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BESSEL BRIDGE REPRESENTATION FOR THE HEAT KERNEL IN HYPERBOLIC SPACE

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ABSTRACT. This article shows a Bessel bridge representation for the transition density of Brownian motion on the Poincaré space. This transition density is also referred to as the heat kernel on the hyperbolic space in differential geometry literature. The representation recovers the well-known closed form expression for the heat kernel on hyperbolic space in dimension three. However, the newly derived bridge representation is different from the McKean kernel in dimension two and from the Gruet's formula in higher dimensions. The methodology is also applicable to the derivation of an analogous Bessel bridge representation for the heat kernel on a Cartan-Hadamard radially symmetric space and for the transition density of the hyperbolic Bessel process.

1. INTRODUCTION

The heat kernel, also known as the fundamental solution for the heat operator, plays a crucial role in various branches of mathematics including analysis, differential geometry, and probability theory. On Euclidean spaces, heat kernels have closed form expression given by the Gaussian kernels, which also serve as the transition density of Euclidean Brownian motions. Deriving, to some extent, analytical expression of the heat kernel on general curved space is more involved if not completely impossible. Symmetry of the underlying space plays an important role. Hyperbolic space is one of the symmetry spaces with constant negative curvature that has an expression for the heat kernel in analytic form. As we shall see throughout the article, due to symmetry, expressions for heat kernels on hyperbolic space depend solely on geodesic distance.

Derivations of the heat kernel on hyperbolic space in closed or quasi-closed forms have been done by various authors. We list a notable few as follows. McKean in [11] presented a quasi-closed form expression (up to an integral), nowadays known as the McKean kernel, for the heat kernel on two dimensional hyperbolic space; see (2.1) below. A detailed derivation of the McKean kernel using Fourier transform, isometries, and eigenvalues and eigenfunctions of the Laplace-Beltrami operator can be found in [1] (Section 2 in Chapter X). The closed form expression for the heat kernel on three dimensional hyperbolic space, see (2.2) below, and the Millson's recursion formula for the higher dimensional hyperbolic heat kernel were reported in [2]. A different proof of Millson's recursion formula based on the relationship

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between the heat kernel and the wave kernel and the explicit formula for the wave kernel on symmetry space was given in [3]. The following expression obtained in [4] for the n-dimensional hyperbolic heat kernel is known as the Gruet's formula

$$p_{\mathbb{H}^n}(t,z,w) = \frac{e^{-(n-1)^2 t/8}}{\pi(2\pi)^{n/2}\sqrt{t}}\Gamma\left(\frac{n+1}{2}\right)\int_0^\infty \frac{e^{(\pi^2-b^2)/2t}\sinh(b)\sin(\pi b/t)}{[\cosh(b)+\cosh(r)]^{(n+1)/2}}db,$$

where $r = d(z,w)$ is the geodesic distance between $z, w \in \mathbb{H}^n$. We refer the interested reader to [10] for a derivation of the Gruet's formula and its relationship to the pricing of Asian options. A probabilistic approach, which is different from the one employed in the current article, of deriving the heat kernel on two dimensional hyperbolic space can also be found in [7]. As closed form expression is concerned, [9] obtained expressions for heat kernels on symmetric spaces of rank 1. Finally, it is worth mentioning that a nice application of the McKean kernel in quantitative finance can be found in [5].

In this article, we prove yet another representation for the heat kernel on hyperbolic space: the Bessel bridge representation in Theorem 1. By working under geodesic polar coordinates, Brownian motion in hyperbolic space is decomposed into a one dimensional process in the radial part and a process on the unit sphere of codimension one. The radial part, also known as the hyperbolic Bessel process, is indeed a Brownian motion with drift. Due to symmetry of hyperbolic space, the drift in the radial part depends only on the geodesic distance. Girsanov's theorem allows us to define an equivalent probability measure on the underlying probability space through a Radon-Nikodym derivative so that, in the new probability measure, the radial process becomes a Bessel process of order n. We then substitute the stochastic integral that results from the Radon-Nikodym derivative with a Riemann integral by applying Ito's formula. The bridge representation of the hyperbolic heat kernel is thus obtained by conditioning on the terminal point of the radial process in the new probability measure. The whole procedure is implemented in the proof of Theorem 1. Similar representation for the transition density of the hyperbolic Bessel process is shown in Theorem 2. With minor modifications, the same procedure is also applicable to the case of general Cartan-Hadamard radially symmetric spaces and the result is summarized in Theorem 3. We remark that, in the one dimensional case, the idea of bridge representation for transition density of a diffusion first appeared, to our knowledge, in [13]; see also [14] for further discussions.

2. The heat kernel on hyperbolic space

Throughout the text, stochastic processes and random variables are assumed defined on the complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,\infty)})$ satisfying the usual conditions. We shall denote the n-dimensional hyperbolic space by $\mathbb{H}^n$ and the associated heat kernel on $\mathbb{H}^n$ between $z, w \in \mathbb{H}^n$ at time t by $p_{\mathbb{H}^n}(t,z,w)$.

2.1. **The hyperbolic space.** Conventionally, hyperbolic spaces are parametrized by two isometrically equivalent models: the half-space model and the ball model. The underlying space in the half-space model is the half plane $\mathbb{R}^n_+ = \{(x_1,\cdots,x_n) : x_n > 0\}$, while the underlying space in the ball model is the open ball $\mathbb{B}_n = \{y = (y_1,\cdots,y_n) : \|y\| < 1\}$. The transformation between the two models can be found for instance in [1] (p. 264). In particular, for $n = 2$, the transformation between the two models is given by the Möbius transform, $T : \mathbb{C}_+ \to \mathbb{B}_2$,

$$w = T(z) = \frac{z-i}{z+i}.$$

In the half-space model, the metric ds^2 and the Laplace-Beltrami are given respectively by

$$\begin{aligned} ds^2 &= \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}, \\ \Delta_M &= x_n^2 \left(\partial_{x_1}^2 + \cdots + \partial_{x_n}^2\right) + (2-n)x_n\partial_n; \end{aligned}$$

whereas in the ball model, the metric is given, in polar coordinates (ρ, θ), $\theta \in S^{n-1}$, by

$$\begin{aligned} ds^2 &= \frac{4}{(1-\rho^2)^2}\left(d\rho^2 + \rho^2 d\theta^2\right), \\ \Delta_M &= \frac{(1-\rho^2)^2}{4}\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\Delta_{S^{n-1}}\right), \end{aligned}$$

where $d\theta^2$ is the Riemann metric and $\Delta_{S^{n-1}}$ the Laplace-Beltrami operator on the standard unit sphere S^{n-1}. Moreover, if we make the transformation $\rho = \tanh\left(\frac{r}{2}\right)$, then (r, θ) becomes the geodesic polar coordinates for $\mathbb{H}^n$. We shall be working primarily in the geodesic polar coordinates $(r, \theta) \in [0, \infty) \times S^{n-1}$ under which the Riemann metric and the Laplace-Beltrami operator of $\mathbb{H}^n$ are given respectively as

$$\begin{aligned} ds^2 &= dr^2 + \sinh^2 r d\theta^2, \\ \Delta_{\mathbb{H}^n} &= \partial_r^2 + (n-1)\coth r\partial_r + \frac{1}{\sinh^2 r}\Delta_{S^{n-1}}. \end{aligned}$$

We remark that the geodesic polar coordinate on $\mathbb{H}^n$ is a global diffeomorphism, henceforth defined as a global coordinate, since hyperbolic space is Cartan-Hadamard.

2.2. **The heat kernel.** Generally speaking, the heat kernel on a differentiable manifold M is a fundamental solution to the (probabilist's) heat operator $\partial_t - \frac{1}{2}\Delta_M$, where Δ_M is the Laplace-Beltrami operator on M. The minimal heat kernel also serves as the transition density of Brownian motion on M. We refer the reader to [6] for expositions of Brownian motions on manifolds and their relationship to the heat kernel. For the reader's reference, we reproduce the heat kernel on the two and three dimensional hyperbolic spaces as follows:

$$p_{\mathbb{H}^2}(z, w, t) = \frac{\sqrt{2}e^{-t/8}}{(2\pi t)^{3/2}} \int_{d(z,w)}^{\infty} \frac{\xi e^{-\frac{\xi^2}{2t}}}{\sqrt{\cosh\xi - \cosh d(z, w)}} d\xi \tag{2.1}$$

and

$$p_{\mathbb{H}^3}(z, w, t) = \frac{e^{-\frac{r^2}{2t}}}{(2\pi t)^{3/2}} e^{-\frac{t}{2}} \frac{r}{\sinh(r)}, \tag{2.2}$$

where $r = d(z, w)$ is the geodesic distance between z and w. Note that the heat kernels given in (2.1) and (2.2) are densities with respect to the volume form.

In the following, we apply Girsanov's theorem to derive an expression for the heat kernel over $\mathbb{H}^n$, for $n \geq 2$, in which the closed form expression (2.2) for $\mathbb{H}^3$ is recovered. Note that since the Laplace-Beltrami operator on $\mathbb{H}^n$ is rotationally invariant, the heat kernel, or equivalently the transition density for the Brownian motion on hyperbolic space, is also rotationally invariant, hence a radial function.

Consider the processes (R_t, Θ_t) governed by the SDEs

$$dR_t = dW_t + \frac{n-1}{2}\coth(R_t)\,dt, \tag{2.3}$$

$$d\Theta_t = \frac{1}{\sinh(R_t)} dZ_t, \tag{2.4}$$

where W_t is a standard one dimensional Brownian motion and Z_t is a Brownian motion on the standard sphere S^{n-1}, independent of W_t. The infinitesimal generator of the process (R_t, Θ_t) is $\frac{1}{2}\Delta_{\mathbb{H}^n}$. Thus, it represents a Brownian motion on $\mathbb{H}^n$ in geodesic polar coordinates. We set the initial condition Θ_0 to be a random variable uniformly distributed on S^{n-1} so that the distribution of Θ_t remains uniformly distributed on S^{n-1} for all t. The main result of the article is given in the following theorem.

Theorem 1 (Bessel bridge representation)**.** *Let $z, w \in \mathbb{H}^n$. The heat kernel $p_{\mathbb{H}^n}(T, z, w)$ on the hyperbolic space $\mathbb{H}^n$ has the following representation:*

$$p_{\mathbb{H}^n}(T,z,w) = e^{-\frac{(n-1)^2 T}{8}} \left(\frac{r}{\sinh r}\right)^{\frac{n-1}{2}} \frac{e^{-\frac{r^2}{2T}}}{(2\pi T)^{\frac{n}{2}}} \tilde{\mathbb{E}}_r \left[e^{-\frac{(n-1)(n-3)}{8}\int_0^T \left[\frac{1}{\sinh^2(R_t)} - \frac{1}{R_t^2}\right] dt} \right], \tag{2.5}$$

where $r = r(z, w)$ is the geodesic distance between z and w. $\tilde{\mathbb{E}}_r[\cdot]$ denotes the conditional expectation $\tilde{\mathbb{E}}[\cdot | R_T = r]$, where R_t is a Bessel process of order n in the $\tilde{\mathbb{P}}$-measure.

Proof. Let (R_t, Θ_t) be the process satisfying (2.3):(2.4) with initial conditions $R_0 = 0$ and Θ_0 being uniformly distributed on S^{n-1}. We start with calculating the expectation of an arbitrary bounded measurable radial function f as

$$\mathbb{E}[f(R_T)] = \int_{S^{n-1}} \int_0^\infty f(r)p(T,r)\sinh^{n-1} r dr d\omega,$$

where $d\omega$ is the volume form on S^{n-1} and $p(T,r) = p_{\mathbb{H}^n}(T,z,w)$, $r = r(z,w)$ denotes the geodesic distance between z and w. For the radial process R_t, define the new measure $\tilde{\mathbb{P}}$ by the Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{\int_0^T h(R_t)dW_t - \frac{1}{2}\int_0^T h^2(R_t)dt}, \tag{2.6}$$

where $h(r) = \frac{n-1}{2}\left[\frac{1}{r} - \coth(r)\right]$. Note that h is a bounded function, in fact, $|h(r)| \le \frac{n-1}{2}$ for all $r \ge 0$. Thus (2.6) is a well-defined change of probability measure. Therefore, Girsanov's theorem implies that, under the measure $\tilde{\mathbb{P}}$, W_t is a Brownian motion with drift h. Moreover, in the $\tilde{\mathbb{P}}$-measure, the SDE for the radial process R_t becomes

$$dR_t = d\tilde{W}_t + \frac{n-1}{2}\frac{dt}{R_t},$$

which is a Bessel process of order n. Therefore, we have

$$\mathbb{E}[f(R_T)] = \tilde{\mathbb{E}}\left[f(R_T)\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\right] = \tilde{\mathbb{E}}\left[f(R_T)e^{-\int_0^T h(R_t)dW_t + \frac{1}{2}\int_0^T h^2(R_t)dt}\right]$$

$$= \tilde{\mathbb{E}}\left[f(R_T)e^{-\int_0^T h(R_t)d\tilde{W}_t - \frac{1}{2}\int_0^T h^2(R_t)dt}\right]. \tag{2.7}$$

We substitute the stochastic integral in (2.7) with a Riemann integral by applying Ito's formula as follows. Let H be an antiderivative of h, i.e., $H' = h$. Apparently, $H(r) = \frac{n-1}{2}\ln\left(\frac{r}{\sinh r}\right)$. Then by applying Ito's formula we have

$$\int_0^T h(R_t)d\tilde{W}_t = H(R_T) - H(0) - \int_0^T \left[\frac{h'(R_t)}{2} + \frac{n-1}{2}\frac{h(R_t)}{R_t}\right] dt.$$

It follows that the exponent of the exponential term in (2.7) becomes

$$\begin{aligned} & -\int_0^T h(R_t)d\tilde{W}_t - \frac{1}{2}\int_0^T h^2(R_t)dt \\ = \; & -H(R_T) + H(0) + \int_0^T \left[\frac{h'(R_t)}{2} + \frac{n-1}{2}\frac{h(R_t)}{R_t} - \frac{h^2(R_t)}{2}\right] dt \\ = \; & \ln\left[\frac{\sinh(R_T)}{R_T}\right]^{\frac{n-1}{2}} - \frac{(n-1)^2}{8}T - \frac{(n-1)(n-3)}{8}\int_0^T \left[\frac{1}{\sinh^2(R_t)} - \frac{1}{R_t^2}\right] dt. \end{aligned}$$

Hence, we have

$$\begin{aligned} & \mathbb{E}[f(R_T)] \\ & = e^{-\frac{(n-1)^2T}{8}}\tilde{\mathbb{E}}\left[f(R_T)\left\{\frac{\sinh(R_T)}{R_T}\right\}^{\frac{n-1}{2}} e^{-\frac{(n-1)(n-3)}{8}\int_0^T\left[\frac{1}{\sinh^2(R_t)} - \frac{1}{R_t^2}\right]dt}\right]. \end{aligned}$$

In particular, when $n = 3$, the last expression has a much simpler form:

$$\mathbb{E}[f(R_T)] = e^{-\frac{T}{2}}\tilde{\mathbb{E}}\left[f(R_T)\frac{\sinh(R_T)}{R_T}\right].$$

Finally, since R_t in $\tilde{\mathbb{P}}$ measure is a Bessel process of order n, we end up with

$$\begin{aligned} & \int_{S^{n-1}}\int_0^\infty f(r)p(T,r)\sinh^{n-1} r dr d\omega = \mathbb{E}[f(R_T)] \\ & = e^{-\frac{(n-1)^2T}{8}}\tilde{\mathbb{E}}\left[f(R_T)\left\{\frac{\sinh(R_T)}{R_T}\right\}^{\frac{n-1}{2}} e^{-\frac{(n-1)(n-3)}{8}\int_0^T\left[\frac{1}{\sinh^2(R_t)} - \frac{1}{R_t^2}\right]dt}\right] \\ & = e^{-\frac{(n-1)^2T}{8}}\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}}\int_{S^{n-1}}\int_0^\infty f(r)\left(\frac{\sinh r}{r}\right)^{\frac{n-1}{2}} \\ & \quad \times \tilde{\mathbb{E}}_r\left[e^{-\frac{(n-1)(n-3)}{8}\int_0^T\left[\frac{1}{\sinh^2(R_t)} - \frac{1}{R_t^2}\right]dt}\right]\frac{2r^{n-1}e^{-\frac{r^2}{2T}}}{(2T)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}dr d\omega \\ & = e^{-\frac{(n-1)^2T}{8}}\int_{S^{n-1}}\int_0^\infty f(r)\left(\frac{r}{\sinh r}\right)^{\frac{n-1}{2}}\frac{e^{-\frac{r^2}{2T}}}{(2\pi T)^{\frac{n}{2}}} \\ & \quad \times \tilde{\mathbb{E}}_r\left[e^{-\frac{(n-1)(n-3)}{8}\int_0^T\left[\frac{1}{\sinh^2(R_t)} - \frac{1}{R_t^2}\right]dt}\right]\sinh^{n-1} r dr d\omega, \end{aligned}$$

where in passing to the penultimate equality we used the probability density p_B of the Bessel process R_t given by

$$p_B(t,r) = \frac{2r^{n-1}e^{-\frac{r^2}{2t}}}{(2t)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}$$

which satisfies the Fokker-Planck equation

$$\partial_t u = \frac{1}{2}\partial_r^2 u - \frac{n-1}{2}\partial_r\left(\frac{u}{r}\right)$$

with initial condition $u(r,0)=\delta(r)$, the Dirac delta function centered at 0. Also note that the normalizing constant $\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ comes from the volume of S^{n-1}. Thus, we obtain the bridge representation (2.5) for the transition density of hyperbolic Brownian motion. □

Remark 1. Note that when $n=3$ the representation (2.5) reduces to

$$p_{\mathbb{H}^3}(T,z,w) = e^{-\frac{T}{2}}\frac{r}{\sinh r}\frac{e^{-\frac{r^2}{2T}}}{(2\pi T)^{\frac{3}{2}}}$$

which coincides with the closed form expression (2.2). However, for $n=2$, (2.5) reads

$$p_{\mathbb{H}^2}(T,z,w) = e^{-\frac{T}{8}}\sqrt{\frac{r}{\sinh r}\frac{e^{-\frac{r^2}{2T}}}{2\pi T}}\tilde{\mathbb{E}}_r\left[e^{\frac{1}{8}\int_0^T\left\{\frac{1}{\sinh^2(R_t)}-\frac{1}{R_t^2}\right\}dt}\right].$$

Notice that

1. (1) This expression is different from the McKean kernel (2.1) or the Gruet's formula in the sense that a) the power of $2\pi T$ is in the correct dimension ($\frac{n}{2}=1$) and b) the "Gaussian" term $e^{-\frac{r^2}{2T}}$ is factored outfront naturally.
2. (2) The integrand in the exponential term, i.e., the function $\phi(x) := \frac{1}{\sinh^2 x}-\frac{1}{x^2}$ is increasing in $[0,\infty)$ with $\lim_{x\to 0^+}\phi(x) = -\frac{1}{3}$ and $\lim_{x\to\infty}\phi(x)=0$. Therefore, ϕ is bounded above by 0 and below by $-\frac{1}{3}$.

As applications of the bridge representation (2.5), a series expansion and an asymptotic expansion in small time for the hyperbolic heat kernel are almost straightforward. For notational simplicity, hereafter in this subsection we shall denote it by

$$g(r) = -\frac{(n-1)(n-3)}{8}\left[\frac{1}{\sinh^2(r)}-\frac{1}{r^2}\right].$$

Note that g is strictly decreasing and $|g(r)| \le \frac{(n-1)(n-3)}{24}$ for all $r>0$.

Corollary 1. *The hyperbolic heat kernel $p_{\mathbb{H}^n}$ has the following series expansion:*

(2.8)

$$\begin{aligned} &p_{\mathbb{H}^n}(T,z,w)\\ &= e^{-\frac{(n-1)^2T}{8}}\left(\frac{r}{\sinh r}\right)^{\frac{n-1}{2}}\frac{e^{-\frac{r^2}{2T}}}{(2\pi T)^{\frac{n}{2}}}e^{\int_0^T g(r_t)dt}\sum_{k=0}^{\infty}\frac{T^k}{k!}\tilde{\mathbb{E}}_r\left[\left(\int_0^1 g(R_{Ts})-g(r_{Ts})ds\right)^k\right], \end{aligned}$$

where r_t, for $t\in[0,T]$, is defined by

$$r_t = g^{-1}\left(\tilde{\mathbb{E}}_r[g(R_t)]\right). \tag{2.9}$$

In other words, $g(r_t)$ is an unbiased estimator for $g(R_t)$ in the Bessel bridge measure.

Proof. It suffices to deal with the conditional expectation term in (2.5)

$$
\begin{aligned}
&\tilde{\mathbb{E}}_r\left[e^{-\frac{(n-1)(n-3)}{8}\int_0^T\left\{\frac{1}{\sinh^2(R_t)}-\frac{1}{R_t^2}\right\}dt}\right]\\
=\;& e^{\int_0^T g(r_t)dt}\tilde{\mathbb{E}}_r\left[e^{\int_0^T\{g(R_t)-g(r_t)\}dt}\right]\\
=\;& e^{\int_0^T g(r_t)dt}\,\tilde{\mathbb{E}}_r\left[\sum_{k=0}^{\infty}\frac{1}{k!}\left(\int_0^T\{g(R_t)-g(r_t)\}\,dt\right)^k\right]\\
=\;& e^{\int_0^T g(r_t)dt}\sum_{k=0}^{\infty}\frac{1}{k!}\,\tilde{\mathbb{E}}_r\left[\left(\int_0^T\{g(R_t)-g(r_t)\}\,dt\right)^k\right]
\end{aligned}
$$

by the dominating convergence theorem since the random variable

$$\int_0^T\{g(R_t)-g(r_t)\}\,dt$$

is bounded. In fact,

$$\left|\int_0^T\{g(R_t)-g(r_t)\}\,dt\right|\leq\frac{(n-1)(n-3)}{12}T\qquad\text{almost surely.}$$

Finally, by making the change of variable $t=Ts$ we obtain the series expansion (2.8). □

We remark that in fact we have the freedom of selecting the deterministic path r_t in the series expansion (2.8). We choose the path as such since it serves as a first order "unbiased estimator" in the small time asymptotic expansion in the corollary that follows.

Corollary 2. *As $T\to 0^+$, the hyperbolic heat kernel $p_{\mathbb{H}^n}$ has the following small time asymptotic expansion up to second order:*

$$
\begin{aligned}
(2.10)\qquad & p_{\mathbb{H}^n}(T,z,w)\\
=\;& e^{-\frac{(n-1)^2T}{8}}\left(\frac{r}{\sinh r}\right)^{\frac{n-1}{2}}\frac{e^{-\frac{r^2}{2T}}}{(2\pi T)^{\frac{n}{2}}}e^{\int_0^T g(r_t)dt}\left\{1+O(T^2)\right\},
\end{aligned}
$$

where r_t is given in (2.9).

Proof. Consider the infinite series on the right hand side of (2.8),

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\frac{T^k}{k!}\tilde{\mathbb{E}}_r\left[\left(\int_0^1 g(R_{Ts})-g(r_{Ts})ds\right)^k\right]\\
=\;& 1+T\,\tilde{\mathbb{E}}_r\left[\int_0^1\{g(R_{Tu})-g(r_{Tu})\}\,du\right]+O(T^2)\\
=\;& 1+O(T^2)
\end{aligned}
$$

by the definition of the path r_t. □

Note that if we choose a different path r_t from (2.9), then the asymptotic expansion in (2.10) is of order T only.

Last, by naïvely choosing r_t as the straight line connecting 0 and r as well as the unbiased estimator (2.9), in Figure 1 we illustrate numerically the accuracy of the asymptotic expansion (2.10), compared with the Gruet's formula. As shown in the plots, the unbiased estimator does a pretty decent job; whereas the straight line approximation is off for high dimensions.

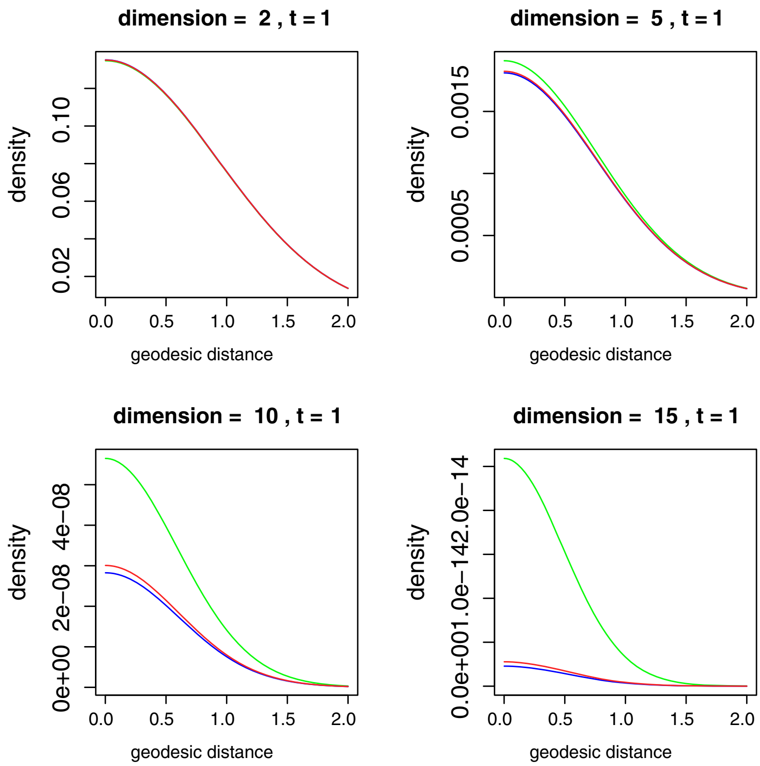

FIGURE 1. Plots of the hyperbolic heat kernel at time 1 in various dimensions. Approximation of r_t in (2.10) is shown by a straight line in green and by the unbiased estimator (2.9) in blue. Gruet's formula is shown in red.

2.3. **Transition density of hyperbolic Bessel process.** The radial part R_t of hyperbolic Brownian motion satisfying (2.3) is also referred to as the hyperbolic Bessel process. Hyperbolic Bessel processes and the calculations of their related moments are extensively explored in recent papers [8] and [12]. By the same token as in Theorem 1, we may as well derive a Bessel bridge representation for the hyperbolic Bessel process. The advantage of the bridge representation is that the expression is consistent across dimensions. However, formulas given in [12] (see Theorem 3.3), obtained by applying the Millson's recursion formula, become more and more intractable as the dimension goes higher.

Theorem 2. *The transition density $p_{HB}(T, x, y)$ of the hyperbolic Bessel process R_t of order n from x to y has the following Bessel bridge representation. For $T > 0$*

and $x \geq 0,\ y > 0,$

$$\begin{aligned}(2.11)\qquad & p_{HB}(T,x,y) \\ = \quad & e^{-\frac{(n-1)^2T}{8}}\left\{\frac{\sinh(y)}{\sinh(x)}\right\}^{\frac{n-1}{2}}\tilde{\mathbb{E}}_x\left[e^{-\frac{(n-1)(n-3)}{8}\int_0^T\left[\frac{1}{\sinh^2(R_t)}-\frac{1}{R_t^2}\right]dt}\middle| R_T=y\right] \\ & \times\frac{e^{-\frac{x^2+y^2}{2T}}}{\sqrt{2\pi T}}\left(\frac{xy}{T}\right)^{\frac{n-1}{2}}\frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}\int_0^\pi e^{\frac{xy}{T}\cos(\xi)}\sin^{n-2}(\xi)d\xi.\end{aligned}$$

Proof. As in the proof of Theorem 1, for any bounded measurable function f, the expectation of $f(R_t)$ conditioned on R_0 can be written as

$$\begin{aligned}&\mathbb{E}[f(R_T)|R_0] \\ &= e^{-\frac{(n-1)^2T}{8}}\tilde{\mathbb{E}}_{R_0}\left[f(R_T)\left\{\frac{\sinh(R_T)}{R_T}\frac{R_0}{\sinh(R_0)}\right\}^{\frac{n-1}{2}}e^{-\frac{(n-1)(n-3)}{8}\int_0^T\left[\frac{1}{\sinh^2(R_t)}-\frac{1}{R_t^2}\right]dt}\right],\end{aligned}$$

where $\tilde{\mathbb{E}}[\cdot]$ is the expectation in the $\tilde{\mathbb{P}}$-measure defined in (2.6), under which R_t is a Bessel process of order n. Recall that the transition density $p_B(t,x,y)$ of the Bessel process of order n from x to y in time t is given by

$$p_B(t,x,y) = \begin{cases} \frac{1}{t}\left(\frac{y}{x}\right)^\nu ye^{-\frac{x^2+y^2}{2t}}I_\nu\left(\frac{xy}{t}\right) & \text{if } x \neq 0; \\ \\ \frac{2y^{n-1}e^{-\frac{y^2}{2t}}}{(2t)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} & \text{if } x = 0, \end{cases}$$

where $\nu = \frac{n}{2}-1$. Hence, the transition density $p_{HB}(t,x,y)$ (from x to y in time t) for the hyperbolic Bessel process (i.e., R_t in the $\mathbb{P}$-measure) has the representation

$$\begin{aligned}& p_{HB}(T,x,y) \\ &= e^{-\frac{(n-1)^2T}{8}}\left\{\frac{\sinh(y)}{y}\frac{x}{\sinh(x)}\right\}^{\frac{n-1}{2}}\tilde{\mathbb{E}}_x\left[e^{-\frac{(n-1)(n-3)}{8}\int_0^T\left[\frac{1}{\sinh^2(R_t)}-\frac{1}{R_t^2}\right]dt}\middle| R_T=y\right] \\ &\quad\times\frac{1}{T}\left(\frac{y}{x}\right)^\nu ye^{-\frac{x^2+y^2}{2T}}I_\nu\left(\frac{xy}{T}\right) \\ &= e^{-\frac{(n-1)^2T}{8}}\left\{\frac{\sinh(y)}{\sinh(x)}\right\}^{\frac{n-1}{2}}\tilde{\mathbb{E}}_x\left[e^{-\frac{(n-1)(n-3)}{8}\int_0^T\left[\frac{1}{\sinh^2(R_t)}-\frac{1}{R_t^2}\right]dt}\middle| R_T=y\right] \\ &\quad\times\frac{e^{-\frac{x^2+y^2}{2T}}}{\sqrt{2\pi T}}\left(\frac{xy}{T}\right)^{\frac{n-1}{2}}\frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}\int_0^\pi e^{\frac{xy}{T}\cos(\xi)}\sin^{n-2}(\xi)d\xi,\end{aligned}$$

where in the last equality we used the following integral representation for the modified Bessel function I_ν

$$I_\nu(z) = \frac{z^\nu}{2^\nu\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^\pi e^{z\cos(\xi)}\sin^{2\nu}(\xi)d\xi.$$

□

In particular, when $n = 3$, (2.11) can be expressed in elementary functions as

$$\begin{aligned} & p_{HB}(t,x,y) \\ = \;& e^{-\frac{t}{2}} \frac{\sinh(y)}{\sinh(x)} \frac{e^{-\frac{x^2+y^2}{2t}}}{\sqrt{2\pi t}} \frac{xy}{t} \frac{1}{2} \int_0^\pi e^{\frac{xy}{t}\cos(\xi)} \sin(\xi) d\xi \\ = \;& \frac{e^{-\frac{t}{2}}}{\sqrt{2\pi t}} \frac{\sinh(y)}{\sinh(x)} e^{-\frac{x^2+y^2}{2t}} \left(e^{\frac{xy}{t}} - e^{-\frac{xy}{t}} \right) \\ = \;& \frac{e^{-\frac{t}{2}}}{\sqrt{2\pi t}} \frac{\sinh(y)}{\sinh(x)} \left(e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}} \right) \end{aligned}$$

which coincides with the formula in [12]. We summarize the result in the following corollary.

Corollary 3. *The transition density p_{HB} of the hyperbolic Bessel process R_t of order 3 has the following closed form expression. For $t > 0$ and $x \geq 0$, $y > 0$,*

$$p_{HB}(t,x,y) = \frac{e^{-\frac{t}{2}}}{\sqrt{2\pi t}} \frac{\sinh(y)}{\sinh(x)} \left(e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}} \right).$$

Notice that, since the conditional expectation term in (2.11) is exactly the same as the one in (2.5), one can easily derive series and small time asymptotic expansions for the transition density of the hyperbolic Bessel process, similarly as the ones in Corollaries 1 and 2. For example, we have

Corollary 4. *As $T \to 0^+$, the transition density p_{HB} of the hyperbolic Bessel process has the following small time asymptotic expansion up to second order:*

$$\begin{aligned} & p_{HB}(T,x,y) \\ = \;& e^{-\frac{(n-1)^2 T}{8}} \left\{ \frac{\sinh(y)}{\sinh(x)} \right\}^{\frac{n-1}{2}} e^{-\frac{(n-1)(n-3)}{8} \int_0^T \left[\frac{1}{\sinh^2(r_t)} - \frac{1}{r_t^2} \right] dt} \\ & \times \frac{e^{-\frac{x^2+y^2}{2T}}}{\sqrt{2\pi T}} \left(\frac{xy}{T} \right)^{\frac{n-1}{2}} \frac{1}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} \int_0^\pi e^{\frac{xy}{T}\cos(\xi)} \sin^{n-2}(\xi) d\xi \; \left\{ 1 + O(T^2) \right\}, \end{aligned}$$

where r_t is given in (2.9)*.*

3. Bridge representation in radially symmetric spaces

Let M be a radially symmetric space that is also Cartan-Hadamard. We recall that a Cartan-Hadamard manifold is a negatively curved, complete and simply connected Riemannian manifold whose exponential map at any given point defines a global diffeomorphism. Consequently, the geodesic polar coordinates at pole are globally defined for such manifolds. We refer the reader to [6] and the references therein for more detailed discussions. In geodesic polar coordinates, the Riemann metric ds^2 and the Laplace-Beltrami operator Δ_M on M can be written respectively as

$$ds^2 = dr^2 + G^2(r) d\theta^2, \tag{3.1}$$

$$\Delta_M = \partial_r^2 + (n-1) \frac{G'(r)}{G(r)} \partial_r + \frac{1}{G^2(r)} \Delta_{S^{n-1}}, \tag{3.2}$$

where r is the geodesic distance and, as before, $d\theta^2$ denotes the standard Riemann metric over the unit sphere S^{n-1}. The radial function G is nonnegative and satisfies

$G(0) = 0$ and $G'(0) = 1$. Conceivably due to symmetry, the heat kernel on such spaces has analogous Bessel bridge representation as for the hyperbolic space with minor modifications. We present the representation in the following theorem but omit its proof since it is almost identical with the proof of Theorem 1.

Theorem 3 (Bessel bridge representation in radially symmetric space)**.** *Let M be an n-dimensional Cartan-Hadamard radial symmetric space with Riemann metric and Laplace-Beltrami operator given by* (3.1) *and* (3.2) *respectively at its pole $z \in M$. Further assume that G satifies the regularity condition $\left|\frac{d}{dr}\ln\frac{G(r)}{r}\right| \le C$ for some $C > 0$. Then, for $w \in M$, the heat kernel $p_M(T,z,w)$ has the following representation:*

$$p_M(T,z,w) = \left\{\frac{r}{G(r)}\right\}^{\frac{n-1}{2}} \frac{e^{-\frac{r^2}{2T}}}{(2\pi T)^{\frac{n}{2}}}\mathbb{E}_r\left[e^{\int_0^T\left(\frac{(n-1)(n-3)}{8}\left\{\frac{1}{R_t^2}-\left(\frac{G'(R_t)}{G(R_t)}\right)^2\right\}-\frac{n-1}{4}\frac{G''(R_t)}{G(R_t)}\right)dt}\right], \tag{3.3}$$

where $r = r(z,w)$ is the geodesic distance between z and w. $\mathbb{E}_r[\cdot]$ denotes the Bessel bridge measure, i.e., the conditional expectation $\mathbb{E}[\cdot|R_T = r]$.

Remark 2. Similarly, in three dimensional case, $n = 3$, the representation (3.3) has the following slightly simpler form:

$$p(T,z,w) = \frac{r}{G(r)}\frac{e^{-\frac{r^2}{2T}}}{(2\pi T)^{\frac{3}{2}}}\tilde{\mathbb{E}}_r\left[e^{-\frac{1}{2}\int_0^T\frac{G''(R_t)}{G(R_t)}dt}\right],$$

where again $\mathbb{E}_r[\cdot]$ denotes the expectation under Bessel bridge measure. Apparently, it recovers $p_{\mathbb{H}^3}$ in (2.2) by setting $G(r) = \sinh r$.

Finally, a direct application of (3.3) is the following expansion in small time of the heat kernel on radially symmetry space:

Corollary 5. *As $T \to 0^+$,*

$$p_M(T,z,w) = \left\{\frac{r}{G(r)}\right\}^{\frac{n-1}{2}} \frac{e^{-\frac{r^2}{2T}}}{(2\pi T)^{\frac{n}{2}}}$$
$$\times\left\{1+\int_0^T\left(\frac{(n-1)(n-3)}{8}\mathbb{E}_r\left[\frac{1}{R_t^2}-\left(\frac{G'(R_t)}{G(R_t)}\right)^2\right]-\frac{n-1}{4}\mathbb{E}_r\left[\frac{G''(R_t)}{G(R_t)}\right]\right)dt+O(T^2)\right\},$$

where R_t is a Bessel bridge of order n connecting 0 and r in time T.

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