PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 5, May 2018, Pages 1921–1935
http://dx.doi.org/10.1090/proc/13869
Article electronically published on December 26, 2017

RHOMBIC TILINGS AND BOTT–SAMELSON VARIETIES

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(Communicated by Patricia Hersh)

ABSTRACT. S. Elnitsky (1997) gave an elegant bijection between rhombic tilings of $2n$-gons and commutation classes of reduced words in the symmetric group on n letters. P. Magyar (1998) found an important construction of the Bott–Samelson varieties introduced by H. C. Hansen (1973) and M. Demazure (1974). We explain a natural connection between S. Elnitsky's and P. Magyar's results. This suggests using tilings to encapsulate Bott–Samelson data (in type A). It also indicates a geometric perspective on S. Elnitsky's bijection. We also extend this construction by assigning desingularizations of Schubert varieties to the zonotopal tilings considered by B. Tenner (2006).

1. INTRODUCTION

Let $X = \mathsf{Flags}(\mathbb{C}^n)$ be the variety of complete flags $\mathbb{C}^0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n$. The group $\mathsf{GL_n}(\mathbb{C})$ acts on the variety X by change of basis, as does its subgroup B of invertible upper triangular matrices and its maximal torus T of invertible diagonal matrices. The T-fixed points are in bijection with permutations w in the symmetric group $\mathfrak{S}_n$: they are the flags $F_\bullet^{(w)}$ defined by $F_k^{(w)} = \langle \vec{e}_{w(1)}, \vec{e}_{w(2)}, \ldots, \vec{e}_{w(k)} \rangle$, where $\vec{e}_i$ is the i-th standard basis vector. The **Schubert variety** X_w is the B-orbit closure of $F_\bullet^{(w)}$.

There is longstanding interest in singularities of Schubert varieties; see, for example, the text by S. Billey–V. Lakshmibai [BL00]. Famously, H. C. Hansen [Han73] and M. Demazure [Dem74] independently presented (in all Lie types) resolutions of singularities $BS^{(i_1,i_2,\ldots,i_{\ell(w)})}$ of X_w, one for each reduced word $s_{i_1}s_{i_2}\cdots s_{i_{\ell(w)}}$ of w. M. Demazure called these resolutions **Bott–Samelson varieties** in reference to a related construction of R. Bott–H. Samelson [BS55]. In more recent work, P. Magyar [Mag98] found an important description of Bott–Samelson varieties.

We propose a canonical connection between P. Magyar's work and the rhombic tilings of S. Elnitsky [Eln97]. P. Magyar [Mag98, §1.3] describes a Bott–Samelson variety as a subvariety of a product of Grassmannians determined by some incidence relations among the vector spaces. Our description in terms of rhombic tilings gives an alternate description of the points in this variety. In this way, tilings graphically encapsulate Bott–Samelson data.

Received by the editors July 13, 2016, and, in revised form, July 6, 2017.
2010 *Mathematics Subject Classification.* Primary 05B45, 05E15, 14M15.
The second author was supported by an NSF Graduate Research Fellowship.
The third author was partially supported by a Simons Foundation Collaboration Grant for Mathematicians.
The fourth author was supported by an NSF grant.

Other ways to present Bott–Samelsons via combinatorial diagrams have been previously described, such as X. Viennot's *heaps* [Vie89]; see also N. Perrin's [Per07] and B. Jones–A. Woo's [JW13]. In addition, R. Vakil [Vak06] introduces *quilts* to describe particular Bott–Samelson resolutions of certain Schubert varieties; these quilts can be seen as deformations of certain Elnitsky tilings. Our connection to the work of Elnitsky however is new and leads to natural generalizations.

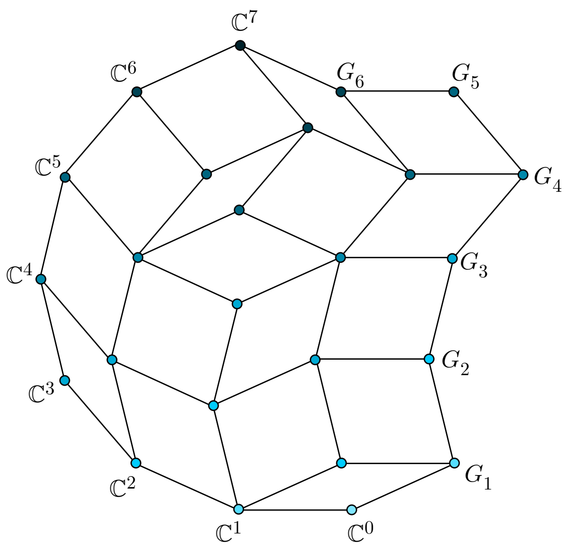


FIGURE 1. The rhombic tiling picture of Bott–Samelson varieties, for the polygon $\mathsf{E}(7456312)$

In order to state the main result of this work, we must introduce the primary objects and notation. We do this briefly, for now, postponing a more thorough treatment to Section 2.

Given a permutation $w \in \mathfrak{S}_n$, the **Elnitsky $2n$-gon** $\mathsf{E}(w)$ has sides labeled, in order, by $1, 2, \ldots, n, w(n), w(n-1), \ldots, w(1)$, in which the first n sides form half of a regular $2n$-gon, and sides with equal labels are parallel and congruent. Figure 1 shows the Elnitsky 14-gon for the permutation $7456312 \in \mathfrak{S}_7$, without edge labels, and this example will be referenced later in this work.

Let $\mathcal{T}(w)$ be the set of **rhombic tilings** of $\mathsf{E}(w)$ in which the rhombi's edges are parallel and congruent to edges of $\mathsf{E}(w)$. The main result of S. Elnitsky's aforementioned work is that the set $\mathcal{T}(w)$ is in bijection with the commutation classes of reduced words for w [Eln97, Theorem 2.2].

For a tiling $T \in \mathcal{T}(w)$, we introduce the notion of a **T-flag** of subspaces of $\mathbb{C}^n$. Starting with the vertex between the edges labeled 1 and $w(1)$, label the vertices of $\mathsf{E}(w)$ in clockwise order by

$$H_0, H_1, \ldots, H_n, G_{n-1}, G_{n-2}, \ldots, G_1.$$

A T-flag is an assignment $\mathcal{V}$ of a linear subspace $V_x \subseteq \mathbb{C}^n$ to each vertex x in the tiling T, subject to the following conditions:

- the dimension of V_x is the minimal path length from H_0 to x along tile edges;

- V_{H_i} is the span of the first i standard basis vectors of $\mathbb{C}^n$; and
- for adjacent vertices x and y in T with y further from H_0, we have $V_x \subset V_y$.

(In Figure 1, we have only labeled the external vertices, identifying V_{H_i} with $\mathbb{C}^i$.) Now we define a parameter space $\mathcal{Z}_T$ for T-flags by

$$\mathcal{Z}_T := \left\{\mathcal{V} = (V_x)_{x\in \mathtt{Vert}(T)} : \mathcal{V} \text{ is a } T\text{-flag}\right\} \subset \prod_{x\in\mathtt{Vert}(T)} \mathsf{Gr}_{\dim(V_x)}(\mathbb{C}^n),$$

where $\mathsf{Gr}_k(\mathbb{C}^n)$ is the Grassmannian of k-dimensional linear subspaces of $\mathbb{C}^n$.

Define the map $\pi : \mathcal{Z}_T \to X$ by forgetting all vector spaces except those associated to the vertices $G_1, G_2, \ldots, G_{n-1}$. In our example, π maps the point depicted in Figure 1 to the complete flag

$$\mathbb{C}^0 \subset V_{G_1} \subset V_{G_2} \subset V_{G_3} \subset V_{G_4} \subset V_{G_5} \subset V_{G_6} \subset \mathbb{C}^7.$$

The following theorem suggests a Schubert-geometric interpretation of tilings of Elnitsky polygons.

Theorem 1.1. *For $T \in \mathcal{T}(w)$, $\mathcal{Z}_T$ is a Bott–Samelson variety, i.e., a desingularization $\pi : \mathcal{Z}_T \to X_w$. Conversely, every Bott–Samelson variety $BS^{(i_1,\ldots,i_{\ell(w)})}$ is canonically isomorphic to $\mathcal{Z}_T$ for some $T \in \mathcal{T}(w)$, where $w = s_{i_1}\ldots s_{i_{\ell(w)}}$ and T is given in an explicit manner by* [Eln97, Theorem 2.2].

A feature of this construction is that it extends naturally to the more general *zonotopal tilings* of the Elnitsky $2n$-gon studied in [Ten06]. That is, for each such zonotopal tiling Z, we construct in Section 4 an analogous parameter space $\mathcal{Z}_Z$ of Z-flags that is again a desingularization of the appropriate Schubert variety. To the best of our knowledge, these *generalized Bott–Samelson varieties* have not previously appeared in the literature. They can be seen in some sense as interpolating between Bott–Samelson resolutions.

As promised, we devote Section 2 to a discussion of the relevant results of Elnitsky and Magyar. The reader who is already familiar with these objects may choose to skip ahead to Section 3, where we prove Theorem 1.1. The remainder of this paper concerns other Bott–Samelson data encoded by tilings. In Section 4, we explain how the hexagon *flips* of [Eln97, Section 3] may be interpreted geometrically. This naturally leads to consideration of zonotopal tilings and generalized Bott–Samelsons. We collect some additional discussion in Section 5; in particular, we explain how coloring rhombi of a tiling describes T-fixed points as well as a standard stratification of a Bott–Samelson variety.

2. Background

2.1. **Elnitsky's polygon.** The symmetric group $\mathfrak{S}_n$ can be generated by the **simple reflections** $\{s_i : 1 \leq i < n\}$, where s_i is the permutation transposing i and $i+1$, while leaving all other elements fixed. Simple reflections satisfy the **Coxeter relations**

$$s_i^2 = e; \tag{Cox.1}$$

$$s_is_j = s_js_i, \text{ when } |i-j| > 1; \text{ and} \tag{Cox.2}$$

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \text{ when } i < n-1. \tag{Cox.3}$$

Any $w \in \mathfrak{S}_n$ can be written (in infinitely many ways) as a product of simple reflections.

The fewest simple reflections needed to represent a permutation w is the **length** of w, denoted $\ell(w)$. A realization of $w \in \mathfrak{S}_n$ as a product of exactly $\ell(w)$ simple reflections is a **reduced word** for w. The collection of all reduced words for w is denoted $R(w)$.

Example 2.1. It is quick to see that

$$s_1s_2s_3s_1 = s_1s_2s_1s_3 = s_2s_1s_2s_3 = 3241 \in \mathfrak{S}_4$$

and that no shorter product can produce this permutation. Thus $\ell(3241) = 4$ and

$$R(3241) = \{s_1s_2s_3s_1, s_1s_2s_1s_3, s_2s_1s_2s_3\}.$$

Note that although $s_1s_2s_1s_3s_1s_1 = 3241$ as well, this product has more than four terms and so is not a reduced word for 3241.

Two reduced words for w are **commutation equivalent**, denoted $\sim$, if they can be obtained from each other using only the second relation (Cox.2) listed above. This notion yields an equivalence relation on the set $R(w)$ of reduced words for w, and we write $C(w) := R(w)/\sim$ for the set of **commutation classes** defined by this equivalence.

Example 2.2. Because $s_3s_1 = s_1s_3$, we have

$$C(3241) = \big\{\{s_1s_2s_3s_1, s_1s_2s_1s_3\}, \{s_2s_1s_2s_3\}\big\}.$$

In [Eln97, Theorem 2.2], Elnitsky gave a bijection between the commutation classes $C(w)$ of reduced words for any $w \in \mathfrak{S}_n$ and the rhombic tilings of a particular $2n$-gon $\mathsf{E}(w)$. As described in Section 1, the **Elnitsky $2n$-gon** $\mathsf{E}(w)$ is a polygon with sides labeled, in order, by $1, 2, \ldots, n, w(n), w(n-1), \ldots, w(1)$. The first half of these labels form half of a regular $2n$-gon, and the remaining sides are oriented so that sides with the same label are parallel and congruent. The set $\mathcal{T}(w)$ consists of the **rhombic tilings** of $\mathsf{E}(w)$ in which all internal edges are also parallel and congruent to edges of $\mathsf{E}(w)$. It is not hard to see that each $T \in T(w)$ consists of exactly $\ell(w)$ rhombi.

The specific orientation of $\mathsf{E}(w)$ does not matter. In fact, one could allow the first n edges to form half of *any* convex $2n$-gon, regardless of angles and side lengths, so long as edges with the same labels are always parallel and congruent, and interior edges in any rhombic tiling are also parallel and congruent to the edges of $\mathsf{E}(w)$.

In this paper, we will orient our polygons so that the "first" edge, labeled 1, is horizontal and at the bottom of the picture, and the "next" edges are labeled in clockwise order. (Note that this is a reflection of the orientation depicted in [Eln97].) We will refer to this particular ordering of the edges, where the edges labeled 1, 2, ..., n are the **first n edges**, in the proof of Theorem 2.4. The vertex between the edges labeled by 1 and $w(1)$ (that is, the node at the counterclockwise end of the first edge) will be called the **source**.

Example 2.3. The Elnitsky 8-gon $\mathsf{E}(3241)$ appears in Figure 2(a), and the two elements of $\mathcal{T}(3241)$ appear in Figure 2(b). Note that, just as $|C(3241)| = 2$, we have $|\mathcal{T}(3241)| = 2$ as well.

We now state the main result of Elnitsky's work. In later discussions, it will be helpful to understand his bijection between tilings and commutation classes, so we briefly present his bijection here. Details of this argument can be found in [Eln97].

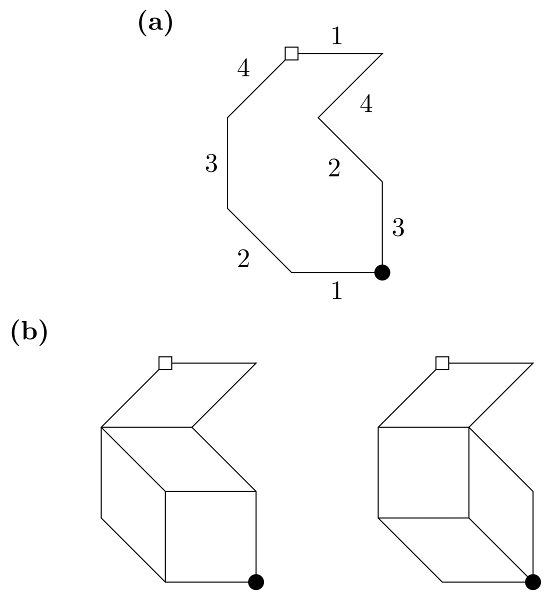


FIGURE 2. The Elnitsky 8-gon for $3241 \in \mathfrak{S}_4$, and its two rhombic tilings. In each figure, the source has been marked with a black dot. For clarity, we use a square to mark the vertex that is halfway around the 8-gon from the source.

Theorem 2.4 ([Eln97, Theorem 2.2])**.** *For any permutation w, there is a bijection between the rhombic tilings $\mathcal{T}(w)$ and the reduced word commutation classes $C(w)$.*

Proof. Fix $w \in \mathfrak{S}_n$.

Consider a tiling $T \in \mathcal{T}(w)$ in which the edges of T that coincide with edges of $\mathsf{E}(w)$ inherit the labels of those edges, and we label the interior edges of T so that parallel edges have the same labels. Let B_0 be the **base boundary** of $\mathsf{E}(w)$, formed by the first n edges of the polygon. Pick any rhombus R_1 of T that shares two edges with B_0. Set $i_1 := d_1 + 1$, where d_1 is the distance from the source to R_1, i.e., the least number of edges between the source and a vertex of R_1. Remove R_1 and define a new boundary, B_1, from B_0 by using the other two edges of R_1 instead. Now repeat this process: pick any rhombus R_2 that shares two edges with B_1; set $i_2 := d_2 + 1$, where d_2 is the minimum distance from the source to R_2; remove R_2 and form a new boundary B_2. Iterating this process an additional $\ell(w) - 2$ times produces $(i_1, i_2, \ldots, i_{\ell(w)})$, for which $s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}}$ represents a commutation class of reduced words for w.

For the other direction of the bijection, consider $\mathbf{i} = (i_1, i_2, \ldots)$ representing a commutation class for w (that is, $s_{i_1} s_{i_2} \cdots$ is a reduced word for w). From this, we construct an ordered tiling of $\mathsf{E}(w)$, as follows. For $k \geq 1$, set $w^{(k)} := s_{i_1} s_{i_2} \cdots s_{i_k}$. For $1 \leq k \leq \ell(w)$, the values $w^{(k)}(i_k)$ and $w^{(k)}(i_k + 1)$ label adjacent edges of the boundary B_{k-1}. Place a rhombus, R_k, so that two of its edges coincide with the edges labeled $w^{(k)}(i_k)$ and $w^{(k)}(i_k + 1)$ in B_{k-1}, and define the new boundary B_k from B_{k-1} by using the other two edges of R_k. $\square$

The procedure in one direction of Elnitsky's bijection is, perhaps, best understood through a large example. After this, we will return to our smaller example, the permutation $3241 \in \mathfrak{S}_4$.

Example 2.5. Consider the tiling $T \in \mathcal{T}(7456312)$ depicted in Figure 1. One way to select the rhombi $\{R_1, R_2, \ldots\}$ described in the proof of Theorem 1.1 is shown in Figure 3, where we have recorded only the subscript k of the rhombus R_k. The labeling in this figure represents the commutation class of the reduced word

$$s_3s_4s_2s_5s_6s_5s_3s_4s_3s_2s_1s_5s_2s_3s_6s_4s_5$$

for the permutation 7456312. Any other such labeling of these tiles would produce a different, but commutation equivalent, reduced word. For example, the labeling obtained by swapping the selections for R_{14} and R_{15}, both of which share two edges with the boundary B_{15}, as indicated in Figure 3, produces the commutation-equivalent reduced word

$$s_3s_4s_2s_5s_6s_5s_3s_4s_3s_2s_1s_5s_2s_6s_3s_4s_5.$$

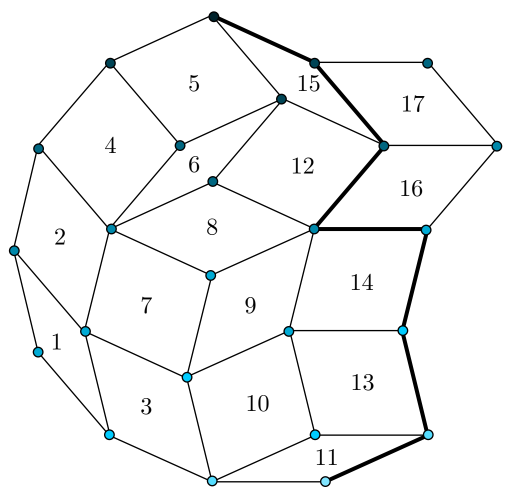
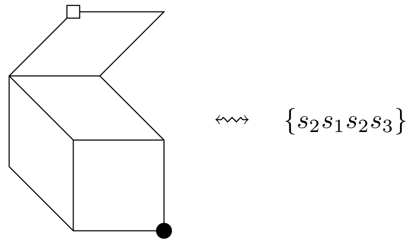


FIGURE 3. A labeling of the rhombi in an element of $\mathcal{T}(7456312)$, corresponding to the reduced word $s_3s_4s_2s_5s_6s_5s_3s_4s_3s_2s_1s_5s_2s_3s_6s_4s_5$ for the permutation 7456312. The boundary B_{15} is indicated by thick line segments.

We conclude this section with a demonstration of Elnitsky's correspondence between rhombic tilings and commutation classes of reduced words.

Example 2.6. The correspondence between rhombic tilings $\mathcal{T}(3241)$ and commutation classes $C(3241)$ is depicted in Figure 4.



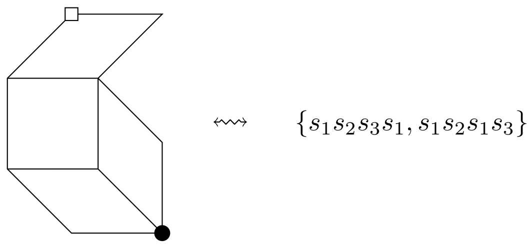


FIGURE 4. Elnitsky's bijection between rhombic tilings and commutations classes, for the permutation $3241 \in \mathfrak{S}_4$.

2.2. **P. Magyar's description of the Bott–Samelson desingularization.** Let $\mathsf{Flags}(\mathbb{C}^n, j)$ denote the variety of partial flags consisting of sequences of subspaces $\mathbb{C}^0 \subset F_1 \subset F_2 \subset \cdots \subset F_{j-1} \subset F_{j+1} \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n$ such that $\dim(F_i) = i$. Let $p_j : \mathsf{Flags}(\mathbb{C}^n) \to \mathsf{Flags}(\mathbb{C}^n, j)$ denote the projection

$$\begin{aligned} &p_j\left(\mathbb{C}^0 \subset F_1 \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n\right) \\ &\quad = \left(\mathbb{C}^0 \subset F_1 \subset \cdots \subset F_{j-1} \subset F_{j+1} \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n\right). \end{aligned}$$

The **fiber product** of two complete flag varieties with respect to $\mathsf{Flags}(\mathbb{C}^n, j)$ is

$$\begin{aligned} &\mathsf{Flags}(\mathbb{C}^n) \times_{\mathsf{Flags}(\mathbb{C}^n,j)} \mathsf{Flags}(\mathbb{C}^n) \\ &\quad = \{(F_\bullet^1, F_\bullet^2) \in \mathsf{Flags}(\mathbb{C}^n) \times \mathsf{Flags}(\mathbb{C}^n) \mid p_j(F_\bullet^1) = p_j(F_\bullet^2)\}, \end{aligned} \tag{1}$$

i.e. it consists of pairs of complete flags such that $F_\bullet^2$ agrees with $F_\bullet^1$ everywhere except possibly on the j-th subspace. P. Magyar [Mag98, Theorem 1] proves that the Bott–Samelson variety of $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ is isomorphic as a B-variety to the fiber product

$$BS^{\mathbf{i}} = F_\bullet^{(id)} \times_{\mathsf{Flags}(\mathbb{C}^n,i_1)} \mathsf{Flags}(\mathbb{C}^n) \times_{\mathsf{Flags}(\mathbb{C}^n,i_2)} \cdots \times_{\mathsf{Flags}(\mathbb{C}^n,i_m)} \mathsf{Flags}(\mathbb{C}^n). \tag{2}$$

This is the definition of a Bott–Samelson variety we use in this paper. Let $\mathbf{i}$ be a reduced word for w. It follows from P. Magyar's isomorphism that the map

$$\begin{aligned} \pi_{\mathbf{i}} : BS^{\mathbf{i}} &\to \mathsf{Flags}(\mathbb{C}^n) \\ (F_\bullet^{(id)}, F_\bullet^1, \ldots, F_\bullet^{\ell(w)}) &\mapsto F_\bullet^{\ell(w)} \end{aligned}$$

is a desingularization of the Schubert variety X_w.

3. PROOF OF THEOREM 1.1

We now show that $\mathcal{Z}_T$ set-theoretically describes the points of $BS^{(i_1,i_2,\ldots,i_{\ell(w)})}$. P. Magyar [Mag98, Theorem 1] describes the points in $BS^{(i_1,i_2,\ldots,i_{\ell(w)})}$ as lists

$(F_\bullet^0, \dots, F_\bullet^{\ell(w)})$ of $\ell(w)+1$ flags where $F_\bullet^0$ is the base flag, and such that $F_\bullet^k$ agrees with $F_\bullet^{k-1}$ everywhere except possibly on the i_k-th subspace. Such a list of flags transparently corresponds in a one-to-one fashion to a point in $\mathcal{Z}_T$:

- Given a T-flag $(V_x)_{x\in \mathrm{Vert}(T)}$ let $F_\bullet^0$ be the base flag which is on the base boundary B_0 and, in general, $F_\bullet^k$ be the flag $(V_x)_{x\in B_k}$. This list of flags is in $BS^{(i_1,i_2,\dots,i_{\ell(w)})}$.
- Let $(F_\bullet^0, \dots, F_\bullet^{\ell(w)})$ be a list of flags in $BS^{(i_1,i_2,\dots,i_{\ell(w)})}$. Given a vertex $x \in B_0$, let $V_x = F_d^0$, where d is the distance from x to H_0. For $k = 1, \dots, \ell(w)$, given a vertex $x \in B_k \setminus B_{k-1}$ let $V_x = F_d^k$, where d is the distance from x to H_0. Then $(V_x)_{x\in \mathrm{Vert}(T)}$ is a point in $\mathcal{Z}_T$.

Suppose that $\mathbf{j} = (j_1, j_2, \dots)$ is commutation equivalent to $\mathbf{i} = (i_1, i_2, \dots)$. It is well known to experts that $BS^{\mathbf{i}}$ and $BS^{\mathbf{j}}$ are isomorphic varieties, but we include a proof for completeness. It suffices to prove this when $\mathbf{j} = (i_1, \dots, i_{k+1}, i_k, \dots, i_{\ell(w)})$ differs from $\mathbf{i}$ only in positions k and $k+1$. The general result then follows by induction. Now, $(F_\bullet^0, \dots, F_\bullet^{\ell(w)})$ is equivalent to a list of subspaces $(V_1, V_2, \dots)$ satisfying:

- $\dim(V_k) = i_k$;
- $\mathbb{C}^{i_1-1} \subset V_1 \subset \mathbb{C}^{i_1+1}$; that is, V_1 is contained in the (i_1+1)-dimensional subspace of $F_\bullet^0$ and contains the (i_1-1)-dimensional subspace of $F_\bullet^0$;
- V_2 is contained in the (i_2+1)-dimensional subspace of $F_\bullet^1$ and contains the (i_2-1)-dimensional subspace of $F_\bullet^1$; and so on.

Since $|i_{k+1} - i_k| > 1$, the (i_k+1)-, (i_k-1)-, $(i_{k+1}+1)$-, and $(i_{k+1}-1)$-dimensional subspaces of $F_\bullet^k$ are precisely the subspaces of $F_\bullet^{k-1}$ with those dimensions. So if a generic element of $BS^{\mathbf{i}}$ is $(V_1, V_2, \dots)$, then a generic element of $BS^{\mathbf{j}}$ is $(V_1, V_2, \dots, V_{k+1}, V_k, \dots)$. That is, the isomorphism by switching factors,

$$\begin{aligned}(3)\quad \tau_k : \mathsf{Gr}_{i_1}(\mathbb{C}^n) \times \cdots \times \mathsf{Gr}_{i_k}(\mathbb{C}^n) \times \mathsf{Gr}_{i_{k+1}}(\mathbb{C}^n) \times \cdots \\ \to \mathsf{Gr}_{i_1}(\mathbb{C}^n) \times \cdots \times \mathsf{Gr}_{i_{k+1}}(\mathbb{C}^n) \times \mathsf{Gr}_{i_k}(\mathbb{C}^n) \times \cdots,\end{aligned}$$

restricts to a canonical isomorphism from $BS^{(i_1,i_2,\dots)}$ to $BS^{(i_1,\dots,i_{k+1},i_k,\dots)}$. In other words, $\mathcal{T}(w)$ indexes Bott–Samelson varieties up to commutation equivalence.

Given $\mathbf{i} = (i_1, i_2, \dots)$ representing a commutation class for w, the inverse map to S. Elnitsky's bijection constructs an ordered tiling T of $\mathsf{E}(w)$. For this T, we have that $\mathcal{Z}_T \cong BS^{\mathbf{i}}$, as desired. (The incidence relations that we obtain for the vector spaces $(V_x)_{x\in \mathrm{Vert}(T)}$ are equivalent to those in P. Magyar [Mag98, §1.3].) □

4. Flips and zonotopal tilings

4.1. **Flips.** Any pair of rhombic tilings of $\mathsf{E}(w)$ is connected by a sequence of hexagon "flips" [Eln97, Section 3]. The effect of a single flip is depicted in Figure 5.

This flip has a geometric interpretation. Let $T, T' \in \mathcal{T}(w)$ be two rhombic tilings that differ by a single flip. Let T_H be the tiling of $\mathsf{E}(w)$ obtained from T (or, equivalently, from T') by erasing the three internal edges by which T and T' differ and placing a hexagonal tile in the flip location. As before, associate vector spaces V_x to each vertex x in T_H, where $\dim(V_x)$ equals the distance from x to the source H_0. The resulting space $\mathcal{Z}_{T_H}$ is similar to a Bott–Samelson variety: instead of being $\ell(w)$-fold iterated $\mathbb{CP}^1$-bundles over the base flag, we replace three

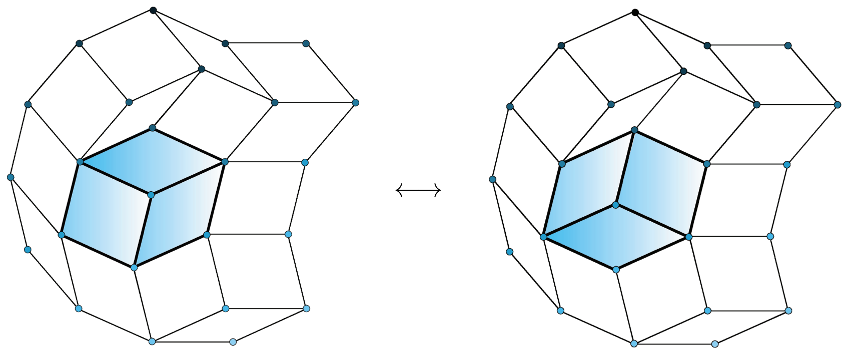


FIGURE 5. Two elements of $\mathcal{T}(7456312)$, related by a hexagon flip.

of these $\mathbb{CP}^1$-bundles (corresponding to either triple of rhombi in the hexagon) by a $\mathsf{Flags}(\mathbb{C}^3)$-bundle. (We describe this variety in more detail below.) We then have

$$\begin{tikzcd} \mathcal{Z}_T \arrow[dr, two heads, "f_T"] & & \mathcal{Z}_{T'} \arrow[dl, two heads, "f_{T'}"'] \\ & \mathcal{Z}_{T_H} & \end{tikzcd}$$

where the two maps are the projections determined by forgetting the vector space attached to the internal vertex of the hexagon.

4.2. **Zonotopal tilings.** The tiling T_H described above is a special case of the "zonotopal" tilings of Elnitsky polygons, which were studied by the third author in [Ten06]. To be precise, a **2-zonotope** is the projection of a regular q-dimensional cube onto the (2-dimensional) plane; equivalently, a 2-zonotope is a centrally symmetric convex polygon. A **zonotopal tiling** of a region is a tiling by 2-zonotopes. Figure 6 shows a zonotopal tiling of $\mathsf{E}(87465312)$ using one octagon, three hexagons, and ten rhombi.

Let $\mathcal{T}_{zono}(w)$ be the collection of zonotopal tilings of $\mathsf{E}(w)$, in which the tiles (2-zonotopes) have sides of length one and edges parallel to edges of $\mathsf{E}(w)$. Because rhombi are a type of 2-zonotope, we have $\mathcal{T}(w) \subseteq \mathcal{T}_{zono}(w)$.

Given a zonotopal tiling $Z \in \mathcal{T}_{zono}(w)$, we can define its corresponding **generalized Bott–Samelson variety** $\mathcal{Z}_Z$ by extending the construction from Section 1. Define a Z-flag to be an assignment $\mathcal{V}$ of a linear subspace $V_x \subseteq \mathbb{C}^n$ to each vertex x in the zonotopal tiling Z, subject to the conditions:

- the dimension of V_x is the minimal path length from H_0 to x along tile edges;
- V_{H_i} is the span of the first i standard basis vectors of $\mathbb{C}^n$; and
- for adjacent vertices x and y in Z with y further from H_0, we have $V_x \subset V_y$.

Now $\mathcal{Z}_Z$ is defined to be the parameter space

$$\mathcal{Z}_Z := \left\{\mathcal{V} = (V_x)_{x\in \mathtt{Vert}(Z)} : \mathcal{V} \text{ is a } Z\text{-flag}\right\} \subset \prod_{x\in\mathtt{Vert}(Z)} \mathsf{Gr}_{\dim(V_x)}(\mathbb{C}^n).$$

Let T be a rhombic tiling that refines Z; $\mathcal{Z}_T$ may be constructed as iterated $\mathbb{CP}^1$-bundles over a point. In the analogous construction of $\mathcal{Z}_Z$, for each $2k$-gon of

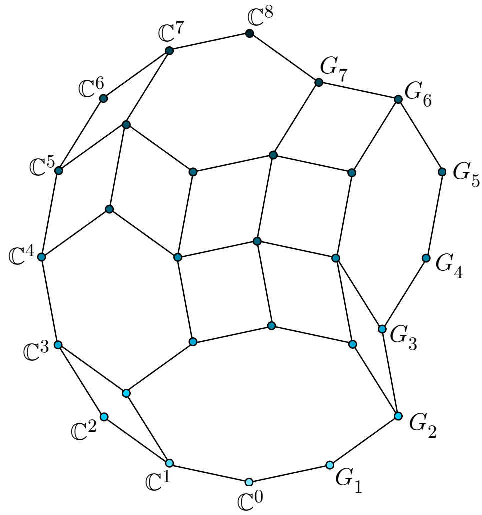


FIGURE 6. A zonotopal tiling for the permutation 87465312

Z, we replace k $\mathbb{CP}^1$-bundles with a $\mathsf{Flags}(\mathbb{C}^k)$-bundle. It therefore follows that the variety $\mathcal{Z}_Z$ is smooth of dimension $\ell(w)$. Define $\pi_Z : \mathcal{Z}_Z \to X_w$ by forgetting all vector spaces except those labeled by the vertices $G_1, G_2, \ldots, G_{n-1}$.

Theorem 4.1. *Given a zonotopal tiling $Z \in \mathcal{T}_{zono}(w)$, its corresponding generalized Bott–Samelson variety $\mathcal{Z}_Z$ together with the map $\pi_Z : \mathcal{Z}_Z \to X_w$ is a resolution of singularities.*

Proof. Let $\pi_T : \mathcal{Z}_T \to X_w$ be a Bott–Samelson resolution where T is any rhombic tiling that refines Z. By [Mag98, Theorem 1], π_T is birational, so let π'_T be its rational inverse. Let $f : \mathcal{Z}_T \twoheadrightarrow \mathcal{Z}_Z$ be the projection determined by forgetting the vector spaces attached to the internal vertices of T that are not vertices of Z. Since f is surjective, the image of π_Z is indeed X_w, and the following commutative diagram implies that $f \circ \pi'_T$ is a rational inverse to π_Z:

$$\begin{tikzcd} \mathcal{Z}_T \arrow[r, two heads, "f"] \arrow[dr, "\pi_T"'] & \mathcal{Z}_Z \arrow[d, "\pi_Z"] \\ & X_w \end{tikzcd}$$

It follows that $\pi_Z : \mathcal{Z}_Z \to X_w$ is also a resolution of singularities. $\square$

There are many characteristics of these zonotopal resolutions on which further study is warranted. For example, it would be interesting to have a characterization of those zonotopal tilings Z that give rise to *small resolutions*.

The zonotopal tilings $\mathcal{T}_{zono}(w)$ of $\mathsf{E}(w)$ have a natural poset structure, as studied by the third author in [Ten06]. The order relation in this poset is given by reverse edge inclusion. Thus the rhombic tilings are the minimal elements in the poset. A pair of rhombic tilings differs by a single hexagon flip if and only if they are covered by a common element. Similarly, one can get a broader sense of how closely two rhombic tilings (equivalently, two commutation classes of reduced words for w) are related by determining their least upper bound in this poset. Geometrically, the relations in the poset $\mathcal{T}_{zono}(w)$ correspond to the projections $\mathcal{Z}_Z \twoheadrightarrow \mathcal{Z}_{Z'}$ between two generalized Bott–Samelsons for X_w.

By [Ten06, Theorem 6.13], the poset of zonotopal tilings of $\mathsf{E}(w)$ has a unique maximal element $\hat{Z}$ exactly in the case that w avoids the patterns 4231, 4312, and 3421. In this case, there is a distinguished $\mathcal{Z}_{\hat{Z}}$ with a projection $\mathcal{Z}_Z \twoheadrightarrow \mathcal{Z}_{\hat{Z}}$ from every other generalized Bott–Samelson. Such permutations have been enumerated by T. Mansour [Man06].

For comparison, consider those Elnitsky polygons all of whose zonotopal tilings contain no hexagonal tiles (equivalently, those polygons with a unique zonotopal tiling). These correspond to 321-avoiding permutations, which are exactly those whose reduced words contain no long braid moves [BJS93, Theorem 2.1] (see also [Ten17, Section 3] for more general results relating pattern avoidance and reduced words). The unique tiling in this case is a deformation of the skew shape associated to the permutation by considering its *Rothe diagram* and removing empty rows and columns. A standard filling orders the tilings in the sense of [Eln97] (and the final paragraph of the proof of Theorem 1.1).

We now have the following result (cf. [Ele15, Remark 3.1], where this fact for ordinary Bott–Samelsons is noted).

Proposition 4.2. *Suppose that $Z \in \mathcal{T}_{zono}(w)$ and that the number of $2i$-sided tiles in Z is t_i, for each $i \geq 1$. Then the Poincaré polynomial of the cohomology ring $H^\star(\mathcal{Z}_Z)$ is*

$$\sum_{k=0}^{\ell(w)} \dim H^{2k}(\mathcal{Z}_Z) q^k = \prod_{i \geq 1} [i]_q!^{t_i},$$

where $[i]_q := 1 + q + q^2 + \cdots + q^{i-1}$ and $[i]_q! := [i]_q[i-1]_q \cdots [1]_q$.

Proof. The variety $\mathcal{Z}_Z$ is constructed as iterated flag bundles over a point, where t_i of the fibrations are by $\mathsf{Flags}(\mathbb{C}^i)$. It is a standard fact (following from the Schubert decomposition of $\mathsf{Flags}(\mathbb{C}^i)$) that the Poincaré polynomial of $H^\star(\mathsf{Flags}(\mathbb{C}^i))$ is $[i]_q!$ (indeed, $[i]_q!$ is the ordinary generating function for $\mathfrak{S}_i$ with each permutation weighted by Coxeter length). The proposition now follows from the Leray–Hirsch theorem (cf. [Hat02, Theorem 4D.1]). □

5. Additional discussion

One may reformulate certain results about $BS^{\mathbf{i}}$ in terms of rhombic colorings; we refer to [Esc16, Section 3.2] for background as well as further references.

Proposition 5.1. *For $T \in \mathcal{T}(w)$, the T-fixed points of $\mathcal{Z}_T$ (under the diagonal action) are in one-to-one correspondence with bipartitions of the rhombi of T.*

Before proving this proposition, we remark that the T-fixed points of the generalized Bott–Samelson variety $\mathcal{Z}_Z$ corresponding to a zonotopal tiling Z do not correspond to bipartitions of the tiles. The Elnitsky polygon $\mathsf{E}(w_0)$ for $w_0 = [n, n-1, \ldots, 1]$ is a zonotope. Let Z_0 be the tiling that only consists of the tile $\mathsf{E}(w_0)$. Then $\mathcal{Z}_{Z_0}$ equals the flag variety $\mathsf{Flags}(\mathbb{C}^n)$ which has $n!$ T-fixed points.

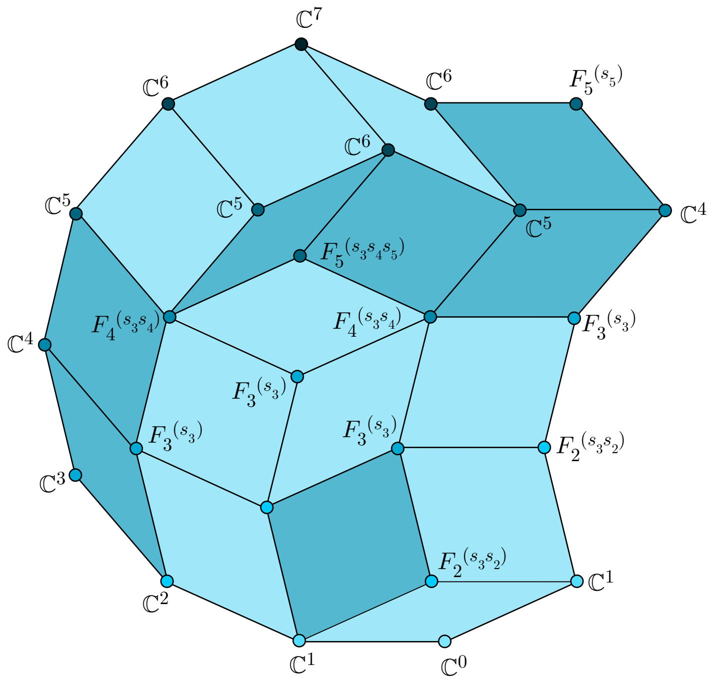


FIGURE 7. A 2-coloring corresponding to a T-fixed point of $\mathcal{Z}_T$.

Proof of Proposition 5.1. Consider a 2-coloring of the rhombi of T representing the bipartition (as shown in Figure 7). There is a unique way to choose $\{V_x\}_{x\in\mathtt{Vert}(T)}$ such that

(1) each V_x is the span of a subset of the standard basis of $\mathbb{C}^n$, and,
(2) for any rhombus, its two vector spaces of common dimension are the same (resp., different) if the rhombus is light-colored (resp., dark-colored).

Since the T-action is diagonal, if $\{V_x\}_{x\in\mathtt{Vert}(T)}$ is a T-fixed point of $\mathcal{Z}_T$, then each V_x must be T-fixed, i.e., each V_x must be spanned by a subset of the standard basis $\{e_1,\ldots,e_n\}$. Using the required containment relations, we can inductively determine V_x for each vertex of T by following an ordering of the rhombi given by a representative of the commutation class of T. At a particular colored rhombus, we make the two vector spaces of common dimension the same (resp., different) if the rhombus is light-colored (resp., dark-colored):

$$\begin{tikzcd}[column sep=small]
 & V_c = V_a \oplus \langle e_b, e_c\rangle \arrow[dl, dash] \arrow[dr, dash] & \\
V_b = V_a \oplus \langle e_b\rangle \arrow[dr, dash] & = (\text{resp.}, \neq) & V_x \arrow[dl, dash] \\
 & V_a &
\end{tikzcd}$$

Conversely, every T-fixed point can be indicated by such a coloring. $\square$

Demazure [Dem74] used the T-fixed points to prove that the image of $BS^{(i_1,i_2,\ldots)}$ under the Bott–Samelson map π is indeed the Schubert variety $X_{s_{i_1}s_{i_2}\ldots}$. These fixed points are also useful in the study of moment polytopes of Bott–Samelson varieties. Bott–Samelson varieties are symplectic manifolds and the T-action is Hamiltonian. Therefore $BS^{\mathbf{i}}$ has a *moment map* $\Phi : BS^{\mathbf{i}} \to \mathbb{R}^n$ and, by [Ati82, GS82], the image of Φ is the convex hull of the image under Φ of the T-fixed points. This polytope is the *moment polytope* of $BS^{\mathbf{i}}$. In [Esc16], the first author studied the moment polytope of the general fiber of the Bott–Samelson resolution. Other uses of the T-fixed points include describing the equivariant cohomology of Bott–Samelson varieties varieties, e.g., [Wil06].

These 2-colorings also correspond to a stratification of $\mathcal{Z}_T$ by smaller Bott–Samelsons. The unique smallest stratum corresponds to the all-light coloring, whereas the unique largest stratum corresponds to the all-dark one.

Proposition 5.2. *Given a 2-coloring C of the rhombi of T, let*

$$\mathcal{S}(C) := \{(V_x)_{x\in\mathtt{Vert}(T)} \,|\, V_x = V_y \textit{ if the rhombus containing } x \textit{ and } y \textit{ is light-colored}\}.$$

The variety $\mathcal{Z}_T$ is stratified by the $\mathcal{S}(C)$ for any 2-coloring C of T, and each $\mathcal{S}(C)$ is a Bott–Samelson variety.

Proof. To verify that the $\mathcal{S}(C)$ give a stratification we must check that they are varieties, their union equals $\mathcal{Z}_T$, and the intersection of two of these varieties is the union of finitely many $\mathcal{S}(C)$. Let $\mathbf{i} = (i_1, i_2, \ldots)$ be a reduced word corresponding to T, as constructed in Theorem 2.4. A 2-coloring C corresponds to the subword $\mathbf{j}$ of $\mathbf{i}$ that uses only the entries of $\mathbf{i}$ coming from dark-colored rhombi. It is straightforward to check that $\mathcal{S}(C)$ is isomorphic to $BS^{\mathbf{j}}$, so the proposed sets are varieties. Since the coloring C_{dark} with all tiles dark-colored gives the stratum $\mathcal{S}(C_{dark}) = \mathcal{Z}_T$, the union of all the strata equals $\mathcal{Z}_T$. Finally, given two colorings C and C', let $C \wedge C'$ be the coloring obtained by making a rhombus light-colored if the rhombus is light-colored in either C or C'. We then have that $\mathcal{S}(C) \cap \mathcal{S}(C') = \mathcal{S}(C \wedge C')$. $\square$

This stratification is used by R. Vakil (together with similar diagrams) [Vak06], for example, to study certain degenerations of *Richardson varieties* in Grassmannians.

In [Eln97], the author extends his main construction to the other Weyl groups of classical Lie type. We suspect that these generalized tilings can be used to describe the Bott–Samelsons for Schubert varieties in parabolic quotients of the associated Lie groups. It seems interesting to us, and potentially useful, to determine if this is the case.

Acknowledgments

We thank Allen Knutson and Alexander Woo for helpful comments. We are very grateful to an anonymous referee for comments that significantly improved the exposition of this paper.

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