regard the Grassmannians $\mathrm{Gr}_1(V)$ and $\mathrm{Gr}_2(V)$ as projective planes. Namely $\mathrm{Gr}_1(V)$ is the projectivization of V and $\mathrm{Gr}_2(V)$ is naturally identified with the projectivization of V^*: to each linear function from $V^* \setminus \{0\}$ one associates its kernel. Each plane $H \in \mathcal{U}$ represents a line in $\mathrm{Gr}_1(V)$, and its f-image is a line in $\mathrm{Gr}_2(V)$ consisting of all planes that contain τ_H.

Thus f is a diffeomorphism between subsets of projective planes, it is defined on the union of an open set of lines, and it maps each of these lines to a line. By the local variant of the fundamental theorem of projective geometry, these properties imply that f is a restriction of a projective map. (In fact, it suffices to assume that there are four independent families of lines that are mapped to lines; see [12].) Thus there exists a linear map $F \colon V \to V^*$ such that f is a restriction of the projectivization of F.

Fix $H \in \mathcal{U}$ and let $\gamma = H \cap \Sigma$. Let $F_H \colon H \to H^*$ be the map given by $F_H(v) = F(v)|_H$ for all $v \in H$. By construction of F for every $v \in \gamma$ the line $\ker F_H(v)$ is parallel to the tangent line to γ at v. Let $J \colon H^* \to H$ be a linear isomorphism which sends every $\alpha \in H^*$ to a vector from its kernel (cf. the proof of Lemma 3.4). Then the map $W = J \circ F_H \colon H \to H$ defines a non-trivial linear vector field on H and γ is tangent to this vector field. As in the proof of Lemma 3.4, it follows that γ is an ellipse centered at 0 and hence $\Phi|_H$ is a Euclidean norm. $\quad \square$

Now we finish the proof of Proposition 3.1 and deduce Theorem 2.

Proof of Proposition 3.1. Let $\mathcal{U} \subset \mathrm{Gr}_2(V)$ be as in Proposition 3.1. In the set-up preceding Lemma 3.5, H was an arbitrary plane from $\mathcal{U}$, hence Lemma 3.5 applies to all planes $H \in \mathcal{U}$. Thus the assumptions of Proposition 3.6 are satisfied and it implies that $\Phi|_H$ is a Euclidean norm for every $H \in \mathcal{U}$. $\quad \square$

Proof of Theorem 2. Let $M = (M^2, \varphi)$ be a monochromatic Finsler manifold, let $V = (V^3, \Phi)$ be a Banach-Minkowski space, and let $f \colon M \to V$ be an isometric embedding. Let $G \colon M \to \mathrm{Gr}_2(V)$ be the Gauss map of f defined by $G(x) = \mathrm{Im}\, d_x f$ for $x \in M$. If the second fundamental form of f is non-degenerate at $p \in M$, then the derivative of G at p is non-degenerate and hence the image of G contains a neighborhood of $G(p)$ in $\mathrm{Gr}_2(V)$. Since φ is monochromatic, this implies that the restrictions of Φ to all planes from this neighborhood are isometric. By Proposition 3.1 this implies that the norm $\varphi|_{T_p M}$ is Euclidean and hence M is Riemannian. $\square$

Remark 3.7. Conversely, Theorem 2 easily implies Proposition 3.1. Indeed, if (V, Φ) and $\mathcal{U}$ are as in Proposition 3.1, then for any surface $M \subset V$ whose tangent planes all belong to $\mathcal{U}$, the induced Finsler metric on M is monochromatic. Suppose that Proposition 3.1 fails and let x, y, z be coordinates in V such that $\mathcal{U}$ contains the plane $\{z = 0\}$. Then a small neighborhood of 0 in the surface $\{z = x^2 + y^2\}$ is a counterexample to Theorem 2.

4. Norms with many Euclidean sections

In this section we finish the proof of Theorem 1' and consider isometric embeddings of Riemannian 2-manifolds into 3-dimensional Banach-Minkowski spaces. It turns out that they are essentially no different from isometric embeddings into Euclidean spaces; see Proposition 4.3. Both results are based on the following lemma which localizes the well-known fact that a normed space is Euclidean if all its 2-dimensional subspaces are.

Lemma 4.1. *Let $V = (V^3, \Phi)$ be a Banach-Minkwoski space and let $\Gamma \subset \mathrm{Gr}_2(V)$ be a set of planes such that the set $\bigcup \Gamma := \bigcup_{H \in \Gamma} H \subset V$ has a non-empty interior. Suppose that for every $H \in \Gamma$ the norm $\Phi|_H$ is Euclidean. Then there exists a Euclidean norm on V whose restriction to every plane from Γ coincides with the restriction of Φ.*

Remark 4.2. Without the assumption that $\bigcup \Gamma$ has a non-empty interior a weaker conclusion holds: There exists a quadratic form Q on V (not necessarily positive definite) such that $Q|_H = \Phi^2|_H$ for every plane $H \in \Gamma$. This can be seen from the proof of Lemma 4.1.

Proof of Lemma 4.1. Consider two cases.

Case 1: There is a line $\ell \in \mathrm{Gr}_1(V)$ contained in at least 3 planes from Γ. Fix such a line ℓ and divide Γ into subsets Γ_0 and Γ_1 where the planes from Γ_0 contain ℓ and those from Γ_1 do not. Pick a vector $v \in \ell \setminus \{0\}$ and define a quadratic form Q on V by $Q(x) = \frac{1}{2} d_v^2 (\Phi^2)(x, x)$ for all $x \in V$. Since Φ^2 is strictly convex, Q is positive definite and hence it is a square of some Euclidean norm. If Φ itself is Euclidean, then $Q = \Phi^2$. This observation applied to the restriction of Φ to a plane $H \in \Gamma_0$ implies that $Q|_H = \Phi^2|_H$ for every such plane.

Now consider a plane $H \in \Gamma_1$. The restriction of Φ^2 to H is a quadratic form which equals Q on at least three lines through the origin (these lines are the intersections of H with the planes from Γ_0). Since a quadratic form on the plane is uniquely determined by its values at three pairwise linearly independent vectors, it follows that $\Phi^2|_H = Q|_H$. This proves the lemma under the assumption of Case 1.

Case 2: *No three planes from* Γ *have a common line.* Pick $H_0 \in \Gamma$ and a vector $v_0 \in V \setminus H_0$ such that $\Phi(v_0) \leq 1$ and $\Phi(-v_0) \leq 1$. Consider the affine plane $H_1 = H_0 + v_0$. Every plane from Γ except H_0 intersects H_1 by an affine line; let Λ denote the set of all such lines. Since Φ is a Euclidean norm on every plane from Γ, Φ^2 is a quadratic polynomial on every line from Λ. By the assumption of Case 2, no two lines from Λ are parallel and no three of them have a common point.

Define a function $F \colon H_1 \to \mathbb{R}$ by

$$F(v) = \Phi^2(v) - \Phi^2(v - v_0), \qquad v \in H_1.$$

The restriction of F to any line from Λ is the difference of two quadratic polynomials and hence a polynomial of degree at most 2. In fact, its degree is no greater than 1. Indeed, the triangle inequality for Φ implies that for every $v \in H_1$,

$$|\Phi(v) - \Phi(v - v_0)| \leq \max\{\Phi(v_0), \Phi(-v_0)\} \leq 1,$$

and therefore

$$|F(v)| = |\Phi(v) - \Phi(v - v_0)| \cdot |\Phi(v) + \Phi(v - v_0)| \leq 2\Phi(v) + 1.$$

This implies that $|F|$ has at most linear growth at infinity, hence its restriction to a line cannot be a degree 2 polynomial.

Thus the restriction of F to any line from Λ is an affine function. Pick two lines $\ell_1, \ell_2 \in \Lambda$. Since they are not parallel, there exist an affine function $\tilde{F} \colon H_1 \to \mathbb{R}$ such that $\tilde{F}|_{\ell_1 \cup \ell_2} = F|_{\ell_1 \cup \ell_2}$. Every line $\ell \in \Lambda \setminus \{\ell_1, \ell_2\}$ intersects $\ell_1 \cup \ell_2$ at two distinct points. An affine function on ℓ is uniquely determined by its values at these two points, hence $\tilde{F}|_\ell = F|_\ell$. Since $\Phi^2|_{H_0}$ is a quadratic form and $\tilde{F}$ is affine, there is a quadratic form $Q \colon V \to \mathbb{R}$ such that $Q|_{H_0} = \Phi^2|_{H_0}$ and $Q(v) = Q(v - v_0) + \tilde{F}(v)$ for all $v \in H_1$. If $v \in \ell \in \Lambda$, then $Q(v) = \Phi^2(v - v_0) + F(v) = \Phi^2(v)$ since $\tilde{F}|_\ell = F|_\ell$ as shown above. By homogeneity it follows that $Q = \Phi^2$ on $\bigcup \Gamma$.

It remains to prove that Q is positive definite. Suppose the contrary and choose $v, w \in V \setminus \{0\}$ such that v is an interior point of $\bigcup \Gamma$ and $Q(w) \leq 0$. Then

$$d_v^2(\Phi^2)(w, w) = (d_v^2 Q)(w, w) = 2Q(w) \leq 0,$$

contrary to the strict convexity of Φ^2. This finishes the proof of Lemma 4.1. $\qquad\square$

Proof of Theorem 1'. Let (V, Φ) and $\mathcal{U}$ be as in Theorem 1', and let $n = \dim V$. First consider the case $n = 3$. By Proposition 3.1 the norm $\Phi|_H$ is Euclidean for every $H \in \mathcal{U}$. Since $\mathcal{U}$ is an open set of planes, $\bigcup \mathcal{U} \setminus \{0\}$ is an open set in V. Hence Lemma 4.1 applied to $\Gamma = \mathcal{U}$ implies the last claim of the theorem. This proves the theorem in dimension 3.

If $n > 3$, one can apply this to any 3-dimensional subspace of V containing at least one plane from $\mathcal{U}$. This implies that $\Phi|_H$ is a Euclidean norm for every $H \in \mathcal{U}$. It remains to prove that there is a Euclidean norm on V extending all these 2-dimensional norms. Fix $v \in \bigcup \mathcal{U} \setminus \{0\}$ and apply the 3-dimensional case to all 3-dimensional subspaces containing v and a fixed plane $H \in \mathcal{U}$ containing v. This yields a Euclidean norm on each of these subspaces. Similarly to the proof of Lemma 4.1, the squares of these norms are restrictions of the quadratic form Q_v defined by $Q_v(x) = \frac{1}{2} d_v^2(\Phi^2)(x, x)$, $x \in V$. Thus $\Phi^2 = Q_v$ in a neighborhood of v. A quadratic form Q_v with this property obviously cannot change if v varies continuously. Since $\mathcal{U}$ is connected, it follows that Q_v is the same quadratic form for all $v \in \bigcup \mathcal{U} \setminus \{0\}$. The square root of this quadratic form is the desired Euclidean norm on V. $\qquad\square$

Now we return to isometric embeddings.

Proposition 4.3. *Let M be a connected Riemannian 2-manifold, let $V = (V^3, \Phi)$ be a Banach-Minkowski space, and let $f \colon M \to V$ be an isometric immersion. Then there exists a Euclidean norm on V such that f is isometric with respect to this norm.*

Proof. Define $\Gamma \subset \mathrm{Gr}_2(V)$ by $\Gamma = \{\mathrm{Im}\, d_p f : p \in M\}$. For every plane $H \in \Gamma$ the restriction $\Phi|_H$ is a Euclidean norm, and we need to prove that there is a Euclidean norm on V extending all these 2-dimensional norms.

If the second fundamental form of f vanishes everywhere, then Γ consists of one plane and the result is trivial. If the second fundamental form at some point $p \in M$ and some vector $v \in T_p M$ is non-zero, then $d_p f(v)$ is an interior point of $\bigcup \Gamma$. In this case the result follows from Lemma 4.1. $\square$

The next corollary asserts that the answer to Question 1.4 is affirmative for all monochromatic metrics.

Corollary 4.4. *Let M^2 be a monochromatic (possibly Riemannian) Finsler manifold, $p \in M$, and $f_i \colon M \to V_i$, $i = 1, 2$, smooth isometric embeddings where V_1 and V_2 are 3-dimensional Banach-Minkowski spaces. Then the second fundamental forms of f_i at p are of the same type: either both are definite, or both are semidefinite, or both are degenerate.*

Proof. If the metric of M is not Riemannian, then by Theorem 2 both second fundamental forms are degenerate. If the metric is Riemannian, then by Proposition 4.3 the norms of V_1 and V_2 can be replaced by Euclidean ones. Then by Gauss' Theorem the type of the second fundamental form is determined by the sign of the Riemannian curvature at p. $\square$

5. A NON-EMBEDDABLE EXAMPLE

The following example describes a class of Finsler metrics on $\mathbb{R}^2$ that do not admit local isometric embeddings to 3-dimensional Banach-Minkowski spaces. The proof of non-embeddability is given in Proposition 5.2.

Example 5.1. Let φ_0 be a Banach-Minkowski norm on $\mathbb{R}^2$ such that it is not Euclidean and moreover there in no open set of $\mathbb{R}^2$ where the restriction of φ_0 equals the restriction of a Euclidean norm.

For $\theta \in \mathbb{R}$, let R^θ denote the rotation of $\mathbb{R}^2$ by angle θ:
$$R^\theta(\xi, \eta) = (\cos\theta \cdot \xi - \sin\theta \cdot \eta, \sin\theta \cdot \xi + \cos\theta \cdot \eta).$$
Define a Finsler metric φ on $\mathbb{R}^2$ by
$$\varphi(x, y, \xi, \eta) = \varphi_0(R^y(\xi, \eta)),$$
where (x, y) and (ξ, η) are coordinates of a point in $\mathbb{R}^2$ and a tangent vector in $T_{(x,y)}\mathbb{R}^2 \cong \mathbb{R}^2$, resp.

Obviously norms φ_0 satisfying the above requirements do exist. For an explicit example, one can take $\varphi_0(\xi, \eta) = \sqrt{\xi^2 + \eta^2} + \sqrt{\xi^2 + 2\eta^2}$ (or, in fact, any analytic formula that defines a non-Euclidean norm).

Proposition 5.2. *Let φ be as in Example 5.1. Then no open subset of $(\mathbb{R}^2, \varphi)$ admits a smooth isometric embedding to a 3-dimensional Banach-Minkowski space.*

Proof. Arguing by contradiction, let $U \subset \mathbb{R}^2$ be an open set, Φ is a Banach-Minkowski norm in $\mathbb{R}^3$ and assume that $f \colon (U, \varphi) \to (\mathbb{R}^3, \Phi)$ is a smooth isometric embedding. Define $M = f(U)$.

By construction φ is monochromatic. Hence by Theorem 2 the second fundamental form of f is degenerate everywhere. Therefore f, regarded as a surface in the Euclidean $\mathbb{R}^3$, is a developable surface. It is well-known that every developable surface in $\mathbb{R}^3$ is a ruled surface, i.e., it is the union of straight line segments; see, e.g., [7, §5-8]. Furthermore the tangent planes to the surface are constant along every such segment.

Consider one of such segments in M parametrized with a constant speed. It can be written as $f(\gamma(t))$ where $\gamma = \gamma(t)$, $t \in (-\varepsilon, \varepsilon)$, is a smooth curve in U. Since the segment is a length-minimizing curve in $(\mathbb{R}^3, \Phi)$, γ is a Finsler geodesic in (U, φ). Let $\gamma(t) = (x(t), y(t))$. Since φ_0 is not a Euclidean norm, its group of self-isometries is discrete. Hence there is a unique continuous family of linear maps $I_t \colon T_{\gamma(0)}U \to T_{\gamma(t)}U$ such that I_0 is the identity and each I_t is an isometry between the Finsler norms at $\gamma(0)$ and $\gamma(t)$. In the standard coordinates this family is given by $I_t = R^{-y(t)}$.

Recall that the tangent plane $T_{f(\gamma(t))}M = \operatorname{Im} d_{\gamma(t)}f$ is constant along the segment. Due to their uniqueness, the isometries I_t correspond to the identity map of this plane, i.e.,

$$(5.1) \qquad\qquad d_{\gamma(t)}f \circ I_t = d_{\gamma(0)}f$$

for all t. Since $f \circ \gamma$ is a constant-speed straight line segment, its velocity vector $\frac{d}{dt}f(\gamma(t)) = d_{\gamma(t)}f(\dot{\gamma}(t))$ does not depend on t. This and (5.1) imply that

$$d_{\gamma(t)}f(\dot{\gamma}(t)) = d_{\gamma(0)}f(\dot{\gamma}(0)) = d_{\gamma(t)}f(I_t(\dot{\gamma}(0)))$$

and therefore

$$(5.2) \qquad\qquad \dot{\gamma}(t) = I_t(\dot{\gamma}(0)) = R^{-y(t)}(\dot{\gamma}(0)).$$

The Euler-Lagrange equations for the geodesic γ are

$$(5.3) \qquad \begin{cases} \frac{d}{dt}\frac{\partial\varphi}{\partial\xi}(\gamma(t), \dot{\gamma}(t)) = \frac{\partial\varphi}{\partial x}(\gamma(t), \dot{\gamma}(t)), \\ \frac{d}{dt}\frac{\partial\varphi}{\partial\eta}(\gamma(t), \dot{\gamma}(t)) = \frac{\partial\varphi}{\partial y}(\gamma(t), \dot{\gamma}(t)), \end{cases}$$

where φ is regarded as a function of variables x, y, ξ, η as in Example 5.1. Since φ is preserved by translations along the x-axis, the first equation in (5.3) imply that $\frac{\partial\varphi}{\partial\xi}(\gamma(t), \dot{\gamma}(t))$ is constant. (One can also see this constant as the Noether integral associated with the group of horizontal translations.) Observe that

$$\frac{\partial\varphi}{\partial\xi}(\gamma(t), \dot{\gamma}(t)) = d_{\dot{\gamma}(t)}(\varphi_0 \circ R^{y(t)})(e_1) = (d_{\dot{\gamma}(0)}\varphi_0)(R^{y(t)}(e_1))$$

due to (5.2), where e_1 is the first coordinate vector of $\mathbb{R}^2$. Since $d_{\dot{\gamma}(0)}\varphi_0 \colon \mathbb{R}^2 \to \mathbb{R}$ is a non-trivial linear map and $R^{y(t)}(e_1)$ belongs to the Euclidean unit circle, it follows that $R^{y(t)}(e_1)$ is constant. Hence $y(t)$ is constant.

Thus γ is a (constant-speed parametrized) horizontal segment in $U \subset \mathbb{R}^2$. Since M contains a segment through every point, it follows that all horizontal segments in U are geodesics of φ. Then the second equation in (5.3) implies that

$$\frac{\partial\varphi}{\partial y}(x, y, 0, 1) = 0$$

for all $(x, y) \in U$. Since φ does not depend on x, it follows that the value

$$\varphi(x, y, 0, 1) = \varphi_0(R^y(e_1)), \qquad (x, y) \in U,$$

is constant. Thus φ_0 is constant on an arc of the Euclidean unit circle and hence it is proportional to the standard Euclidean norm on an open subset of $\mathbb{R}^2$. This contradicts the requirements of Example 5.1. □

Acknowledgments

References

[1] H. Auerbach, S. Mazur, and S. Ulam, *Sur une propriété caractéristique de l'ellipsoïde* (French), Monatsh. Math. Phys. **42** (1935), no. 1, 45–48, DOI 10.1007/BF01733278. MR1550413

[2] S. Banach, *Théorie des opérations linéaires*, Monografie Matematyczne, vol. 1, Warszawa, 1932.

[3] N. Bartelmeß and V. S. Matveev, *Monochromatic metrics are generalized Berwald*, preprint, arXiv:1705.05721, to appear in J. Diff. Geom. Appl.

[4] David Bao, *On two curvature-driven problems in Riemann-Finsler geometry*, Finsler geometry, Sapporo 2005—in memory of Makoto Matsumoto, Adv. Stud. Pure Math., vol. 48, Math. Soc. Japan, Tokyo, 2007, pp. 19–71. MR2389251

[5] D. Burago and S. Ivanov, *Isometric embeddings of Finsler manifolds* (Russian, with Russian summary), Algebra i Analiz **5** (1993), no. 1, 179–192; English transl., St. Petersburg Math. J. **5** (1994), no. 1, 159–169. MR1220494

[6] Dmitri Burago and Sergei Ivanov, *On intrinsic geometry of surfaces in normed spaces*, Geom. Topol. **15** (2011), no. 4, 2275–2298, DOI 10.2140/gt.2011.15.2275. MR2862157

[7] Manfredo P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1976. Translated from the Portuguese. MR0394451

[8] Richard J. Gardner, *Geometric tomography*, 2nd ed., Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, New York, 2006. MR2251886

[9] M. L. Gromov, *On a geometric hypothesis of Banach* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), 1105–1114. MR0217566

[10] Peter M. Gruber, *Convex and discrete geometry*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 336, Springer, Berlin, 2007. MR2335496

[11] S. Kakutani, *Some characterizations of Euclidean space*, Jap. J. Math. **16** (1939), 93–97. MR0000895

[12] Walter Prenowitz, *The characterization of plane collineations in terms of homologous families of lines*, Trans. Amer. Math. Soc. **38** (1935), no. 3, 564–599, DOI 10.2307/1989815. MR1501829

[13] A. C. Thompson, *Minkowski geometry*, Encyclopedia of Mathematics and its Applications, vol. 63, Cambridge University Press, Cambridge, 1996. MR1406315

[14] Zhongmin Shen, *On Finsler geometry of submanifolds*, Math. Ann. **311** (1998), no. 3, 549–576, DOI 10.1007/s002080050200. MR1637939

St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences, Fontanka 27, St. Petersburg 191023, Russia – and –Saint Petersburg State University, 7/9 Universitetskaya emb., St. Petersburg 199034, Russia

Email address: svivanov@pdmi.ras.ru