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A NOTE ON GROTHENDIECK'S STANDARD CONJECTURES OF TYPE C^+ AND D

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Abstract. Grothendieck conjectured in the sixties that the even Künneth projector (with respect to a Weil cohomology theory) is algebraic and that the homological equivalence relation on algebraic cycles coincides with the numerical equivalence relation. In this note we extend these celebrated conjectures from smooth projective schemes to the broad setting of smooth proper dg categories. As an application, we prove that Grothendieck's conjectures are invariant under homological projective duality. This leads to a proof of Grothendieck's original conjectures in the case of intersections of quadrics and linear sections of determinantal varieties. Along the way, we also prove the case of quadric fibrations and intersections of bilinear divisors.

1. Introduction and statement of results

Let k be a base field of characteristic zero. Given a smooth projective k-scheme X and a Weil cohomology theory H^*, let us denote by $\pi^i_X\colon H^*(X) \to H^*(X)$ the ith Künneth projector, by $Z^*(X)_\mathbb{Q}$ the $\mathbb{Q}$-vector space of algebraic cycles on X, and by $Z^*(X)_\mathbb{Q}/_{\sim \mathrm{hom}}$ and $Z^*(X)_\mathbb{Q}/_{\sim \mathrm{num}}$ the quotients with respect to the homological and numerical equivalence relations, respectively. Following Grothendieck [4] (see also Kleiman [6,7]), the standard conjecture[1] of type C^+, denote by $C^+(X)$, asserts that the even Künneth projector $\pi^+_X := \sum_i \pi^{2i}_X$ is algebraic, and the standard conjecture of type D, denoted by $D(X)$, asserts that $Z^*(X)_\mathbb{Q}/_{\sim \mathrm{hom}} = Z^*(X)_\mathbb{Q}/_{\sim \mathrm{num}}$. Thanks to the work of Kleiman [7] and Lieberman [15], and to the fact that $D(X \times X) \Rightarrow C^+(X)$ (see [1, Thm. 5.4.2.1]), the conjecture $C^+(X)$, resp. $D(X)$, holds in the case where X is of dimension ≤ 2, resp., ≤ 4, and also for abelian varieties. In addition to these cases, the aforementioned important conjectures remain wide open.

A *dg category* $\mathcal{A}$ is a category enriched over complexes of k-vector spaces; see §2.1. Every (dg) k-algebra A naturally gives rise to a dg category with a single object. Another source of example is provided by schemes since the category of perfect complexes $\mathrm{perf}(X)$ of every quasi-compact quasi-separated k-scheme X admits a canonical dg enhancement[2] $\mathrm{perf}_\mathrm{dg}(X)$. As explained in §§2.3-2.4, given a smooth

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[1]The standard conjecture of type C^+ is also usually known as the *sign conjecture*. If the even Künneth projector is algebraic, then the odd Künneth projector $\pi^-_X := \sum_i \pi^{2i+1}_X$ is also algebraic.

[2]When X is quasi-projective this dg enhancement is unique; see Lunts–Orlov [16, Thm. 2.12].

proper dg category $\mathcal{A}$ in the sense of Kontsevich (see §2.1), the standard conjectures of type C^+ and D admit noncommutative analogues $C^+_{\mathrm{nc}}(\mathcal{A})$ and $D_{\mathrm{nc}}(\mathcal{A})$.

Theorem 1.1. *Given a smooth projective k-scheme X, we have the following equivalences*[3] *of conjectures $C^+(X) \Leftrightarrow C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$ and $D(X) \Leftrightarrow D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$.*

Theorem 1.1 extends Grothendieck's standard conjectures of type C^+ and D from schemes to dg categories. Making use of this noncommutative viewpoint, we now prove Grothendieck's original conjectures in the case of quadric fibrations.

Theorem 1.2 (Quadric fibrations)**.** *Let $q\colon Q \to S$ be a flat quadric fibration of relative dimension d, with Q a smooth projective k-scheme. Whenever the dimension of S is ≤ 2, resp., ≤ 4, d is even, and the discriminant divisor of q is smooth, the conjecture $C^+(Q)$, resp., $D(Q)$, holds.*

Remark 1.3. A "geometric" proof of Theorem 1.2 can be obtained by combining the aforementioned work of Kleiman and Lieberman, with Vial's computation (see [21, Thm. 4.2 and Cor. 4.4]) of the rational Chow motive of Q.

Making use of Theorem 1.1, we now prove that Grothendieck's conjectures are invariant under homological projective duality (=HPD). Let X be a smooth projective k-scheme equipped with a line bundle $\mathcal{O}_X(1)$; we write $X \to \mathbb{P}(V)$ for the associated morphism, where $V := H^0(X, \mathcal{O}_X(1))^*$. Assume that the triangulated category $\mathrm{perf}(X)$ admits a Lefschetz decomposition $\langle \mathbb{A}_0, \mathbb{A}_1(1), \ldots, \mathbb{A}_{i-1}(i-1)\rangle$ with respect to $\mathcal{O}_X(1)$ in the sense of Kuznetsov [13, Def. 4.1]. Following [13, Def. 6.1] and [11, §2.4], let $(Y;\mathcal{F})$ be the HP-dual of X (Y stands for a projective k-scheme and $\mathcal{F}$ for a coherent sheaf of $\mathcal{O}_Y$-algebras), let $\mathcal{O}_Y(1)$ be the HP-dual line bundle, and let $Y \to \mathbb{P}(V^*)$ be the morphism associated to $\mathcal{O}_Y(1)$. Given a generic linear subspace $L \subset V^*$, consider the linear sections $X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp)$ and $(Y_L;\mathcal{F}_L)$, where $Y_L := Y \times_{\mathbb{P}(V^*)} \mathbb{P}(L)$ and $\mathcal{F}_L$ stands for the restriction of $\mathcal{F}$ to Y_L.

Theorem 1.4 (HPD-invariance)**.** *Let X and $(Y;\mathcal{F})$ be as above. Assume that X_L is smooth, that the dg category $\mathrm{perf}_{\mathrm{dg}}(Y_L;\mathcal{F}_L)$ is smooth, that $\dim(X_L) = \dim(X) - \dim(L)$, that $\dim(Y_L) = \dim(Y) - \dim(L^\perp)$, and that the conjecture $C^+_{\mathrm{nc}}(\mathbb{A}_0^{\mathrm{dg}})$, resp., $D_{\mathrm{nc}}(\mathbb{A}_0^{\mathrm{dg}})$, holds, where $\mathbb{A}_0^{\mathrm{dg}}$ stands for the dg enhancement of $\mathbb{A}_0$ induced from $\mathrm{perf}_{\mathrm{dg}}(X)$. Under these assumptions, we have the equivalence $C^+(X_L) \Leftrightarrow C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(Y_L;\mathcal{F}_L))$, resp., $D(X_L) \Leftrightarrow D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(Y_L;\mathcal{F}_L))$.*

Remark 1.5.

(i) Conjectures $C^+_{\mathrm{nc}}(\mathbb{A}_0^{\mathrm{dg}})$ and $D_{\mathrm{nc}}(\mathbb{A}_0^{\mathrm{dg}})$ hold, in particular, whenever the triangulated category $\mathbb{A}_0$ admits a full exceptional collection.

(ii) When $\mathcal{F} = \mathcal{O}_Y$, we write Y and Y_L instead of $(Y;\mathcal{O}_Y)$ and $(Y_L;\mathcal{O}_{Y_L})$; see Example 1.10, Theorem 1.11 and Remark 1.12 below. In this case, thanks to Theorem 1.1, the conjectures $C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(Y_L;\mathcal{O}_{Y_L})) = C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(Y_L))$ and $D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(Y_L;\mathcal{O}_{Y_L})) = D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(Y_L))$ are equivalent to $C^+(Y_L)$ and $D(Y_L)$.

To the best of the author's knowledge, Theorem 1.4 is new in the literature. As a first application, it provides us with an alternative (noncommutative) formulation of Grothendieck's original conjectures. Here are two "antipodal" examples; many more can be found in the survey [11].

[3] Thanks to Theorem 1.1, the assumption that Grothendieck's standard conjectures of type C^+ and D hold for every smooth projective k-scheme can now be removed from [17, Thm. 1.7].

Example 1.6 (Veronese–Clifford duality)**.** Let W be a k-vector space of dimension d, and let X be the associated projective space $\mathbb{P}(W)$ equipped with the double Veronese embedding $\mathbb{P}(W) \to \mathbb{P}(S^2W)$. By construction, we have a flat quadric fibration $q\colon Q \to \mathbb{P}(S^2W^*)$, where Q stands for the universal quadric in $\mathbb{P}(W)$. As proved in [12, Thm. 5.4], the HP-dual $(Y;\mathcal{F})$ of X is given by $(\mathbb{P}(S^2W^*);\mathcal{F})$, where $\mathcal{F}$ stands for the sheaf of even Clifford algebras associated to q. Moreover, given a generic linear subspace $L \subset S^2W^*$, the linear section X_L corresponds to the (smooth) intersection of the $\dim(L)$ quadric hypersurfaces in $\mathbb{P}(W)$ parametrized by L, and $(Y_L;\mathcal{F}_L)$ is given by $(\mathbb{P}(L);\mathcal{F}_L)$. Making use of Theorem 1.4, we hence conclude that $C^+(X_L) \Leftrightarrow C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}(L);\mathcal{F}_L))$ and $D(X_L) \Leftrightarrow D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}(L);\mathcal{F}_L))$.

By solving the preceding noncommutative standard conjectures, we hence prove Grothendieck's original standard conjectures in the case of intersections of quadrics.

Theorem 1.7 (Intersections of quadrics)**.** *Whenever the dimension of L is ≤ 3, resp., ≤ 5, d is even, and the discriminant division of q is smooth, the conjecture $C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}(L);\mathcal{F}_L))$, resp., $D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}(L);\mathcal{F}_L))$, holds. Consequently, Grothendieck's original standard conjecture $C^+(X_L)$, resp., $D(X_L)$, also holds.*

Remark 1.8. As mentioned by Grothendieck in [4, page 197], the standard conjecture of Lefschetz type $B(X)$ holds for smooth complete intersections. Since this conjecture implies the standard conjectures of type C^+ and D (the implication $B(X) \Rightarrow D(X)$ uses in an essential way the Hodge index theorem; see [6, Thm. 4.1 and Prop. 5.1]), we hence obtain an alternative "geometric" proof of Theorem 1.7.

Example 1.9 (Grassmannian–Pfaffian duality)**.** Let W be a k-vector space of dimension 6, and let X be the associated Grassmannian $\mathrm{Gr}(2,W)$ equipped with the Plücker embedding $\mathrm{Gr}(2,W) \to \mathbb{P}(\bigwedge^2 W)$. As proved in [14, Thm. 1], the HP-dual $(Y;\mathcal{F})$ of X is given by $(\mathrm{Pf}(4,W^*);\mathcal{F})$, where $\mathrm{Pf}(4,W^*) \subset \mathbb{P}(\bigwedge^2 W^*)$ is the singular Pfaffian variety and $\mathcal{F}$ a certain coherent sheaf of $\mathcal{O}_{\mathrm{Pf}(4,W^*)}$-algebras.[4] Moreover, given a generic linear subspace $L \subset \bigwedge^2 W^*$ of dimension 7, the linear section X_L corresponds to a curve of genus 8, and $(Y_L;\mathcal{F}_L)$ is given by $(\mathrm{Pf}(4,W^*)_L;\mathcal{F}_L)$, where $\mathrm{Pf}(4,W^*)_L$ is a (singular) cubic 5-fold; see [14, §10]. Making use of Theorem 1.4, we hence obtain equivalences of conjectures $C^+(X_L) \Leftrightarrow C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Pf}(4,W^*)_L;\mathcal{F}_L))$ and $D(X_L) \Leftrightarrow D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Pf}(4,W^*)_L;\mathcal{F}_L))$. Since Grothendieck's standard conjectures of type C^+ and D are well known in the case of curves, we then conclude that the preceding noncommutative standard conjectures also hold.

As a second application, Theorem 1.4 with $\mathcal{F} = \mathcal{O}_Y$ (see Remark 1.5(ii)) shows us that whenever X_L, resp., Y_L, is of dimension ≤ 2, the conjecture $C^+(Y_L)$, resp., $C^+(X_L)$, holds. Similarly, whenever X_L, resp., Y_L, is of dimension ≤ 4, the conjecture $D(Y_L)$, resp., $D(X_L)$, holds. Here is an illustrative example; many more can be found in the survey [11].

Example 1.10 (Determinantal duality)**.** Let U and V be two k-vector spaces of dimensions m and n, respectively, with $m \leq n$, W the tensor product $U \otimes V$, and

[4]The sheaf $\mathcal{F}$ is isomorphic to a matrix algebra on the smooth locus of $\mathrm{Pf}(4,W^*)$. Consequently, whenever $\mathrm{Pf}(4,W^*)_L$ is contained in the smooth locus of $\mathrm{Pf}(4,W^*)$, we conclude from Theorem 1.1 that the conjectures $C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Pf}(4,W^*)_L;\mathcal{F}_L))$ and $D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Pf}(4,W^*)_L;\mathcal{F}_L))$ are equivalent to $C^+(\mathrm{Pf}(4,W^*)_L)$ and $D(\mathrm{Pf}(4,W^*)_L)$, respectively.

$0 < r < m$ an integer. Following Bernardara–Bolognesi–Faenzi [3, §3], consider the determinantal variety $\mathcal{Z}^r_{m,n} \subset \mathbb{P}(W)$, resp., $\mathcal{W}^r_{m,n} \subset \mathbb{P}(W^*)$, defined as the locus of those matrices $V \to U^*$, resp., $V^* \to U$, with rank at most r, resp., with corank at least r. For example, $\mathcal{Z}^1_{m,n}$ are the classical Segre varieties. As explained in loc. cit., $\mathcal{Z}^r_{m,n}$ and $\mathcal{W}^r_{m,n}$ admit (Springer) resolutions of singularities $X := \mathcal{X}^r_{m,n}$ and $Y := \mathcal{Y}^r_{m,n}$, respectively. Moreover, as proved in [3, Thm. 3.5], Y is the HP-dual of X. Making use of Theorem 1.4, we hence conclude that $C^+(X_L) \Leftrightarrow C^+(Y_L)$ and $D(X_L) \Leftrightarrow D(Y_L)$ for every generic linear subspace $L \subset W$ of codimension c.

The linear section X_L has dimension $r(m+n-r)-c-1$ and the linear section Y_L has dimension $r(m-n-r)+c-1$. Therefore, by combining the aforementioned work of Kleiman and Lieberman with Example 1.10, we prove Grothendieck's standard conjectures in the case of linear sections of determinantal varieties.

Theorem 1.11 (Linear sections of determinantal varieties)**.** *Let X_L and Y_L be as in Example* 1.10*. Whenever $r(m+n-r)-c-1$ is ≤ 2, resp., ≤ 4, the conjecture $C^+(Y_L)$, resp., $D(Y_L)$, holds. Whenever $r(m-n-r)+c-1$ is ≤ 2, resp., ≤ 4, the conjecture $C^+(X_L)$, resp., $D(X_L)$, holds.*

Remark 1.12 (Dimension). Note that Theorem 1.11 infinitely furnishes us with many examples of smooth projective k-schemes of arbitrary (high) dimension which satisfy Grothendieck's standard conjectures. For example, consider the case of square matrices, i.e., $m = n$. Choose integers n, r, and c (as above) such that $c < nr$ and $-r^2 + c - 1 = 2$. Under these choices, Y_L has dimension 2 and X_L has dimension $2rn-(r^2+c+1) > 2$. Moreover, thanks to Theorem 1.11, the conjectures $C^+(X_L)$ and $D(X_L)$ hold. Now, note that if we replace n by $n' > n$ and keep r and c, we obtain a higher-dimensional k-scheme X'_L of dimension $2rn' - (r^2+c+1)$ for which the conjectures $C^+(X'_L)$ and $D(X'_L)$ still hold.

Theorem 1.1 allows us to easily extend Grothendieck's original conjectures from schemes X to (smooth proper) stacks $\mathcal{X}$ by setting $C^+(\mathcal{X}) := C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X}))$ and $D(\mathcal{X}) := D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X}))$. We now prove these extended conjectures in the case of bilinear divisors. Let W be a k-vector space of dimension d, and let $\mathcal{X}$ be the associated smooth proper Deligne–Mumford stack $(\mathbb{P}(W) \times \mathbb{P}(W))/\mu_2$ equipped with the map $\mathcal{X} \to \mathbb{P}(S^2W), ([w_1],[w_2]) \mapsto [w_1 \otimes w_2 + w_2 \otimes w_1]$. Given a generic linear subspace $L \subset S^2W^*$, the linear section $\mathcal{X}_L$ corresponds to the intersection of the $\dim(L)$ bilinear divisors in $\mathcal{X}$ parametrized by L.

Theorem 1.13 (Intersections of bilinear divisors)**.** *Assume that $\mathcal{X}_L$ has the* expected dimension, i.e., *that the codimension of $\mathcal{X}_L$ in $\mathcal{X}$ is equal to the codimension of $L^\perp$ in S^2W. Assume also that the dimension of L is ≤ 3, resp., ≤ 5, and that d is odd or that the dimension of L is ≤ 3 and that d is even. In these cases, the conjecture $C^+(\mathcal{X}_L)$, resp., $D(\mathcal{X}_L)$, holds.*

Remark 1.14. Voevodsky conjectured that the smash-nilpotence equivalence relation coincides with the numerical equivalence relation. The corresponding analogues of Theorems 1.1-1.2, 1.4, 1.7, and 1.11 were established in [2].

2. Preliminaries

2.1. **Dg categories.** For a survey on dg categories consult Keller's ICM talk [5]. Let $\mathcal{C}(k)$ be the category of complexes of k-vector spaces. A *dg category* $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$, and a *dg functor* $F\colon \mathcal{A} \to \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$.

Let $\mathrm{dgcat}(k)$ be the category of (small) dg categories and dg functors. Recall from [5, §3.8] that a *derived Morita equivalence* is a dg functor which induces an equivalence on derived categories. Following Kontsevich [8–10], a dg category $\mathcal{A}$ is called *smooth* if it is perfect as a bimodule over itself and *proper* if $\sum_i \dim H^i\mathcal{A}(x,y) < \infty$ for any pair of objects (x,y). Examples include the dg categories of perfect complexes $\mathrm{perf}_{\mathrm{dg}}(X)$ associated to smooth proper k-schemes X.

2.2. **Noncommutative motives.** For a book on noncommutative motives consult [19]. Recall from [19, §4.1] the construction of the category of noncommutative Chow motives $\mathrm{NChow}(k)_{\mathbb{Q}}$. By construction, this rigid symmetric monoidal category comes equipped with a $\otimes$-functor $U(-)_{\mathbb{Q}}\colon \mathrm{dgcat}_{\mathrm{sp}}(k) \to \mathrm{NChow}(k)_{\mathbb{Q}}$ defined on smooth proper dg categories. Moreover, $\mathrm{Hom}_{\mathrm{NChow}(k)_{\mathbb{Q}}}(U(\mathcal{A})_{\mathbb{Q}}, U(\mathcal{B})_{\mathbb{Q}}) = K_0(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})_{\mathbb{Q}}$. Recall from [17, Thm. 9.2] that periodic cyclic homology gives rise to a $\mathbb{Q}$-linear $\otimes$-functor $HP^{\pm}\colon \mathrm{NChow}(k)_{\mathbb{Q}} \to \mathrm{Vect}_{\mathbb{Z}/2}(k)$ with values in finite-dimensional $\mathbb{Z}/2$-graded k-vector spaces. The category of *noncommutative homological motives* $\mathrm{NHom}(k)_{\mathbb{Q}}$ is defined as the idempotent completion of the quotient $\mathrm{NChow}(k)_{\mathbb{Q}}/\mathrm{Ker}(HP^{\pm})$. Given a rigid symmetric monoidal category $\mathcal{C}$, its $\mathcal{N}$-ideal is defined as follows ($\mathrm{tr}(g\circ f)$ stands for the categorical trace of $g \circ f$):

$$\mathcal{N}(a,b) := \{f \in \mathrm{Hom}_{\mathcal{C}}(a,b) \mid \forall g \in \mathrm{Hom}_{\mathcal{C}}(b,a) \text{ we have } \mathrm{tr}(g\circ f) = 0\}\,.$$

Under this notation, the category of *noncommutative numerical motives* $\mathrm{NNum}(k)_{\mathbb{Q}}$ is defined as the idempotent completion of the quotient $\mathrm{NChow}(k)_{\mathbb{Q}}/\mathcal{N}$.

2.3. **Noncommutative standard conjecture of type C^+.** Given a smooth proper dg category $\mathcal{A}$, consider the Künneth projector $\pi^+_{\mathcal{A}}\colon HP^{\pm}(\mathcal{A}) \to HP^{\pm}(\mathcal{A})$. Following[5] [17], the conjecture $C^+_{\mathrm{nc}}(\mathcal{A})$ asserts that $\pi^+_{\mathcal{A}}$ is *algebraic*, i.e., that there exists an endomorphism $\underline{\pi}^+_{\mathcal{A}} \in \mathrm{End}_{\mathrm{NChow}(k)_{\mathbb{Q}}}(U(\mathcal{A})_{\mathbb{Q}})$ such that $HP^{\pm}(\underline{\pi}^+_{\mathcal{A}}) = \pi^+_{\mathcal{A}}$.

2.4. **Noncommutative standard conjecture of type D.** Given a smooth proper dg category $\mathcal{A}$, consider the $\mathbb{Q}$-vector spaces $K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim \mathrm{hom}}$ and $K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim \mathrm{num}}$ defined as $\mathrm{Hom}_{\mathrm{NHom}(k)_{\mathbb{Q}}}(U(k)_{\mathbb{Q}}, U(\mathcal{A})_{\mathbb{Q}})$ and $\mathrm{Hom}_{\mathrm{NNum}(k)_{\mathbb{Q}}}(U(k)_{\mathbb{Q}}, U(\mathcal{A})_{\mathbb{Q}})$. Following [17], the conjecture $D(\mathcal{A})$ asserts that $K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim \mathrm{hom}} = K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim \mathrm{num}}$.

2.5. **Orbit categories.** Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a $\mathbb{Q}$-linear symmetric monoidal additive category and let $\mathcal{O} \in \mathcal{C}$ be a $\otimes$-invertible object. The *orbit category* $\mathcal{C}/_{-\otimes\mathcal{O}}$ has the same objects as $\mathcal{C}$ and morphisms $\mathrm{Hom}_{\mathcal{C}/_{-\otimes\mathcal{O}}}(a,b) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(a, b\otimes \mathcal{O}^{\otimes n})$. Given objects a,b,c and morphisms $\underline{f} = \{f_n\}_{n\in\mathbb{Z}}$ and $\underline{g} = \{g_n\}_{n\in\mathbb{Z}}$, the ith component of $\underline{g}\circ\underline{f}$ is defined as $\sum_n (g_{i-n}\otimes \mathcal{O}^{\otimes n})\circ f_n$. The canonical functor $\iota\colon \mathcal{C} \to \mathcal{C}/_{-\otimes\mathcal{O}}$, given by $a \mapsto a$ and $f \mapsto \underline{f} = \{f_n\}_{n\in\mathbb{Z}}$, where $f_0 = f$ and $f_n = 0$ if $n \neq 0$, is endowed with an isomorphism $\iota \circ (-\otimes \mathcal{O}) \Rightarrow \iota$ and is 2-universal among all such functors. Finally, the category $\mathcal{C}/_{-\otimes\mathcal{O}}$ is $\mathbb{Q}$-linear, additive, and inherits from $\mathcal{C}$ a symmetric monoidal structure making ι symmetric monoidal.

3. Proof of Theorem 1.1

We start by proving the equivalence $C^+(X) \Leftrightarrow C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$. The implication $C^+(X) \Rightarrow C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$ was proved in [17, Thm. 1.3]. Hence, we will prove solely the converse implication. Since k is of characteristic zero, all the (classical) Weil cohomology theories H^* are equivalent; see [1, §3.4.2]. Therefore, in the proof we can (and will) make use solely of de Rham cohomology theory H^*_{dR}.

[5]In loc. cit. we used the notation $C_{\mathrm{nc}}(\mathcal{A})$ instead of $C^+_{\mathrm{nc}}(\mathcal{A})$.

Let us denote by $\mathrm{Chow}(k)_{\mathbb{Q}}$ the classical category of Chow motives. By construction, this rigid symmetric monoidal category comes equipped with a (contravariant) $\otimes$-functor $\mathfrak{h}(-)_{\mathbb{Q}}\colon \mathrm{SmProj}(k)^{\mathrm{op}} \to \mathrm{Chow}(k)_{\mathbb{Q}}$ defined on smooth projective k-schemes. As explained in [19, Thm. 4.3], there exists a $\mathbb{Q}$-linear, fully faithful, $\otimes$-functor Φ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{SmProj}(k)^{\mathrm{op}} \arrow[r, "X\mapsto \mathrm{perf}_{\mathrm{dg}}(X)"] \arrow[d, "\mathfrak{h}(-)_{\mathbb{Q}}"'] & \mathrm{dgcat}_{\mathrm{sp}}(k) \arrow[dd, "U(-)_{\mathbb{Q}}"] \\
\mathrm{Chow}(k)_{\mathbb{Q}} \arrow[d, "\iota"'] & \\
\mathrm{Chow}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)} \arrow[r, "\Phi"'] & \mathrm{NChow}(k)_{\mathbb{Q}}
\end{tikzcd}$$

where $\mathrm{Chow}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}$ stands for the orbit category with respect to the Tate motive $\mathbb{Q}(1)$. Consider the following composition:

$$\mathrm{Chow}(k)_{\mathbb{Q}} \xrightarrow{\iota} \mathrm{Chow}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)} \xrightarrow{\Phi} \mathrm{NChow}(k)_{\mathbb{Q}} \xrightarrow{HP^{\pm}} \mathrm{Vect}_{\mathbb{Z}/2}(k)\,. \tag{3.1}$$

Consider also the category $\mathrm{Vect}_{\mathbb{Z}}(k)$ of finite-dimensional $\mathbb{Z}$-graded k-vector spaces and the associated "2-perioditization" $\otimes$-functor:

$$\mathrm{Vect}_{\mathbb{Z}}(k) \longrightarrow \mathrm{Vect}_{\mathbb{Z}/2}(k) \qquad \{V_i\}_{i\in\mathbb{Z}} \mapsto \left(\bigoplus_{i\,\mathrm{even}} V_i, \bigoplus_{i\,\mathrm{odd}} V_i\right). \tag{3.2}$$

Given any smooth (projective) k-scheme X, the Hochschild–Kostant–Rosenberg theorem identifies $HP^{+}(\mathrm{perf}_{\mathrm{dg}}(X))$ and $HP^{-}(\mathrm{perf}_{\mathrm{dg}}(X))$ with $\bigoplus_{i\,\mathrm{even}} H^i_{dR}(X)$ and $\bigoplus_{i\,\mathrm{odd}} H^i_{dR}(X)$, respectively. Consequently, (3.1) reduces to the composition of

$$H^*_{dR}\colon \mathrm{Chow}(k)_{\mathbb{Q}} \longrightarrow \mathrm{Vect}_{\mathbb{Z}}(k) \qquad \mathfrak{h}(X)_{\mathbb{Q}} \mapsto \bigoplus_{i=0}^{2\dim(X)} H^i_{dR}(X) \tag{3.3}$$

with the above functor (3.2). Assume now the conjecture $C^{+}_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$. Since Φ is an equivalence of categories, there then exists an endomorphism $\underline{\pi}^{+} = \{\pi^{+}_n\}_{n\in\mathbb{Z}}$ of $\iota(\mathfrak{h}(X)_{\mathbb{Q}})$ in the orbit category $\mathrm{Chow}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}$ whose image under the composed functor $HP^{\pm}\circ\Phi$ agrees with the homomorphism of $\mathbb{Z}/2$-graded k-vector spaces:

$$\pi^{+}\colon \left(\bigoplus_{i\,\mathrm{even}} H^i_{dR}(X), \bigoplus_{i\,\mathrm{odd}} H^i_{dR}(X)\right) \stackrel{(\mathrm{id},0)}{\longrightarrow} \left(\bigoplus_{i\,\mathrm{even}} H^i_{dR}(X), \bigoplus_{i\,\mathrm{odd}} H^i_{dR}(X)\right).$$

Note that π^{+} is the image of the even Künneth projector $\pi^{+}_X := \sum_i \pi^{2i}_X$ under (3.2). Note also that $H^*_{dR}(\pi^{+}_n)$ is an homomorphism of degree $-2n$. The preceding considerations, combined with the construction of the orbit category $\mathrm{Chow}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}$, then allows us to conclude that $H^*_{dR}(\pi^{+}_0) = \pi^{+}_X$ and $H^*_{dR}(\pi^{+}_n) = 0$ if $n \neq 0$. Consequently, the even Künneth projector π^{+}_X is algebraic and conjecture $C^{+}(X)$ holds.

Let us now prove the equivalence $D(X) \Leftrightarrow D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$. The implication $D(X) \Rightarrow D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$ was proved in [17, Thm. 1.5]. Hence, we will prove solely the converse implication. Recall from [17, page 645] the construction of the following commutative square:

$$
\begin{tikzcd}
Z^*(X)_{\mathbb{Q}}/_{\sim\mathrm{hom}} \arrow[r, two heads] \arrow[d] & K_0(\mathrm{perf}_{\mathrm{dg}}(X))_{\mathbb{Q}}/_{\sim\mathrm{hom}} \arrow[d] \\
Z^*(X)_{\mathbb{Q}}/_{\sim\mathrm{num}} \arrow[r, "\simeq"'] & K_0(\mathrm{perf}_{\mathrm{dg}}(X))_{\mathbb{Q}}/_{\sim\mathrm{num}}
\end{tikzcd}
\tag{3.4}
$$

Note that the conjecture $D(X)$, resp., $D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$, is equivalent to the injectivity of the vertical left-hand side, resp., right-hand side, homomorphism. Assume now conjecture $D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$, i.e., that the vertical right-hand side homomorphism in (3.4) is injective. By construction, the vertical left-hand side homomorphism in (3.4) is a diagonal (matrix) homomorphism:

$$Z^*(X)_{\mathbb{Q}}/_{\sim\mathrm{hom}} = \bigoplus_{n=0}^{\dim(X)} Z^n(X)_{\mathbb{Q}}/_{\sim\mathrm{hom}} \to \bigoplus_{n=0}^{\dim(X)} Z^n(X)_{\mathbb{Q}}/_{\sim\mathrm{num}} = Z^*(X)_{\mathbb{Q}}/_{\sim\mathrm{num}}\,.$$

Therefore, in order to prove conjecture $D(X)$ it suffices to show that the following homomorphisms are injective:

$$Z^n(X)_{\mathbb{Q}}/_{\sim\mathrm{hom}} \longrightarrow K_0(\mathrm{perf}_{\mathrm{dg}}(X))_{\mathbb{Q}}/_{\sim\mathrm{hom}}, \qquad 0 \leq n \leq \dim(X)\,. \tag{3.5}$$

As explained above, we have (3.1) = (3.2) $\circ$ (3.3). This implies that the classical category of homological motives $\mathrm{Hom}(k)_{\mathbb{Q}}$ agrees with the idempotent completion of the quotient $\mathrm{Chow}(k)_{\mathbb{Q}}/\mathrm{Ker}((3.1))$. Moreover, the induced functor $\mathrm{Hom}(k)_{\mathbb{Q}} \to \mathrm{NHom}(k)_{\mathbb{Q}}$ is faithful. Under the identifications

$$\mathrm{Hom}_{\mathrm{Hom}(k)_{\mathbb{Q}}}(\mathfrak{h}(\mathrm{Spec}(k))_{\mathbb{Q}}, \mathfrak{h}(X)_{\mathbb{Q}}(n)) \simeq Z^n(X)_{\mathbb{Q}}/_{\sim\mathrm{hom}}\,,$$

$$\mathrm{Hom}_{\mathrm{NHom}(k)_{\mathbb{Q}}}(U(k)_{\mathbb{Q}}, U(\mathrm{perf}_{\mathrm{dg}}(X))_{\mathbb{Q}} \simeq K_0(\mathrm{perf}_{\mathrm{dg}}(X))_{\mathbb{Q}}/_{\sim\mathrm{hom}}\,,$$

where $\mathfrak{h}(X)_{\mathbb{Q}}(n)$ stands for $\mathfrak{h}(X)_{\mathbb{Q}} \otimes \mathbb{Q}(1)^{\otimes n}$, the homomorphisms (3.5) correspond to the homomorphisms induced by the functor $\mathrm{Hom}(k)_{\mathbb{Q}} \to \mathrm{NHom}(k)_{\mathbb{Q}}$:

$$\mathrm{Hom}_{\mathrm{Hom}(k)_{\mathbb{Q}}}(\mathfrak{h}(\mathrm{Spec}(k))_{\mathbb{Q}}, \mathfrak{h}(X)_{\mathbb{Q}}(n)) \longrightarrow \mathrm{Hom}_{\mathrm{NHom}(k)_{\mathbb{Q}}}(U(k)_{\mathbb{Q}}, U(\mathrm{perf}_{\mathrm{dg}}(X))_{\mathbb{Q}})\,.$$

Since these latter homomorphisms are injective, we hence conclude that the conjecture $D(X)$ holds. This finishes the proof.

4. Proof of Theorem 1.2

As proved in [12, Thm. 4.2], the category $\mathrm{perf}(Q)$ admits a semiorthogonal decomposition $\langle \mathrm{perf}(S;\mathcal{F}), \mathrm{perf}(S)_1, \ldots, \mathrm{perf}(S)_d \rangle$, where $\mathcal{F}$ stands for the sheaf of even Clifford algebras associated to q and $\mathrm{perf}(S)_i := q^*\mathrm{perf}(S) \otimes \mathcal{O}_{Q/S}(i)$. Note that $\mathrm{perf}(S)_i \simeq \mathrm{perf}(S)$. As explained in [19, §2.4.1], this semiorthogonal decomposition gives rise to a direct sum decomposition in the category $\mathrm{NChow}(k)_{\mathbb{Q}}$:

$$U(\mathrm{perf}_{\mathrm{dg}}(Q))_{\mathbb{Q}} \simeq U(\mathrm{perf}_{\mathrm{dg}}(S;\mathcal{F}))_{\mathbb{Q}} \oplus U(\mathrm{perf}_{\mathrm{dg}}(S))_{\mathbb{Q}} \oplus \cdots \oplus U(\mathrm{perf}_{\mathrm{dg}}(S))_{\mathbb{Q}}\,.$$

Making use of the definition of the noncommutative standard conjectures of type C^+ and D, we hence obtain the following equivalence of conjectures:

$$C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(Q)) \Leftrightarrow C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(S;\mathcal{F})) + C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(S)), \tag{4.1}$$

$$D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(Q)) \Leftrightarrow D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(S;\mathcal{F})) + D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(S))\,. \tag{4.2}$$

Since d is even and the discriminant division of q is smooth, the category $\mathrm{perf}(S;\mathcal{F})$ is equivalent (via a Fourier–Mukai functor) to $\mathrm{perf}(\widetilde{S};\widetilde{\mathcal{F}})$, where $\widetilde{S}$ is the discriminant double cover of S and $\widetilde{\mathcal{F}}$ is a sheaf of Azumaya algebras over $\widetilde{S}$; see [12, Prop. 3.13]. This implies that $\mathrm{perf}_{\mathrm{dg}}(S;\mathcal{F})$ is derived Morita equivalent to $\mathrm{perf}_{\mathrm{dg}}(\widetilde{S};\widetilde{\mathcal{F}})$. Using the fact that $U(-)_{\mathbb{Q}}$ inverts Morita equivalences and also the isomorphism between $U(\mathrm{perf}_{\mathrm{dg}}(\widetilde{S};\widetilde{\mathcal{F}}))_{\mathbb{Q}}$ and $U(\mathrm{perf}_{\mathrm{dg}}(\widetilde{S}))_{\mathbb{Q}}$ established in [20, Thm. 2.1], we hence conclude that the right-hand side of (4.1), resp., (4.2), reduces to $C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\widetilde{S}))+C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(S))$, resp., to $D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\widetilde{S}))+D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(S))$. Finally, since the dimension of S is ≤ 2, resp., ≤ 4, the aforementioned work of Kleiman and Lieberman, combined with Theorem 1.1, implies conjecture $C^+(Q)$, resp., $D(Q)$.

5. Proof of Theorem 1.4

The proof is similar for the noncommutative standard conjectures of type C^+ and type D. Therefore, we will prove solely the first case.

By definition of the Lefschetz decomposition $\langle \mathbb{A}_0, \mathbb{A}_1(1), \ldots, \mathbb{A}_{i-1}(i-1)\rangle$, we have a chain of admissible triangulated subcategories $\mathbb{A}_{i-1}\subseteq \cdots \subseteq \mathbb{A}_1 \subseteq \mathbb{A}_0$ and $\mathbb{A}_r(r) := \mathbb{A}_r \otimes \mathcal{O}_X(r)$. Note that $\mathbb{A}_r(r)\simeq \mathbb{A}_r$. Let $\mathfrak{a}_r$ be the right orthogonal complement to $\mathbb{A}_{r+1}$ in $\mathbb{A}_r$; these are called the *primitive subcategories* in [13, §4]. Note that we have semiorthogonal decompositions:

$$\mathbb{A}_r = \langle \mathfrak{a}_r, \mathfrak{a}_{r+1}, \ldots, \mathfrak{a}_{i-1}\rangle, \qquad 0\leq r\leq i-1\,. \tag{5.1}$$

As proved in [13, Thm. 6.3] and [11, §2.4], the category $\mathrm{perf}(Y;\mathcal{F})$ admits an HP-dual Lefschetz decomposition $\langle \mathbb{B}_{j-1}(1-j), \mathbb{B}_{j-2}(2-j), \ldots, \mathbb{B}_0\rangle$ with respect to $\mathcal{O}_Y(1)$. As above, we have a chain of admissible subcategories $\mathbb{B}_{j-1}\subseteq \mathbb{B}_{j-2}\subseteq \cdots \subseteq \mathbb{B}_0$. Moreover, the primitive subcategories coincide (via a Fourier–Mukai functor) with those of $\mathrm{perf}(X)$ and we have semiorthogonal decompositions:

$$\mathbb{B}_r = \langle \mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_{\dim(V)-r-2}\rangle, \qquad 0\leq r\leq j-1\,. \tag{5.2}$$

Furthermore, the assumptions $\dim(X_L)=\dim(X)-\dim(L)$ and $\dim(Y_L)=\dim(Y)-\dim(L^\perp)$ imply the existence of semiorthogonal decompositions

$$\mathrm{perf}(X_L) = \langle \mathbb{C}_L, \mathbb{A}_{\dim(V)}(1), \ldots, \mathbb{A}_{i-1}(i-\dim(V))\rangle\,, \tag{5.3}$$

$$\mathrm{perf}(Y_L;\mathcal{F}_L) = \langle \mathbb{B}_{j-1}(\dim(L^\perp)-j), \ldots, \mathbb{B}_{\dim(L^\perp)}(-1), \mathbb{C}_L\rangle\,, \tag{5.4}$$

where $\mathbb{C}_L$ is a common triangulated category. Let us denote by $\mathbb{C}_L^{\mathrm{dg}}$, $\mathbb{A}_r^{\mathrm{dg}}$, and $\mathfrak{a}_r^{\mathrm{dg}}$ the dg enhancement of $\mathbb{C}_L$, $\mathbb{A}_r$, and $\mathfrak{a}_r$ induced from $\mathrm{perf}_{\mathrm{dg}}(X_L)$. Similarly, let us denote by $\mathbb{C}_L^{\mathrm{dg}'}$ and $\mathbb{B}_r^{\mathrm{dg}}$ the dg enhancement of $\mathbb{C}_L$ and $\mathbb{B}_r$ induced from $\mathrm{perf}_{\mathrm{dg}}(Y_L;\mathcal{F}_L)$. Note that since X_L is a smooth proper k-scheme and $\mathrm{perf}_{\mathrm{dg}}(Y_L;\mathcal{F}_L)$ a smooth proper dg category, all the preceding dg categories are smooth and proper; see [2, Lem. 2.1]. As explained in [19, §2.4.1], the above semiorthogonal decompositions (5.3)–(5.4) give rise to the direct sums decompositions in the category $\mathrm{NChow}(k)_{\mathbb{Q}}$:

$$U(\mathrm{perf}_{\mathrm{dg}}(X_L))_{\mathbb{Q}} \simeq U(\mathbb{C}_L^{\mathrm{dg}})_{\mathbb{Q}} \oplus U(\mathbb{A}^{\mathrm{dg}}_{\dim(V)})_{\mathbb{Q}} \oplus \cdots \oplus U(\mathbb{A}^{\mathrm{dg}}_{i-1})_{\mathbb{Q}}\,,$$

$$U(\mathrm{perf}_{\mathrm{dg}}(Y_L;\mathcal{F}_L))_{\mathbb{Q}} \simeq U(\mathbb{B}^{\mathrm{dg}}_{j-1})_{\mathbb{Q}} \oplus \cdots \oplus U(\mathbb{B}^{\mathrm{dg}}_{\dim(L^\perp)})_{\mathbb{Q}} \oplus U(\mathbb{C}_L^{\mathrm{dg}'})_{\mathbb{Q}}\,.$$

Making use of the definition of the noncommutative standard conjecture of type C^+, we hence obtain the following equivalences of conjectures:

$$C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X_L)) \Leftrightarrow C^+_{\mathrm{nc}}(\mathbb{C}^{\mathrm{dg}}_L) + C^+_{\mathrm{nc}}(\mathbb{A}^{\mathrm{dg}}_{\dim(V)}) + \cdots + C^+_{\mathrm{nc}}(\mathbb{A}^{\mathrm{dg}}_{i-1})\,, \tag{5.5}$$

$$C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(Y_L;\mathcal{F}_L)) \Leftrightarrow C^+_{\mathrm{nc}}(\mathbb{B}^{\mathrm{dg}}_{j-1}) + \cdots + C^+_{\mathrm{nc}}(\mathbb{B}^{\mathrm{dg}}_{\dim(L^\perp)}) + C^+_{\mathrm{nc}}(\mathbb{C}^{\mathrm{dg}'}_L)\,. \tag{5.6}$$

On the one hand, since the conjecture $C^+_{\mathrm{nc}}(\mathbb{A}^{\mathrm{dg}}_0)$ holds, we conclude from the semi-orthogonal decompositions (5.1)–(5.2) that the conjectures $C^+_{\mathrm{nc}}(\mathbb{A}^{\mathrm{dg}}_r)$ and $C^+_{\mathrm{nc}}(\mathbb{B}^{\mathrm{dg}}_r)$ hold for every r. This implies that the right-hand side of (5.5), resp., (5.6), reduces to $C^+_{\mathrm{nc}}(\mathbb{C}^{\mathrm{dg}}_L)$, resp., $C^+_{\mathrm{nc}}(\mathbb{C}^{\mathrm{dg}'}_L)$. On the other hand, since the composed functor $\mathrm{perf}(X_L) \to \mathbb{C}_L \to \mathrm{perf}(Y_L;\mathcal{F}_L)$ is of Fourier–Mukai type, the dg categories $\mathbb{C}^{\mathrm{dg}}_L$ and $\mathbb{C}^{\mathrm{dg}'}_L$ are derived Morita equivalent. Using the fact that the functor $U(-)_{\mathbb{Q}}$ inverts derived Morita equivalences, this implies that $C^+_{\mathrm{nc}}(\mathbb{C}^{\mathrm{dg}}_L) \Leftrightarrow C^+_{\mathrm{nc}}(\mathbb{C}^{\mathrm{dg}'}_L)$. Finally, since X_L is a smooth projective k-scheme, the proof now follows from the equivalence $C^+(X_L) \Leftrightarrow C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X_L))$ of Theorem 1.1.

6. Proof of Theorem 1.7

Similarly to the proof of Theorem 1.2, since the fibration $q\colon Q \to \mathbb{P}(S^2W^*)$ is of relative dimension $d-2$, d is even, and the discriminant divisor of q is smooth, the conjecture $C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}(L);\mathcal{F}_L))$, resp., $D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}(L);\mathcal{F}_L))$, reduces to conjecture $C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\widetilde{\mathbb{P}(L)}))$, resp., $D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\widetilde{\mathbb{P}(L)}))$. The proof then follows from the assumption that the dimension of L is ≤ 3, resp., ≤ 5, from the aforementioned work of Kleiman and Lieberman, and from Theorem 1.1.

7. Proof of Theorem 1.13

The proof is similar for the noncommutative standard conjectures of type C^+ and type D. Therefore, we will prove solely the first case.

Let us assume first that $\dim(L) \leq 3$ and that d is odd. Recall from [18, §8] the construction of a certain smooth projective double cover Y of $\mathbb{P}(L)$. Since by assumption $\dim(L) \leq 3$, the conjecture $C^+(Y) \Leftrightarrow C^+_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(Y))$ holds.

(i) When $\mathrm{codim}(\mathcal{X}_L) > d$, we have a semiorthogonal decomposition

$$\mathrm{perf}(Y) = \langle \mathrm{perf}(\mathcal{X}_L), {}^\perp\mathrm{perf}(\mathcal{X}_L)\rangle\,,$$

where ${}^\perp\mathrm{perf}(\mathcal{X}_L)$ stands for the left orthogonal to $\mathrm{perf}(\mathcal{X}_L)$ in $\mathrm{perf}(Y)$; see [18, Thm. 1.1 and Prop. 1.2]. Similarly to the proof of Theorem 1.4, we then conclude that the noncommutative Chow motive $U(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X}_L))_{\mathbb{Q}}$ is a direct summand of $U(\mathrm{perf}_{\mathrm{dg}}(Y))_{\mathbb{Q}}$. By definition of the noncommutative standard conjecture of type C^+, this implies that the conjecture $C^+(\mathcal{X}_L)$ holds.

(ii) When $\mathrm{codim}(\mathcal{X}_L) = d$, the category $\mathrm{perf}(\mathcal{X}_L)$ is equivalent (via a Fourier-Mukai functor) to $\mathrm{perf}(Y)$; see [18, Thm. 1.1 and Prop. 1.2]. Similarly to the proof of Theorem 1.4, we then conclude that $U(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X}_L))_{\mathbb{Q}} \simeq U(\mathrm{perf}_{\mathrm{dg}}(Y))_{\mathbb{Q}}$. This implies that the conjecture $C^+(\mathcal{X}_L) \Leftrightarrow C^+(Y)$ holds.

(iii) When $\mathrm{codim}(\mathcal{X}_L) < d$, we have a semiorthogonal decomposition

$$\mathrm{perf}(\mathcal{X}_L) = \langle \mathrm{perf}(Y), \mathcal{E}_1, \ldots, \mathcal{E}_n\rangle\,,$$

where $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are certain exceptional objects; see [18, Thm. 1.1 and Props. 1.2 and 5.16]. Similarly to the proof of Theorem 1.4, we then conclude that $U(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X}_L))_{\mathbb{Q}} \simeq U(\mathrm{perf}_{\mathrm{dg}}(Y))_{\mathbb{Q}} \oplus U(k)_{\mathbb{Q}}^{\oplus n}$. This implies that $C^+(\mathcal{X}_L) \Leftrightarrow C^+(Y)$. In particular, the conjecture $C^+(\mathcal{X}_L)$ holds.

Let us now assume that $\dim(L) \leq 3$ and that d is even. Recall from [18, §9] the construction of a certain smooth projective double cover Y of $\mathbb{P}(L)$. Since by assumption $\dim(L) \leq 3$, the conjecture $C^+(Y)$ holds.

(i) When $\mathrm{codim}(\mathcal{X}_L) > d$, we have a semiorthogonal decomposition

$$\mathrm{perf}(Y) = \langle \mathrm{perf}(\mathcal{X}_L), {}^{\perp}\mathrm{perf}(\mathcal{X}_L) \rangle ;$$

see [18, Thm. 1.1 and Prop. 1.3]. Similarly to the above item (i), this implies that the conjecture $C^+(\mathcal{X}_L)$ holds.

(ii) When $d/2 < \mathrm{codim}(\mathcal{X}_L) \leq d$, we have semiorthogonal decompositions

$$\mathrm{perf}(\mathcal{X}_L) = \langle \mathbb{C}_L, \mathcal{E}_1, \ldots, \mathcal{E}_n \rangle, \qquad \mathrm{perf}(Y) = \langle \mathbb{C}_L, {}^{\perp}\mathbb{C}_L \rangle ,$$

where $\mathbb{C}_L$ is a common triangulated category; see [18, Thm. 1.1 and Props. 1.3 and 5.16]. Similarly to the proof of Theorem 1.4, we then conclude that $U(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X}_L))_{\mathbb{Q}} \simeq U(\mathbb{C}_L^{\mathrm{dg}})_{\mathbb{Q}} \oplus U(k)_{\mathbb{Q}}^{\oplus n}$ and that $U(\mathbb{C}_L^{\mathrm{dg}})_{\mathbb{Q}}$ is a direct summand of $U(\mathrm{perf}_{\mathrm{dg}}(Y))_{\mathbb{Q}}$. This implies that $C^+(\mathcal{X}_L) \Leftrightarrow C^+_{\mathrm{nc}}(\mathbb{C}_L^{\mathrm{dg}})$ and that the conjecture $C^+_{\mathrm{nc}}(\mathbb{C}_L^{\mathrm{dg}})$ holds. Consequently, the conjecture $C^+(\mathcal{X}_L)$ also holds.

(iii) When $\mathrm{codim}(\mathcal{X}_L) \leq d/2$, we have a semiorthogonal decomposition

$$\mathrm{perf}(\mathcal{X}_L) = \langle \mathrm{perf}(Y), \mathcal{E}_1, \ldots, \mathcal{E}_n \rangle ,$$

where $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are certain exceptional objects; see [18, Thm. 1.1 and Props. 1.3 and 5.16]. Similarly to the above item (iii), this implies conjecture $C^+(\mathcal{X}_L)$.

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