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THE GROMOV-HAUSDORFF HYPERSPACE OF NONNEGATIVELY CURVED 2-SPHERES

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ABSTRACT. We study topological properties of the Gromov-Hausdorff metric on the set of isometry classes of nonnegatively curved 2-spheres.

1. INTRODUCTION

The Gromov-Hausdorff (GH) distance is ubiquitous in studying families of Riemannian metrics with lower curvature bounds. The simplest scenario is when all the metrics in the family live on the same manifold. We call any set of isometry classes of metrics on closed C^∞ manifold N equipped with the GH distance a *GH hyperspace of* N.

A metric is *intrinsic* if the distance between any two points is the infimum of lengths of curves joining the points. Any C^∞ Riemannian metric is intrinsic, and this property is preserved under GH limits. For $\kappa \in \mathbb{R}$ let $\mathcal{M}^{\mathrm{GH}}_{\mathrm{curv}\ge\kappa}(N)$ be the GH hyperspace of intrinsic metrics of curvature $\ge \kappa$ on N. Let $\mathcal{M}^{\mathrm{GH}}_{\sec\ge\kappa}(N)$, $\mathcal{M}^{\mathrm{GH}}_{\sec>\kappa}(N)$ be the GH hyperspaces of C^∞ Riemannian metrics on N of sectional curvatures $\ge \kappa$, $> \kappa$, respectively. Topological properties of these GH hyperspaces are largely a mystery which is why it is more common to give $\mathcal{M}^{\mathrm{GH}}_{\sec>\kappa}(N)$ the C^∞ topology resulting in a stratified space whose strata are Hilbert manifolds [8].

Our starting point is that for $N = S^2$ and $\kappa = 0$ the above GH hyperspaces can be identified with the $O(3)$-quotients of certain hyperspaces of $\mathbb{R}^3$; see Theorem 1.1 below. This is made possible by the convex surface theory.

A *hyperspace of* $\mathbb{R}^3$ is a set of compacta of $\mathbb{R}^3$ equipped with the Hausdorff metric. A *convex body* is a compact convex set with nonempty interior. The boundary of any convex body in $\mathbb{R}^3$ inherits an intrinsic metric of nonnegative curvature, which we call the *boundary metric.* A metric that is isometric to the distance function of a C^∞ Riemannian metric is *intrinsically* C^∞*.* The *Steiner point* is a way to assign a center to any convex compactum in $\mathbb{R}^3$ that is continuous, Iso($\mathbb{R}^3$)-invariant, and Minkowski linear, and in fact, these properties characterize the Steiner point [18, Theorem 3.3.3]. We shall work with the following hyperspaces of $\mathbb{R}^3$:

$$\mathcal{K} = \{\text{convex compacta in } \mathbb{R}^3\},$$

$$\mathcal{K}_s = \{\text{convex compacta in } \mathbb{R}^3 \text{ with Steiner point at the origin}\},$$

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$$\mathcal{K}_s^{k\leq l} = \{D \in \mathcal{K}_s \text{ with } k \leq \dim(D) \leq l\},$$

$$\mathcal{B}_p = \{\text{convex bodies } D \in \mathcal{K}_s \text{ with } C^\infty \text{ boundary of } \sec > 0\},$$

$$\mathcal{B}_d = \{\text{convex bodies } D \in \mathcal{K}_s \text{ with intrinsically } C^\infty \text{ boundary metrics}\},$$

$$\mathcal{B}^{k,\alpha} = \{C^{k,\alpha} \text{ convex bodies in } \mathcal{K}_s\} \quad \text{and} \quad \mathcal{B}^k = \mathcal{B}^{k,0}.$$

One purpose of this paper is to give an exposition of fundamental (but not widely known) results of convex surface theory, which easily imply the following.

Theorem 1.1. *The map $\mathcal{K}_s^{2\leq 3}/O(3) \to \mathcal{M}^{\mathrm{GH}}_{\mathrm{curv}\geq 0}(S^2)$ that assigns to the congruence class of a convex compactum the isometry class of its boundary surface is a homeomorphism which restricts to homeomorphisms $\mathcal{B}_d/O(3) \to \mathcal{M}^{\mathrm{GH}}_{\sec\geq 0}(S^2)$ and $\mathcal{B}_p/O(3) \to \mathcal{M}^{\mathrm{GH}}_{\sec>0}(S^2)$.*

Here the *boundary surface of a* 2*-dimensional convex compactum* K is the double of K along the boundary with the induced intrinsic metric.

Consider the *Hilbert cube* $Q = [-1,1]^\omega$ and its *radial interior*

$$\Sigma = \{(t_i)_{i\in\omega} \text{ in } Q \,:\, \sup_{i\in\omega} |t_i| < 1\}.$$

Here ω is the set of nonnegative integers, and the superscript ω refers to the product of countably many copies of a space. We have a canonical inclusion $\Sigma^\omega \subset Q^\omega$. Note that Q^ω and Q are homeomorphic.

This paper is a sequel to [6] where the author used convex geometry and infinite dimensional topology to determine the homeomorphism types of $\mathcal{K}_s$, $\mathcal{K}_s^{2\leq 3}$, $\mathcal{B}_p$, and also derive a number of properties of their $O(3)$-quotients. In particular, in [6, Section 6] we isolated some conditions on a hyperspace $\mathcal{D}$ with $\mathcal{B}_p \subseteq \mathcal{D} \subset \mathcal{B}^{1,1}$ that give the conclusion of Theorem 1.2 below with $\mathcal{B}_d$ replaced by $\mathcal{D}$. The conditions hold, e.g., if $\mathcal{D} \setminus \mathcal{B}_p$ is σ-compact, which includes the case $\mathcal{D} = \mathcal{B}_p$. Here we verify the conditions for $\mathcal{D} = \mathcal{B}_d$.

Theorem 1.2. *If E is a subset of $Q^\omega \setminus \Sigma^\omega$ homeomorphic to suspension of the real projective plane, then there is a homeomorphism $h\colon \mathcal{K}_s^{2\leq 3} \to Q^\omega \setminus E$ with $h(\mathcal{B}_d) = \Sigma^\omega$.*

The new ingredient, stated in Theorem 1.3 below, follows from a version of the Cheeger-Gromov compactness theorem.

Theorem 1.3. *$\mathcal{M}^{\mathrm{GH}}_{\sec\geq 0}(S^2) \setminus \mathcal{M}^{\mathrm{GH}}_{\sec>0}(S^2)$ is an F_σ subset of $\mathcal{M}^{\mathrm{GH}}_{\sec\geq 0}(S^2)$ and also is a countable intersection of σ-compact sets.*

Theorem 1.2 together with results in [6] yield a number of topological properties for the quotients $\mathcal{K}_s/O(3)$, $\mathcal{B}_p/O(3)$, $\mathcal{B}_d/O(3)$, and hence for the corresponding GH hyperspaces, as summarized below.

Theorem 1.4. *Let $M = \mathcal{M}^{\mathrm{GH}}_{\mathrm{curv}\geq 0}(S^2)$ and M_0 be the GH hyperspace of the isometry classes in M represented by metrics with trivial isometry groups. Let X be $\mathcal{M}^{\mathrm{GH}}_{\sec\geq 0}(S^2)$ or $\mathcal{M}^{\mathrm{GH}}_{\sec>0}(S^2)$, and let $X_0 = X \cap M_0$. Then*

(1) *M is a locally compact Polish absolute retract.*
(2) *X is an absolute retract that is neither Polish nor locally compact.*
(3) *Any σ-compact subset of X has empty interior.*
(4) *X is homotopy dense in M, i.e., any continuous map $Q \to M$ can be uniformly approximated by a continuous map with image in X.*

(5) *M_0 is open in M.*
(6) *If L is the product of $[0,1)$ and any locally finite simplicial complex that is homotopy equivalent to $BO(3)$, then there is a homeomorphism $M_0 \to L \times Q^\omega$ that takes X_0 onto $L \times \Sigma^\omega$.*
(7) *The pairs (M_0, X_0) and $(Q^\omega, \Sigma^\omega)$ are locally homeomorphic, i.e., each point of M_0 has a neighborhood $U \subset M_0$ such that some open embedding $h\colon U \to Q^\omega$ takes $U \cap X_0$ onto $h(U) \cap \Sigma^\omega$.*
(8) *M_0, X_0 are dense but not homotopy dense in M, X, respectively.*
(9) *$\mathcal{M}^{\mathrm{GH}}_{\sec\geq\kappa}(S^2)$ and $\mathcal{M}^{\mathrm{GH}}_{\sec>\kappa}(S^2)$ are weakly contractible for every $\kappa > 0$.*

Let us supply some context for various items in Theorem 1.4 :

(1)–(2) We refer to [7] for background on absolute retracts (AR) and absolute neighborhood retracts (ANR), and only mention here some basic facts. Any open subset of an ANR is an ANR. Being an AR is equivalent to being a contractible ANR. Any ANR is locally contractible, i.e., any neighborhood U of every point contains a neighborhood V of the same point such that the inclusion $V \to U$ is null-homotopic. Any ANR is homotopy equivalent to a CW complex.

(4) Another definition of a homotopy dense subset $A \subset B$ is that there is a homotopy $h\colon B \times [0,1] \to B$ with $h(b,0) = b$ and $h(b,t) \in A$ for $t > 0$. The two definitions are equivalent when B is an ANR [5, Exercise 10 in Section 1.2].

(5)–(7) The Slice Theorem for compact Lie group actions [9, Corollary II.5.5] implies that M_0 is open in M and the restriction of the orbit map $\mathcal{K}^{2\leq 3}_s \to \mathcal{K}^{2\leq 3}_s/O(3)$ to the principal orbit $\mathring{\mathcal{K}}^{2\leq 3}_s$ is a principal $O(3)$-bundle whose base is homeomorphic to M_0. Similarly, X_0 is the base of a principal $O(3)$-bundle. By [6, Lemma 8.2] the principal orbit $\mathring{\mathcal{B}}_p$ for the $O(3)$-action on $\mathcal{K}_s$ is homotopy dense in $\mathcal{K}_s$, and hence the total spaces of the above principal bundles are contractible. Thus M_0, X_0 are homotopy equivalent to $BO(3)$, the Grassmanian of 3-planes in $\mathbb{R}^\omega$. The claims (6)–(7) follow from the main results of [6] and Theorem 1.2.

(8) has a curious interpretation that there is no continuous "destroy the symmetry map" that would instantly push M into M_0, or X into X_0.

(9) The contractibility of these GH hyperspaces follow from the contractibility of $\mathcal{M}^{\mathrm{GH}}_{\sec>0}(S^2)$ and a rescaling argument.

In [6] the reader can find a number of open questions about the above GH hyperspaces, disguised as $O(3)$-orbit spaces of hyperspaces of $\mathbb{R}^3$. For example, it is unknown whether $\mathcal{M}^{\mathrm{GH}}_{\mathrm{curv}\geq 0}(S^2)$ is a Q-manifold, which by Theorem 1.4 is equivalent to the following.

Question 1.5. Is $\mathcal{M}^{\mathrm{GH}}_{\mathrm{curv}\geq 0}(S^2)$ topologically homogeneous?

A space is *topologically homogeneous* if its homeomorphism group acts transitively.

Theorem 1.1 is proven in Section 2 while the other main results are justified in Section 3. In Section 4 we offer some remarks about the hyperspace $\mathcal{B}_d$ whose structure is still quite mysterious.

2. Spaces on convex surfaces

In this section we review some fundamental properties of convex surfaces and prove Theorem 1.1.

Two subsets of $\mathbb{R}^3$ are *δ-congruent* if some isometry of $\mathbb{R}^3$ takes one subset within the δ-neighborhood of the other one; if $\delta = 0$ we call the subsets *congruent.* A homeomorphism $f\colon (A, d_A) \to (B, d_B)$ of metric spaces is a *δ-isometry* if $|d_B(f(x), f(y)) - d_A(x, y)| < \delta$ for any $x, y \in A$. If δ is small we use the terms *nearly congruent* and *nearly isometric.*

A *convex surface* is either the boundary of a convex body $B \subset \mathbb{R}^3$ or the double DK of a 2-dimensional convex compactum $K \subset \mathbb{R}^3$ along the identity map of ∂K, each with the induced intrinsic metric. We refer to these two alternatives as the *nondegenerate* and the *degenerate* convex surfaces, call their intrinsic metrics the *boundary metrics*, and say that they *bound* B, K, respectively. With this definition any convex surface is homeomorphic to S^2.

The intrinsic metric on a degenerate surface DK can be canonically approximated by the boundary metric of the right cylinder with base K and small height.

Each convex surface bounds a unique convex compactum in $\mathbb{R}^3$ which has dimension 2 if the surface is degenerate and dimension 3 otherwise. If two such convex compacta K_1, K_2 are Hausdorff close, then the corresponding convex surfaces are nearly isometric. (For nondegenerate convex surfaces this is proved in [10, Lemma 10.2.7] and the degenerate case reduces to the nondegenerate one by approximating DK with the cylinder as above.)

Alexandrov, see [1] or [3, pp. 112 and 399] showed that an intrinsic metric isometric to a 2-sphere of nonnegative curvature if and only if it is isometric to a convex surface. Pogorelov proved in [17] that any two isometric convex surfaces are congruent, even though his argument is commonly described as very complicated, and we hesitate to rely on it. An easier proof of this result was found by Volkov [20], see [3, Section 12.1] for a reprint and [11, Section 5.2] for an exposition of Volkov's stability theorem which we discuss below.

Each nondegenerate convex surface has another metric obtained by restricting the distance function on $\mathbb{R}^3$; we call the metric *extrinsic.* If Σ_1, Σ_2 are nondegenerate convex surfaces with intrinsic metrics ρ_1, ρ_2, and extrinsic metrics d_1, d_2, and if $f\colon (\Sigma_1, \rho_1) \to (\Sigma_2, \rho_2)$ is an ε-isometry, then Volkov stability theorem states that $f\colon (\Sigma_1, d_1) \to (\Sigma_2, d_2)$ is a $C_1\varepsilon^{\beta}$-isometry where C_1 depends onto on diameters of ρ_1, ρ_2 and β is a positive universal constant. This easily implies that Σ_1, Σ_2 are nearly congruent, e.g., according to [2, Theorem 2.2] any δ-isometry between compacta in $\mathbb{R}^n$ can be approximated by the restriction of an isometry of $\mathbb{R}^n$ with the additive error at most $C_2\sqrt{\delta}$ where C_2 depends only on n and the diameters of the compacta.

To extend the result to the case of a degenerate surface DK we replace it with a nearby right cylinder with base K, and then apply Volkov's theorem.

If the isometry classes of two convex surfaces are GH close, then the surfaces are nearly isometric, e.g., by the Perelman stability theorem [16]. (A less heavy-handed argument is as follows. For a convex surface Σ we denote its isometry class by $[\Sigma]$. If Σ_i, Σ are convex surfaces such that the sequence $[\Sigma_i] \to [\Sigma]$ in the GH metric, then up to congruence $\bigcup_i \Sigma_i$ has compact closure in $\mathbb{R}^3$ and any limit point of the sequence Σ_i with respect to the Hausdorff metric is congruent to Σ, which by the above gives the desired near isometry of Σ and Σ_i for large i.)

The map $r\colon \mathcal{K} \to \mathcal{K}_s$ given by $r(D) = D - s(D)$ where s is the Steiner point descends to a homeomorphism of orbit spaces $\mathcal{K}/\mathrm{Iso}(\mathbb{R}^3) \to \mathcal{K}_s/O(3)$; see [6, Section 4]. Note that the homeomorphism is dimension preserving.

A $C^{k,\alpha}$ *convex body* is a convex body whose boundary is a $C^{k,\alpha}$ submanifold of $\mathbb{R}^n$. A function is $C^{k,\alpha}$ if its kth partial derivatives are α-Hölder for $\alpha \in (0,1]$ and continuous for $\alpha = 0$. As usual C^k means $C^{k,0}$.

Lemma 2.1. *Any convex body $D \in \mathcal{B}_d$ has $C^{1,1}$ boundary, that is, C^∞ at points of intrinsically positive curvature. In particular, if the boundary metric is intrinsically C^∞ of positive sectional curvature, then $D \in \mathcal{B}_p$.*

Proof. The last statement was proved much earlier by Pogorelov and Nirenberg (independently). The boundary ∂D is the image of an isometric embedding of the distance function of a C^∞ nonnegatively curved metric g on S^2. Improving on Nirenberg's method Guan-Li [13] and Hong-Zuily [14] independently proved that any C^∞ nonnegatively curved metric of S^2 admits a $C^{1,1}$ isometric embedding into $\mathbb{R}^3$ that is C^∞ at points of positive curvature, and moreover the embedding is the limit of a sequence of C^∞ isometric embeddings of positively curved metrics on S^2. By the Hadamard theorem, see e.g., [19, Chapter 2], the image of an isometric embedding of positively curved sphere bounds a convex body, and hence the same is true for the limiting $C^{1,1}$ isometric embedding that induces g. The limiting convex body is congruent to D by the above mentioned results of Pogorelov and Volkov. □

The above discussion proves Theorem 1.1.

3. Proofs of main results

Proof of Theorem 1.3. For integers $k \geq 2$, $l \geq 1$ let $Q_l^k \subset \mathcal{M}^{\text{GH}}_{\sec\geq 0}(S^2)$ be the subset consisting of isometry classes of metrics whose sectional curvature vanishes somewhere, the diameter is in $[0, l]$, the injectivity radius is at least $1/l$, and the C^0 norms of the curvature tensor and of every covariant derivative of the curvature tensor of orders $1, \dots, k$ is at most l. Its closure $\bar{Q}_l^k$ in $\mathcal{M}^{\text{GH}}_{\text{curv}\geq 0}(S^2)$ is compact and disjoint from $\mathcal{M}^{\text{GH}}_{\sec>0}(S^2)$ because for each $\alpha \in (0,1)$ any sequence in Q_l^k subconverges in the $C^{k,\alpha}$ topology to an isometry class of a $C^{k+1,\alpha}$ Riemannian manifold, see e.g. [4, Theorem 2.2], and since $k \geq 2$ the sectional curvature must vanish in the limit. For each k we clearly have

$$\mathcal{M}^{\text{GH}}_{\sec\geq 0}(S^2) \setminus \mathcal{M}^{\text{GH}}_{\sec>0}(S^2) = \bigcup_{l\geq 1} \bar{Q}_l^k \cap \mathcal{M}^{\text{GH}}_{\sec\geq 0}(S^2)$$

which is F_σ in $\mathcal{M}^{\text{GH}}_{\sec\geq 0}(S^2)$. The σ-compact set $\bigcup_{l\in\omega} \bar{Q}_l^k$ in $\mathcal{M}^{\text{GH}}_{\text{curv}\geq 0}(S^2)$

- consists of the isometry classes of $C^{k+1,\alpha}$ Riemannian manifolds,
- contains $\mathcal{M}^{\text{GH}}_{\sec\geq 0}(S^2) \setminus \mathcal{M}^{\text{GH}}_{\sec>0}(S^2)$,
- and is disjoint from $\mathcal{M}^{\text{GH}}_{\sec>0}(S^2)$.

Thus $\mathcal{M}^{\text{GH}}_{\sec\geq 0}(S^2) \setminus \mathcal{M}^{\text{GH}}_{\sec>0}(S^2)$ equals $\bigcap_{k\geq 2} \bigcup_{l\geq 1} \bar{Q}_l^k$ as claimed. □

Now the results of [6] can be put together to yield what we claimed in the introduction. To justify this we are going to use some infinite dimensional topology terminology that can be found in [6, Section 3].

Proof of Theorem 1.2. First we show that $\mathcal{B}_d$ is homeomorphic to Σ^ω. By Lemma 2.1 we have $\mathcal{B}_p \subset \mathcal{B}_d \subset \mathcal{B}^{1,1}$, hence [6, Lemmas 6.1–6.3] show that $\mathcal{B}_d$ is an AR with SDAP and also σZ.

The $O(3)$-orbit maps from $\mathcal{B}_d$ and $\mathcal{B}_p$ onto the sets of congruence classes are continuous and proper. Taking preimage of a proper continuous map preserves being F_σ and being σ-compact so preimages $\mathcal{B}_p \setminus \mathcal{B}_p$ is F_σ in $\mathcal{B}_d$ and also is a countable intersection of σ-compact sets. Hence [6, Lemmas 6.6 and 6.9] imply that $\mathcal{B}_d \in \mathcal{M}_2$ and $\mathcal{B}_d$ is strongly $\mathcal{M}_2$-universal. These properties imply that $\mathcal{B}_d$ is homeomorphic to Σ^ω.

Then the pair $(\mathcal{K}_s^{2\le 3}, \mathcal{B}_d)$ is $(\mathcal{M}_0, \mathcal{M}_s)$-absorbing by [6, Lemma 7.1].

Also [6, Lemma 5.2] shows that $\mathcal{K}_s^{2\le 3}$ is homeomorphic to the complement in Q^ω of a Z-set homeomorphic to the suspension SRP^2 over RP^2. Since Σ^ω is convex and dense in Q^ω, it is also homotopy dense in Q^ω; see [5, Exercise 13 in 1.2]. Hence every compact subset of $Q^\omega \setminus \Sigma^\omega$ is a Z-set. If E is as in the statement of Theorem 1.2, then by the knotting of Z-sets in Q-manifolds [5, Theorem 1.1.25] the set $Q^\omega \setminus E$ can be taken to $\mathcal{K}_s^{2\le 3}$ by some homeomorphism of Q^ω. The pair $(Q^\omega \setminus E, \Sigma^\omega)$ is $(\mathcal{M}_0, \mathcal{M}_s)$-absorbing by [6, Lemma 7.2]. Now the uniqueness of absorbing pairs [6, Lemma 7.2] proves Theorem 1.2 for $\mathcal{B}_d$. The same argument works for $\mathcal{B}_p$. □

Proof of Theorem 1.4. The statements (1)-(8) of Theorem 1.4 were proved in [6, Section 8–9] for the $O(3)$-quotients of an arbitrary $O(3)$-invariant hyperspace X that is locally homeomorphic to Σ^ω and such that $\mathcal{B}_p \subset X \subset \mathcal{K}_s$. The statement (9) was explained in [6, Question (g) of Section 1]. □

4. Remarks on the structure of $\mathcal{B}_d$

The hyperspace $\mathcal{B}_d$, which is the main object of his paper, is not well understood, e.g., we suspect that $\mathcal{B}_d$ is not convex but cannot yet prove it. This section is to shed some light on the properties of $\mathcal{B}_d$.

The moral of Theorem 1.3 is that the awkward features of $\mathcal{B}_d$ disappear in $\mathcal{B}_d/O(3)$, as they should because the condition of being intrinsically C^∞ makes much more sense in $\mathcal{M}^{\mathrm{GH}}_{\sec\ge 0}(S^2)$.

Recall that $\mathcal{B}_p \subset \mathcal{B}^\infty \subset \mathcal{B}_d \subset \mathcal{B}^{1,1}$. It turns that $\mathcal{B}_d \setminus \mathcal{B}^\infty$ is quite large.

Lemma 4.1. *$\mathcal{B}_d \setminus \mathcal{B}^2$ is dense in $\mathcal{K}_s$.*

Proof. The convex surface $x_3 = f(x_1, x_2) = r^3$, where $r = \sqrt{x_1^2 + x_2^2}$, is $C^{1,1}$ but not C^2. Its boundary metric is intrinsically C^∞ because the components of the metric tensor induced on the graph of $f\colon \mathbb{R}^n \to \mathbb{R}$ are $g_{ij} = \delta_{ij} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}$ and for f as above we have $\frac{\partial f}{\partial x_i} = 3rx_i$.

A small neighborhood of the origin in this surface can be patched as in [12] at any point of positive curvature of every C^∞ convex surface to produce a convex surface that has positive curvature everywhere except near one point where the surface is a portion of the graph of f near the origin. The conditions of Ghomi's patching theorem are satisfied because in the x_1, x_2 local coordinates any positively curved surface lies above the graph of $g(x_1, x_2) = kr^2$ for some $k > 0$, and hence $kr^2 > r^3$ for small r. Thus $\mathcal{B}_p$ lies in the closure of $\mathcal{B}_d \setminus \mathcal{B}^2$ in $\mathcal{K}_s$, and the claim follows by noting that by Schneider's regularization $\mathcal{B}_d$ is dense in $\mathcal{K}_s$, see e.g. [6, Section 4]. □

In Theorem 1.2 we show that $\mathcal{B}_d$ is homeomorphic to Σ^ω. The same is true for any hyperspace $\mathcal{B}_p \subseteq \mathcal{D} \subset \mathcal{B}^{1,1}$ such that $\mathcal{D} \setminus \mathcal{B}_p$ is σ-compact [6, Theorem 6.10]. Perhaps this conclusion holds for any naturally occurring hyperspace $\mathcal{D}$ with

$\mathcal{B}_p \subseteq \mathcal{D} \subset \mathcal{B}^1$, and while thinking on this problem one wants an example of a hyperspace that is not homeomorphic to Σ^ω.

In [6, Theorem 6.11] one finds a hyperspace $\mathcal{D}$ with $\mathcal{B}_p \subset \mathcal{D} \subset \mathcal{B}^{1,1}$ such that $\mathcal{D} \setminus \mathcal{B}_p$ embeds into the Cantor set, $\mathcal{B}_p$ is open in $\mathcal{D}$, and $\mathcal{D}$ is not topologically homogeneous, and in particular, not homeomorphic to Σ^ω. We improve this example as follows.

Proposition 4.2. *There is a hyperspace $\mathcal{D}$ with $\mathcal{B}_p \subset \mathcal{D} \subset \mathcal{B}_d \cap \mathcal{B}^{2,1}$ such that $\mathcal{D} \setminus \mathcal{B}_p$ embeds into the Cantor set, $\mathcal{B}_p$ is open in $\mathcal{D}$, and $\mathcal{D}$ is not topologically homogeneous.*

Proof. A slight modification of an example in [15] gives a 3-dimensional convex body whose boundary is C^∞ except at one point p where it is $C^{2,1}$ but not C^3, and such that the boundary metric is intrinsically C^∞. The curvature vanishes at p and is positive elsewhere. Any slight smooth perturbation at a boundary point of positive curvature gives a body with the same properties, and in particular, there is a path of such metrics, so by the proof of [6, Theorem 6.11] we can pick $\mathcal{D} \setminus \mathcal{B}_p$ to be a subset of a Cantor set inside this path. □

References

[1] A. D. Aleksandrov, *Vnutrennyaya Geometriya Vypuklyh Poverhnosteĭ* (Russian), OGIZ, Moscow-Leningrad,] 1948. MR0029518

[2] P. Alestalo, D. A. Trotsenko, and J. Väisälä, *Isometric approximation*, Israel J. Math. **125** (2001), 61–82, DOI 10.1007/BF02773375. MR1853806

[3] A. D. Alexandrov, *A. D. Alexandrov selected works. Part II*, Chapman & Hall/CRC, Boca Raton, FL, 2006. Intrinsic geometry of convex surfaces; Edited by S. S. Kutateladze; Translated from the Russian by S. Vakhrameyev. MR2193913

[4] Michael T. Anderson, *Ricci curvature bounds and Einstein metrics on compact manifolds*, J. Amer. Math. Soc. **2** (1989), no. 3, 455–490, DOI 10.2307/1990939. MR999661

[5] T. Banakh, T. Radul, and M. Zarichnyi, *Absorbing sets in infinite-dimensional manifolds*, Mathematical Studies Monograph Series, vol. 1, VNTL Publishers, L′viv, 1996. MR1719109

[6] I. Belegradek, *Deformation spaces of convex bodies up to congruence.* arXiv:1705.01220.

[7] Karol Borsuk, *Theory of retracts*, Monografie Matematyczne, Tom 44, Państwowe Wydawnictwo Naukowe, Warsaw, 1967. MR0216473

[8] Jean-Pierre Bourguignon, *Une stratification de l'espace des structures riemanniennes* (French), Compositio Math. **30** (1975), 1–41. MR0418147

[9] Glen E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York-London, 1972. Pure and Applied Mathematics, Vol. 46. MR0413144

[10] Dmitri Burago, Yuri Burago, and Sergei Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001. MR1835418

[11] Yu. D. Burago and S. Z. Shefel′, *The geometry of surfaces in Euclidean spaces [MR1039818 (91d:53004)]*, Geometry, III, Encyclopaedia Math. Sci., vol. 48, Springer, Berlin, 1992, pp. 1–85, 251–256, DOI 10.1007/978-3-662-02751-6_1. MR1306734

[12] Mohammad Ghomi, *Strictly convex submanifolds and hypersurfaces of positive curvature*, J. Differential Geom. **57** (2001), no. 2, 239–271. MR1879227

[13] Pengfei Guan and Yan Yan Li, *The Weyl problem with nonnegative Gauss curvature*, J. Differential Geom. **39** (1994), no. 2, 331–342. MR1267893

[14] J. Hong and C. Zuily, *Isometric embedding of the 2-sphere with nonnegative curvature in* $\mathbf{R}^3$, Math. Z. **219** (1995), no. 3, 323–334, DOI 10.1007/BF02572368. MR1339708

[15] Joseph A. Iaia, *The Weyl problem for surfaces of nonnegative curvature*, Geometric analysis and nonlinear partial differential equations (Denton, TX, 1990), Lecture Notes in Pure and Appl. Math., vol. 144, Dekker, New York, 1993, pp. 213–220. MR1207184

[16] Vitali Kapovitch, *Perelman's stability theorem*, Surveys in differential geometry. Vol. XI, Surv. Differ. Geom., vol. 11, Int. Press, Somerville, MA, 2007, pp. 103-136, DOI 10.4310/SDG.2006.v11.n1.a5. MR2408265

[17] A. V. Pogorelov, *Odnoznačnaya opredelennost′ obščih vypuklyh poverhnosteĭ* (Russian), Monografii Instituta Matematiki, vyp. II, Akad. Nauk Ukrainskoĭ SSR, Kiev, 1952. MR0063681

[18] Rolf Schneider, *Convex bodies: the Brunn-Minkowski theory*, Second expanded edition, Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, 2014. MR3155183

[19] Michael Spivak, *A comprehensive introduction to differential geometry. Vol. III*, 2nd ed., Publish or Perish, Inc., Wilmington, Del., 1979. MR532832

[20] Ju. A. Volkov, *An estimate of the deformation of a convex surface as a function of the change in its intrinsic metric* (Russian), Ukrain. Geometr. Sb. Vyp. **5–6** (1968), 44–69. MR0283734

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