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ON THE ISOLATION PHENOMENA OF EINSTEIN MANIFOLDS—SUBMANIFOLDS VERSIONS

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ABSTRACT. In this paper, we study the isolation phenomena of Einstein manifolds from the viewpoint of submanifolds theory. First, for locally strongly convex Einstein affine hyperspheres we prove a rigidity theorem and as its direct consequence we establish a unified affine differential geometric characterization of the noncompact symmetric spaces $\mathrm{E}_{6(-26)}/\mathrm{F}_4$ and $\mathrm{SL}(m,\mathbb{R})/\mathrm{SO}(m)$, $\mathrm{SL}(m,\mathbb{C})/\mathrm{SU}(m)$, $\mathrm{SU}^*(2m)/\mathrm{Sp}(m)$ for each $m \geq 3$. Second and analogously, for Einstein Lagrangian minimal submanifolds of the complex projective space $\mathbb{C}P^n(4)$ with constant holomorphic sectional curvature 4, we prove a similar rigidity theorem and as its direct consequence we establish a unified differential geometric characterization of the compact symmetric spaces $\mathrm{E}_6/\mathrm{F}_4$ and $\mathrm{SU}(m)/\mathrm{SO}(m)$, $\mathrm{SU}(m)$, $\mathrm{SU}(2m)/\mathrm{Sp}(m)$ for each $m \geq 3$.

1. Introduction

In this paper we study Einstein manifolds which appear as hypersurfaces or generally the submanifolds of some ambient spaces. Recall that there are plenty of investigations about the rigidity phenomena for Einstein manifolds; for related references we refer to [1, 3, 9, 10, 15, 18, 21, 28, 29, 31], among many others. For example, in [3], S. Brendle proved that compact Einstein manifolds with nonnegative isotropic curvature are locally symmetric, which generalizes Tachibana's result in [29]: compact Einstein manifolds with nonnegative curvature operator are locally symmetric. In [28], Singer also proved a rigidity theorem about positive Einstein metric with small $L^{n/2}$-norm of the Weyl tensor for dimension $n = 2m \geq 4$ and some additional assumptions. Itoh and Satoh [15] then generalized the result of [28], in particular, it is proved that for a closed connected oriented Einstein manifold (M^n, g) with positive scalar curvature R and of unit volume $\mathrm{Vol}(M) = 1$, there is a constant $C(n)$, depending only on n such that if the $L^{n/2}$-norm $\|W\|_{L^{n/2}} < C(n)R$, then $W = 0$ so that (M^n, g) is a finite isometric quotient of the standard n-sphere of unit volume, whereas in [9] and [31], the various very interesting rigidity theorems are derived for Einstein 4-manifolds with positive curvature.

The aim of this paper is to establish two theorems, both of which state that if an Einstein submanifold is located in a special ambient space, then its Weyl conformal

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curvature tensor has an optimal pointwise estimation. To present the results more precisely, we will consider compact and/or noncompact Einstein manifolds which can be realized as the submanifolds of one of the two ambient spaces. Our proof of the results are strongly dependent on the two remarkable achievements in local differential geometry. The first one is in affine differential geometry about the finally complete classification of locally strongly convex affine hypersurfaces with parallel *cubic form* (also named *Fubini-Pick form*) by Hu, Li and Vrancken [13] (cf. also [11, 12] for its preparatory stage). The second one is the updated complete classification of Lagrangian submanifolds in complex projective space with parallel second fundamental form by Dillen, Li, Vrancken and Wang [6] and [20]. We point out that Lagrangian submanifolds in complex projective space with parallel second fundamental form and without the Euclidean factor were classified in the 1980s by H. Naitoh and M. Takeuchi in a series of papers [23–26], but their original proof relies heavily on the theory of Lie groups and symmetric spaces.

Now, we first consider locally strongly convex affine hyperspheres in a unimodular $(n+1)$-dimensional affine space $\mathbb{R}^{n+1}$ such that their affine metrics are Einstein. As the main result we will prove:

Theorem 1.1. *Let $x : M^n \to \mathbb{R}^{n+1}$ $(n \geq 4)$ be a locally strongly convex affine hypersphere with affine mean curvature L_1 such that its affine metric is Einstein. Then we have $JR \leq 0$ and the Weyl conformal curvature tensor W of M^n satisfies*

$$\sum (W_{ijkl})^2 \leq -(n+1)JR, \tag{1.1}$$

where J and R denote the Pick invariant and scalar curvature of M^n, respectively. Moreover, the equality holds identically if and only if one of the following cases occurs:

(i) *$J = 0$ and M^n is affinely equivalent to a hyperquadric;*
(ii) *$R = 0$, $J \neq 0$ and M^n is affinely equivalent to the flat hyperbolic affine hypersphere $Q(1,n):\ x_1 x_2 \cdots x_{n+1} = 1$ in $\mathbb{R}^{n+1}$;*
(iii) *$n = \frac{1}{2}m(m+1) - 1$, $m \geq 3$, and M^n is affinely equivalent to the standard embedding of the noncompact symmetric space $\mathrm{SL}(m,\mathbb{R})/\mathrm{SO}(m)$ into $\mathbb{R}^{n+1}$;*
(iv) *$n = m^2 - 1$, $m \geq 3$, and M^n is affinely equivalent to the standard embedding of the noncompact symmetric space $\mathrm{SL}(m,\mathbb{C})/\mathrm{SU}(m)$ into $\mathbb{R}^{n+1}$;*
(v) *$n = 2m^2 - m - 1$, $m \geq 3$, and M^n is affinely equivalent to the standard embedding of the noncompact symmetric space $\mathrm{SU}^*(2m)/\mathrm{Sp}(m)$ into $\mathbb{R}^{n+1}$;*
(vi) *$n = 26$, and M^n is affinely equivalent to the standard embedding of the noncompact symmetric space $\mathrm{E}_{6(-26)}/\mathrm{F}_4$ into $\mathbb{R}^{27}$.*

Accordingly, if the equality sign in (1.1) *holds, then hypersurfaces as stated in (iii), (iv), (v) and (vi) give all Einstein affine hyperspheres whose affine metrics are not of constant sectional curvatures.*

Remark 1.1. It is well known from [30] (see also [17]) that a locally strongly convex affine hypersphere with constant affine sectional curvature is affinely equivalent to either a hyperquadric or the flat $Q(1,n)$. Moreover, our recent result in [14] shows that up to dimension 4 the cases (i) and (ii) of Theorem 1.1 exhaust all Einstein hyperspheres, i.e., an Einstein hypersphere for $n \leq 4$ must be of constant affine sectional curvature. Therefore, related to Theorem 1.1, we would raise an interesting problem of *how to classify all locally strongly convex Einstein affine hyperspheres in* $\mathbb{R}^{n+1}$. At the moment the problem for $n \geq 5$ remains open.

To introduce our second result, we denote by $\mathbb{C}P^n(4)$ the complex projective space that is equipped with the Fubuni-Study metric of constant holomorphic sectional curvature 4. We will consider Lagrangian minimal submanifolds of $\mathbb{C}P^n(4)$ such that their induced metrics are Einstein. As the main result we will prove:

Theorem 1.2. *Let $x : M^n \to \mathbb{C}P^n(4)$ $(n \geq 4)$ be a Lagrangian minimal submanifold with Einstein induced metric; then the Weyl conformal curvature tensor W of M^n satisfies*

$$\sum (W_{ijkl})^2 \geq \tfrac{n+1}{n(n-1)} SR, \tag{1.2}$$

where S and R denote the squared-norm of the second fundamental form and the scalar curvature of M^n, respectively. Moreover, the equality holds identically if and only if one of the following cases occurs:

- (i) *$R = n(n-1)$ and M^n is congruent with the totally geodesic standard embedding of the real projective space $\mathbb{R}P^n$ into $\mathbb{C}P^n(4)$;*
- (ii) *$R = 0$ and M^n is congruent with the standard embedding of the flat Clifford torus T^n into $\mathbb{C}P^n(4)$;*
- (iii) *$n = \frac{1}{2}m(m+1) - 1$, $m \geq 3$, and M^n is congruent with the standard embedding of the compact symmetric space $\mathrm{SU}(m)/\mathrm{SO}(m)$ into $\mathbb{C}P^n(4)$;*
- (iv) *$n = m^2 - 1$, $m \geq 3$, and M^n is congruent with the standard embedding of the compact symmetric space $\mathrm{SU}(m)$ into $\mathbb{C}P^n(4)$;*
- (v) *$n = 2m^2 - m - 1$, $m \geq 3$, and M^n is congruent with the standard embedding of the compact symmetric space $\mathrm{SU}(2m)/\mathrm{Sp}(m)$ into $\mathbb{C}P^n(4)$;*
- (vi) *$n = 26$, and M^n is congruent with the standard embedding of the compact symmetric space $\mathrm{E}_6/\mathrm{F}_4$ into $\mathbb{C}P^{26}(4)$.*

Accordingly, if the equality sign in (1.2) *holds, then submanifolds as stated in (iii), (iv), (v) and (vi) give all Einstein Lagrangian minimal submanifolds which are not of constant sectional curvature.*

Remark 1.2. It is known from Ejiri [8] and then more rigorously by Li-Zhao [19] (see also Chen and Ogiue [5]) that, for each $n \geq 2$, a Lagrangian, minimal submanifold with constant sectional curvature c in $\mathbb{C}P^n(4)$ is totally geodesic or flat $(c = 0)$, so that it is congruent to either $\mathbb{R}P^n$ or T^n. Trying to generalize this result, from Theorem 1.2 and similar to that in Remark 1.1, we would raise the problem: *How do we classify all Lagrangian minimal Einstein submanifolds in $\mathbb{C}P^n(4)$?*

2. Einstein affine hyperspheres and proof of Theorem 1.1

2.1. Basic facts of equiaffine hypersurfaces.

In this section, we briefly review the theory of local equiaffine hypersurfaces; for details we refer to Chapter 2 of [17] (cf. also [27]). Let $\mathbb{R}^{n+1}$ be the equiaffine space equipped with its canonical flat connection and a parallel volume element, defined by the determinant det. Let M^n be a connected and smooth n-dimensional manifold, and $x : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex hypersurface immersion. We choose an equiaffine frame field $\{x; e_1, e_2, \dots, e_n, e_{n+1}\}$ on M^n, such that

$$\det [e_1, e_2, \dots, e_n, e_{n+1}] = 1,$$

$$e_1, e_2, \dots, e_n \in T_xM, \quad G_{ij} := G(e_i, e_j) = \delta_{ij}, \quad e_{n+1} = Y,$$

where G and Y denote the (Blaschke-Berwald) affine metric and the equiaffine normal vector field of M^n, respectively.

Denote by B the equiaffine Weingarten form of $x : M^n \to \mathbb{R}^{n+1}$. The eigenvalues of B relative to G are called the affine principal curvatures of $x(M)$, and are denoted by $\lambda_1, \ldots, \lambda_n$. Then the equiaffine mean curvature is defined by $L_1 = \frac{1}{n}\sum_{i=1}^n \lambda_i$. An affine hypersurface is an affine hypersphere if and only if $\lambda_1 = \cdots = \lambda_n = \text{const}$.

Denote by R_{ijkl}, R_{ij} the components of Riemannian curvature tensor and Ricci tensor with respect to the affine metric, respectively, and by R the affine scalar curvature. Let A_{ijk} and $A_{ijk,l}$ be the components of the Fubini-Pick form A and its covariant derivative with respect to the Levi-Civita connection of the affine metric. Then, we have the following structural equations (cf. Section 2.5 of [17]):

$$A_{ijk,l} - A_{ijl,k} = \tfrac{1}{2}(\delta_{ik}B_{jl} + \delta_{jk}B_{il} - \delta_{il}B_{jk} - \delta_{jl}B_{ik}), \tag{2.1}$$

$$R_{ijkl} = \sum_m (A_{iml}A_{jmk} - A_{imk}A_{jml}) + \tfrac{1}{2}(\delta_{ik}B_{jl} + \delta_{jl}B_{ik} - \delta_{il}B_{jk} - \delta_{jk}B_{il}), \tag{2.2}$$

$$R_{ij} = \sum_{k,l} A_{ikl}A_{jkl} + \tfrac{n-2}{2}B_{ij} + \tfrac{n}{2}L_1\delta_{ij}, \tag{2.3}$$

$$\sum_i A_{iij} = 0,\ 1 \le j \le n, \tag{2.4}$$

$$J = \tfrac{1}{n(n-1)}\sum (A_{ijk})^2, \quad \chi = J + L_1, \quad \chi = \tfrac{1}{n(n-1)}R, \tag{2.5}$$

where J and χ are called the Pick invariant and normalized affine scalar curvature, respectively.

2.2. **Affine hyperspheres with constant scalar curvature.**

In this section, we consider n-dimensional locally strongly convex affine hyperspheres with constant affine mean curvature L_1 and constant affine scalar curvature R. Since our concern is the case that the affine metric is Einstein, and that locally strongly convex affine hyperspheres with constant affine sectional curvature have been classified in [30], we will assume that $n \ge 4$.

First of all, we prove the following result which modifies the statement in [16].

Lemma 2.1. *Let $x : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex affine hypersphere with constant affine scalar curvature R; then we have*

$$\sum (A_{ijk,l})^2 + \sum (W_{ijkl})^2 + (n+1)JR \le 0, \tag{2.6}$$

where the equality holds identically if and only if (M^n, G) is Einstein.

Proof. Choose as in Section 2.1 a local equiaffine frame field $\{x; e_1, \ldots, e_{n+1}\}$ along M^n. Since $x(M)$ is an affine hypersphere, (2.1) - (2.3) yield immediately that

$$A_{ijk,l} = A_{ijl,k}, \tag{2.7}$$

$$R_{ijkl} = \sum_m (A_{iml}A_{jmk} - A_{imk}A_{jml}) + L_1(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{2.8}$$

$$R_{ij} = \sum_{k,l} A_{ikl}A_{jkl} + (n-1)L_1\delta_{ij}. \tag{2.9}$$

From (2.4) and (2.7), and applying the Ricci identity, we calculate the Laplacian of the Fubini-Pick form A to get

$$\begin{aligned}\Delta A_{ijk} &= \sum_l A_{ijk,ll} = \sum_l A_{ijl,kl} \\ &= \sum_l A_{ijl,lk} + \sum_l A_{ijr}R_{rlkl} + \sum_l A_{irl}R_{rjkl} + \sum_l A_{rjl}R_{rikl} \\ &= \sum_l A_{ijr}R_{rlkl} + \sum_l A_{irl}R_{rjkl} + \sum_l A_{rjl}R_{rikl}.\end{aligned} \tag{2.10}$$

It follows that the Laplacian of the Pick invariant J can be calculated by

$$\begin{aligned}\Delta J &= \tfrac{1}{n(n-1)}\Delta\left(\sum (A_{ijk})^2\right) \\ &= \tfrac{2}{n(n-1)}\left[\sum (A_{ijk,l})^2 + \sum A_{ijk}A_{ijk,ll}\right] \\ &= \tfrac{2}{n(n-1)}\Big[\sum (A_{ijk,l})^2 + \sum A_{ijk}A_{ijr}R_{rlkl} \\ &\qquad + \sum (A_{ijk}A_{irl} - A_{ijl}A_{irk})R_{rjkl}\Big].\end{aligned} \tag{2.11}$$

Following [16], we substitute (2.8) and (2.9) into (2.11) to get

$$\tfrac{1}{2}n(n-1)\Delta J = \sum (A_{ijk,l})^2 + \sum (R_{ij})^2 + \sum (R_{ijkl})^2 - (n+1)RL_1. \tag{2.12}$$

Using the decomposition

$$\begin{aligned}R_{ijkl} = W_{ijkl} &+ \tfrac{1}{n-2}(\delta_{ik}R_{jl} + \delta_{jl}R_{ik} - \delta_{il}R_{jk} - \delta_{jk}R_{il}) \\ &- \tfrac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),\end{aligned} \tag{2.13}$$

and the computations

$$\sum (R_{ijkl})^2 = \sum (W_{ijkl})^2 + \tfrac{4}{n-2}\sum (R_{ij})^2 - \tfrac{2R^2}{(n-1)(n-2)}, \tag{2.14}$$

we can rewrite (2.12) as

$$\begin{aligned}\tfrac{1}{2}n(n-1)\Delta J = \sum (A_{ijk,l})^2 &+ \sum (W_{ijkl})^2 + \tfrac{n+2}{n-2}\sum (R_{ij})^2 \\ &- \tfrac{2R^2}{(n-1)(n-2)} - (n+1)RL_1.\end{aligned} \tag{2.15}$$

This, combining with $R = n(n-1)(J + L_1)$ and the inequality

$$\sum (R_{ij})^2 \geq \tfrac{1}{n}R^2, \tag{2.16}$$

where the equality holds if and only if the affine metric is Einstein, gives that

$$\tfrac{1}{2}n(n-1)\Delta J \geq \sum (A_{ijk,l})^2 + \sum (W_{ijkl})^2 + (n+1)JR. \tag{2.17}$$

Then, noting that $R = \text{const.}$ implies that $J = \text{const.}$, the assertion (2.6) follows immediately. □

As an immediately consequence of Lemma 2.1, we have

Theorem 2.1. *Let $x : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex affine hypersphere with nonnegative constant affine scalar curvature; then it is locally affinely equivalent to an open part of either one of the hyperquadrics, or the flat hyperbolic affine hypersphere $Q(1, n) :\ x_1x_2\cdots x_{n+1} = 1$, where $(x_1, \ldots, x_{n+1})$ is the coordinate of $\mathbb{R}^{n+1}$.*

Proof. If $R =$ const. is nonnegative, then from (2.6) we have either $J = 0$, or that $J \neq 0$, $R = 0$ and M^n is a hyperbolic affine hypersphere. If $J = 0$, the Maschke-Pick-Berwald theorem (cf. Theorem 2.13 in [17]) shows that M^n is locally affinely equivalent to one of the hyperquadrics. In the latter case, from [16] (cf. Theorem 3.8 of [17]), M^n is locally affinely equivalent to $Q(1, n)$. □

2.3. Proof of Theorem 1.1.

Assume that $x : M^n \to \mathbb{R}^{n+1}$ is a locally strongly convex Einstein affine hypersphere; then from Lemma 2.1, we have

$$\sum (A_{ijk,l})^2 + \sum (W_{ijkl})^2 + (n+1)JR = 0. \tag{2.18}$$

Therefore, we have the inequality

$$\sum (W_{ijkl})^2 \leq -(n+1)JR, \tag{2.19}$$

and the equality sign in (2.19) holds identically if and only if

$$A_{ijk,l} = 0, \quad 1 \leq i, j, k, l \leq n, \tag{2.20}$$

namely that $x : M^n \to \mathbb{R}^{n+1}$ has parallel Fubini-Pick (cubic) form. If it is the latter case, then we can apply the Classification Theorem of [13] to see that only the following three cases can occur:

(1) $J = 0$ and M^n are affinely equivalent to a hyperquadric.

(2) M^n is obtained either as the Calabi product of a lower dimensional hyperbolic affine hypersphere with parallel cubic form and a point, or the Calabi product of two lower dimensional hyperbolic affine hyperspheres both with parallel cubic form.

In this case, according to the results of [12] (cf. [4,7]), (M^n, G) is a Riemannian manifold with Euclidean factor, i.e., it can be regarded either as $(I \times M_1, c_1 dt^2 + c_2 h_1)$ or $(I \times M_1 \times M_2, c_1 dt^2 + c_2 h_1 + c_3 h_2)$, where c_1, c_2, c_3 are constant, thus the Ricci curvature $\mathrm{Ric}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 0$. Then, the Einstein condition implies that $R = 0$ and M^n is locally affinely equivalent to $Q(1, n)$.

(3) M^n is affinely equivalent to one of the standard embeddings of the noncompact symmetric spaces: $\mathrm{SL}(m, \mathbb{R})/\mathrm{SO}(m)$, $\mathrm{SL}(m, \mathbb{C})/\mathrm{SU}(m)$, $\mathrm{SU}^*(2m)/\mathrm{Sp}(m)$ for each $m \geq 3$ and $\mathrm{E}_{6(-26)}/\mathrm{F}_4$, with dimensions $\frac{1}{2}m(m+1)-1$, m^2-1, $2m^2-m-1$ and 26, respectively. We claim that these hyperbolic affine hypersurfaces are all of Einstein affine metrics. Indeed, from [2] and its proof of Proposition 4.1, we see that the standard embeddings of $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$, $\mathrm{SL}(3, \mathbb{C})/\mathrm{SU}(3)$, $\mathrm{SU}^*(6)/\mathrm{Sp}(3)$ and $\mathrm{E}_{6(-26)}/\mathrm{F}_4$ produce Einstein affine hyperspheres. Generally, the Einstein property of these examples corresponding to $m \geq 4$ can be verified directly by using (2.9) and the computations of $\{A_{ijk}\}$, i.e., components of the difference tensor K in [13].

This completes the proof of Theorem 1.1. □

3. Lagrangian minimal submanifolds in $\mathbb{C}P^n(4)$ and proof of Theorem 1.2

3.1. $\mathbb{C}P^n(4)$ and its Lagrangian minimal submanifolds.

In this section, we briefly review the theory of Lagrangian submanifolds in the complex projective space; for details we refer to [5] and [19]. Let $\mathbb{C}P^n(4)$ denote the n-dimensional complex projective space that is equipped with the canonical Fubini-Study metric g of constant holomorphic sectional curvature 4, and J denotes its almost complex structure.

Now, we suppose that M^n is a Lagrangian submanifold of $\mathbb{C}P^n(4)$. That means that the almost complex structure J of $\mathbb{C}P^n(4)$ carries each tangent space of M^n into its corresponding normal space. We choose a local orthonormal frame field $\{e_1, e_2, \ldots, e_{2n}\}$ for $\mathbb{C}P^n(4)$ such that, restricted to M^n, the vectors $e_1, e_2, \ldots, e_n$ are tangent to M^n and $e_{n+i} = Je_i$ for $1 \le i \le n$. In follows we shall make use of the indices convention: $1 \le i, j, k, \ldots \le n$.

Let h^k_{ij} denote the components of the second fundamental form h of $M^n \hookrightarrow \mathbb{C}P^n(4)$ with respect to the frame field $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$, namely we assume that $h(e_i, e_j) = \sum h^k_{ij} Je_k$. Then we have the totally symmetric

$$h^k_{ij} = h^j_{ik} = h^i_{kj}. \tag{3.1}$$

From now on we further suppose that M^n is minimal; then

$$\sum_i h^k_{ii} = 0. \tag{3.2}$$

Denote by $R_{ijkl}, R_{ij}, h^l_{ij,k}$ respectively the components of Riemannian curvature tensor, Ricci tensor and the covariant derivative ∇h of the second fundamental form of M^n. Then, we have the following equations:

$$h^l_{ij,k} = h^l_{ik,j}, \tag{3.3}$$

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \sum (h^m_{ik} h^m_{jl} - h^m_{il} h^m_{jk}), \tag{3.4}$$

$$R_{ij} = (n-1)\delta_{ij} - \sum h^l_{ik} h^l_{jk}, \tag{3.5}$$

$$R = n(n-1) - S, \quad S = \sum (h^k_{ij})^2, \tag{3.6}$$

where S and R denote the squared-norm of the second fundamental form and the scalar curvature of M^n, respectively.

3.2. **Proof of Theorem 1.2.**

We begin with the following result.

Lemma 3.1. *Let $x : M^n \to \mathbb{C}P^n(4)$ $(n \ge 4)$ be a Lagrangian minimal submanifold with Einstein induced metric; then we have*

$$\sum (h^l_{ij,k})^2 - \sum (W_{ijkl})^2 + \tfrac{n+1}{n(n-1)} SR = 0. \tag{3.7}$$

Proof. Using (3.1)–(3.3) and the Ricci identity, we have

$$\begin{aligned}
\Delta h^k_{ij} &= \sum_l h^k_{ij,ll} = \sum_l h^k_{il,jl} \\
&= \sum_l h^k_{il,lj} + \sum_{l,r} h^k_{ir} R_{rljl} + \sum_{l,r} h^k_{rl} R_{rijl} + \sum_{l,r} h^r_{il} R_{rkjl} \\
&= \sum_{l,r} h^k_{ir} R_{rljl} + \sum_{l,r} h^k_{rl} R_{rijl} + \sum_{l,r} h^r_{il} R_{rkjl}.
\end{aligned} \tag{3.8}$$

From the computation of the Laplacian of $\frac{1}{2}S$, with using (3.1), (3.4)–(3.6), (3.8) and (2.14), we obtain

$$\begin{aligned}\tfrac{1}{2}\Delta S &= \sum (h^k_{ij,l})^2 + \sum (h^i_{rj}h^i_{kl} - h^k_{ij}h^r_{il})R_{krjl} + \sum h^k_{ij}h^k_{ir}R_{rj}\\ &= \sum (h^k_{ij,l})^2 - \sum (R_{ij})^2 - \sum (R_{ijkl})^2 + (n+1)R\\ &= \sum (h^k_{ij,l})^2 - \sum (W_{ijkl})^2 + \tfrac{n+1}{n(n-1)}SR.\end{aligned} \tag{3.9}$$

Noting that $R = \text{const.}$ implies that $S = \text{const.}$, then the assertion (3.7) follows. □

Next, from (3.7) we have

$$\sum (W_{ijkl})^2 = \tfrac{n+1}{n(n-1)}SR + \sum (h^k_{ij,l})^2 \geq \tfrac{n+1}{n(n-1)}SR, \tag{3.10}$$

and the equality sign in (3.10) holds identically if and only if

$$h^k_{ij,l} = 0, \quad 1 \leq i, j, k, l \leq n, \tag{3.11}$$

namely that $x : M^n \to \mathbb{C}P^n(4)$ has parallel second fundamental form.

If it is the latter case, then we can apply the Classification Theorem of [6] to see that only the following three cases can occur:

(1) M^n is totally geodesic. In this case, according to the result of [5, 8, 19], M^n is congruent to $\mathbb{R}P^n$, which has constant sectional curvature 1.

(2) M^n is obtained as the Calabi product of a lower dimensional Lagrangian submanifold with parallel second fundamental form and a point, or the Calabi product of two lower dimensional Lagrangian submanifolds with parallel second fundamental form. In this case, according to [20], M^n has a Euclidean factor so that its Ricci curvature $\text{Ric} = 0$ and therefore we have $R = 0$. It follows that $\sum (W_{ijkl})^2 = 0$ and M^n becomes a minimal Lagrangian submanifold with constant sectional curvature 0. Then, according to Ejiri [8], Li and Zhao [19], M^n is congruent to the flat Clifford torus T^n in $\mathbb{C}P^n(4)$.

(3) M^n is congruent to one of the standard embeddings of the following compact symmetric spaces: $\mathrm{SU}(m)/\mathrm{SO}(m)$, $\mathrm{SU}(m)$, $\mathrm{SU}(2m)/\mathrm{Sp}(m)$ for each $m \geq 3$ and $\mathrm{E}_6/\mathrm{F}_4$, with dimensions $\frac{1}{2}m(m+1) - 1$, $m^2 - 1$, $2m^2 - m - 1$ and 26, respectively. We claim that these Lagragian submanifolds are indeed minimal and are all of Einstein induced metrics. In fact, from [22] and the statement after its proof of Theorem 1 there, we see that the standard embeddings of $\mathrm{SU}(3)/\mathrm{SO}(3)$, $\mathrm{SU}(3)$, $\mathrm{SU}(6)/\mathrm{Sp}(3)$ and $\mathrm{E}_6/\mathrm{F}_4$ into $\mathbb{C}P^n(4)$ produce Einstein Lagrangian minimal submanifolds. Generally, the Einstein property of these examples corresponding to $m \geq 4$ can be verified directly by using (3.5) and the computation of $\{h^k_{ij}\}$. We remark that the computation of $\{h^k_{ij}\}$ is tedious and a little complicated, which however can be carried just following the procedure outlined in Section 10 of [6].

We have completed the proof of Theorem 1.2. □

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